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Quantum cosmology and recollapse

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The wave function of the Universe defined by the Hartle-Hawking proposal is studied for two minisuperspace models, the Friedmann-Robertson-Walker universe with a massive scalar field and the anisotropic Kantowski-Sachs cosmology. Solutions of the Wheeler-DeWitt equation are evaluated numerically. We also examine the trajectories to which these correspond in the classical limit. Attention is focused on the fact that most classical trajectories recollapse to a singularity. They will add an oscillatory component to the wave function in the region of small three-volume where it is predominantly exponential. They will thus modify the original boundary conditions. The Kantowski-Sachs universes generically evolve from isotropy during expansion to increasing anisotropy during recollapse, ending as a black-hole interior. We comment, therefore, on the arrow of time naturally induced by the Hartle-Hawking proposal.

I. INTRODUCTION

Observations of the cosmic microwave background radiation demonstrate that our Universe was initially remarkably homogeneous and isotropic. This makes our Universe very special in the class of all possible cosmological models[1] (inflation alleviates, but does not solve, this problem[2]). It has been suggested by Hawking,[3] however, that these very special initial conditions are a natural consequence of the quantum state of the Universe being described by a path integral over compact Euclidean geometries.

Of course, the quantum state or wave function of the Universe is exceedingly difficult to calculate. It is possible, however, to gain some insight into its behavior by studying simplified models with a restricted number of degrees of freedom, for example, the homogeneous and isotropic minisuperspace model. Indeed, this latter model has been studied extensively and perturbative treatments about it[4] have shown that it provides a good approximation to the full wave function of the universe at small "initial" volumes. It can sometimes prove difficult in such models, however, to distinguish between properties which are general, and those introduced by the imposition of a given symmetry.

In this paper we review some of the implications of the Hartle-Hawking proposal for the Friedmann-Robertson-Walker (FRW) minisuperspace model with a massive scalar field. By way of comparison, we also investigate a less symmetric minisuperspace model with an added anisotropic degree of freedom, the Kantowski-Sachs (KS) universe. We concentrate on the fate of the universe in this formulation, an emphasis which motivates the choice of this second model. KS cosmologies provide a far more realistic picture of a collapsing universe than is possible with a FRW model.[5] In the final stages of recollapse, the KS metric corresponds to the analytically extended region in the interior of a black hole.

In Sec. II we briefly introduce the necessary formalism in which to enunciate the Hartle-Hawking proposal. The two models are then studied in Sec. III with their respective wave functions. They have been calculated numerically using boundary conditions in minisuperspace given by the Hartle-Hawking proposal. In Sec. IV we examine the classical universes corresponding to the respective wave functions. In Secs. V—VII we study the implications of the Hartle-Hawking proposal for the occurrence of singularities in quantum cosmology, the anisotropy of the universe, and its arrow of time.

II. THE HARTLE-HAWKING PROPOSAL

Quantum cosmology is the study of the quantum evolution of a three-surface S representing the universe at a given time. From the wave function Ψ one can obtain the probability that S will have evolved from an initial surface S_i.

In analogy to the quantum-mechanical treatment of a point particle, it is possible to define Ψ by a Feynman path integral

$$\Psi[S,S_i]=\int_C d[g_{\mu\nu}]d[\phi]e^{-I_E}\,, \tag{2.1}$$

where C is a class of four-geometries and matter fields which match the initial surface S_i to the final S. I_E is the Euclidean action for a given four-geometry and $d[g_{\mu\nu}]$ and $d[\phi]$ define the measure of the path integral.

The wave function obeys a differential equation

$$H\Psi=0\,, \tag{2.2}$$

where H is the Hamiltonian operator obtained from the Lagrangian of the classical theory, with the conjugate momenta replaced by the appropriate derivatives.

Describing a four-geometry by the metric

$$ds^2=-(N^2-N_iN^i)dt^2+2N_idx^idt+h_{ij}dx^idx^j\,, \tag{2.3}$$

where N and N_i are, respectively, the lapse and shift functions and h_{ij} is the metric describing the three-surface S, we can split Eq. (2.1) into two separate parts

$$N_i H^i \Psi = 0 \tag{2.4}$$

and

$$N H^0 \Psi = 0 \ . \tag{2.5}$$

The first corresponds to the invariance of Ψ under diffeomorphisms in the three-surface S. The second part, the Wheeler-DeWitt equation, is the evolution equation for Ψ on the space of three-geometries. It is the gravitational analog of the zero-energy Schrödinger equation.

This formulation is not complete, however, without the imposition of boundary conditions to specify Ψ uniquely. Hartle and Hawking[6] have made a proposal for these by restricting the class C in (2.1) to compact four-geometries with regular matter fields. This corresponds to the assertion that "the universe does not have any boundaries in space or time (at least in the Euclidean regime)."[3] For a more detailed review of this proposal refer to Refs. 6 and 7.

III. TWO MINISUPERSPACE MODELS

The implications of this proposal are difficult to investigate because of the infinite dimensionality of superspace, so we restrict our attention to only two or three degrees of freedom. With few exceptions, previous work in quantum cosmology has been based around simple FRW models with homogeneous three-surfaces of topology S^3. We contrast this by considering a minisuperspace model based on the Kantowski-Sachs (KS) cosmology, homogeneous three-surfaces with topology $S^1 \times S^2$. This extra nonperturbative degree of freedom (anisotropy) holds out the hope of a more realistic insight into the ultimate fate of the universe in this framework.

We will describe the interaction of gravity with a minimally coupled homogeneous massive scalar field by the Lorentzian action:

$$I = \int_M d^4x (L_g + L_m) + \frac{1}{8\pi G}\int_{\partial M} d^3x\, h^{1/2} K \ , \tag{3.1}$$

where K is the second fundamental form on the boundary ∂M,

$$L_g = \frac{m_p{}^2}{16\pi}(-g)^{1/2}\,{}^4R \tag{3.2}$$

is the Einstein-Hilbert Lagrangian, and we take the matter Lagrangian

$$L_m = -\tfrac{1}{2}(-g)^{1/2}\left[\partial_\mu \frac{\phi}{2\pi\sigma}\partial^\mu \frac{\phi}{2\pi\sigma} + m^2\left[\frac{\phi}{2\pi\sigma}\right]^2\right] , \tag{3.3}$$

where $\sigma = 1/m_p$.

(i) FRW model: The homogeneous and isotropic minisuperspace model based on the FRW metric

$$ds^2 = -N^2(t)dt^2 + a^2(t) d\Omega_{(3)}{}^2 \tag{3.4}$$

should by now be familiar. The study of this simple model has been very significant in the development of quantum cosmology and its many encouraging features have been generalized, a process justified by linear perturbative treatment about it.[4]

The Wheeler-DeWitt equation, $H\Psi = 0$, becomes

$$\frac{1}{2}\left[\frac{\partial^2}{\partial a^2} + \frac{\partial}{a\partial a} - \frac{1\partial^2}{a^2\partial\phi^2} - a^2 + a^4 m^2\phi^2\right]\Psi = 0 \ . \tag{3.5}$$

Using the semiclassical approximation to the path integral (2.1), we can evaluate the wave function for small three-geometries[7] and obtain

$$\Psi \approx C \exp\left[-\frac{1}{3m^2\phi^2}[1 - (1 - m^2\phi^2 a^2)^{3/2}]\right] , \tag{3.6}$$

where C is the prefactor and can be taken to be unity. At zero volume, we then have the boundary condition $\Psi = 1$.

In the more convenient coordinates

$$x = a \sinh\phi, \quad y = a \cosh\phi \tag{3.7}$$

the differential operator is diagonalized and Eq. (3.5) becomes amenable to straightforward numerical solution. The boundary conditions are reduced to having $\Psi = 1$ on the "light cone" ($x = \pm y$) of minisuperspace. The initial data for Ψ are given on two characteristics of the WD equation and this is sufficient to specify the wave function uniquely, that is, the normal derivative is not necessary.

Figure 1 illustrates the wave function for the FRW model with $m = 5$ integrated with a second-order leapfrog algorithm (outlined in a somewhat different context in Ref. 8). A small note of caution should be mentioned because the leapfrog algorithm will be a valid approximation only when we observe the condition $V(x,y) \ll 1/\delta x^2$, where δx is the grid step size. Numerical instabilities will appear and grow exponentially otherwise. In the oscillatory regions, leapfrog is also known to exhibit an aliasing instability: high-frequency modes pile up with wavelength equal to the grid size δx (Ref. 9).

Figure 1 is in fact a careful repetition of Fig. 2 in Ref. 10 and several minor differences due to these considerations are apparent.

(ii) KS model: The second minisuperspace model considered corresponds to a Kantowski-Sachs cosmology with a massive scalar field. The three-surfaces are homogeneous and have topology $S^1 \times S^2$. They are the boundary of a four-manifold with metric

$$ds^2 = -N^2(t)dt^2 + a^2(t)dr^2 + b(t) d\Omega_{(2)}{}^2 \ , \tag{3.8}$$

where $d\Omega_{(2)}{}^2$ is the metric on the two-sphere and r is identified periodically. The Wheeler-DeWitt equation becomes

$$\left[\frac{a\partial^2}{2b^2\partial a^2} + \frac{1\partial}{ab\partial b} - \frac{1\partial^2}{b\partial a\partial b} + \frac{1\partial^2}{2ab^2\partial\phi^2} + V\right]\Psi = 0 \tag{3.9}$$

with

$$V = \frac{a}{2} - \frac{ab^2 m^2\phi^2}{2} \ .$$

The wave function for the KS model has also been

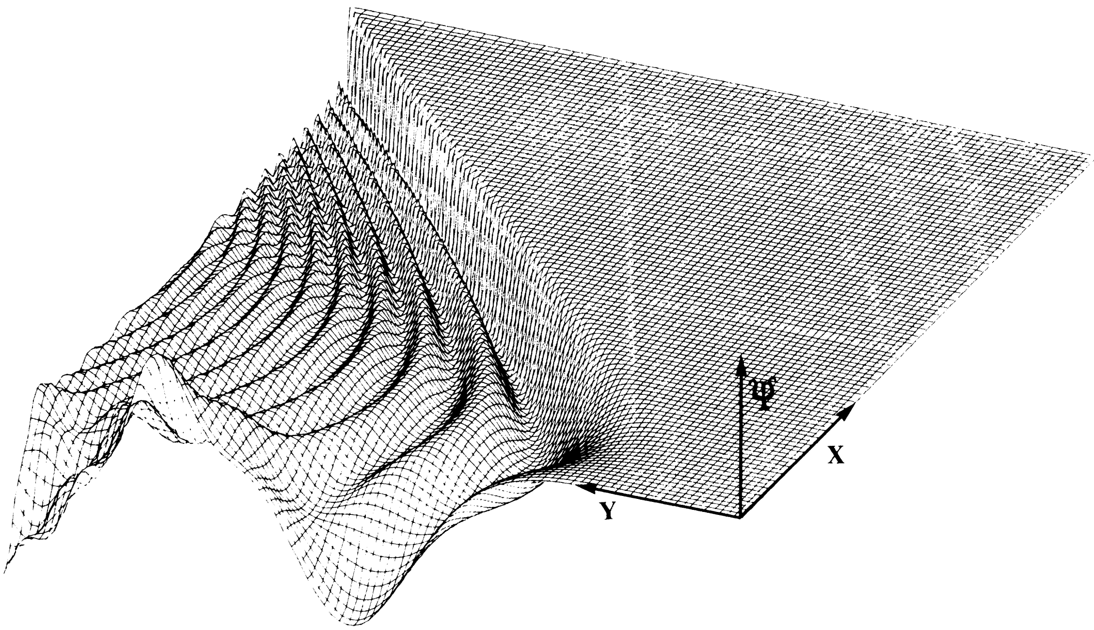

FIG. 1. The wave function for the FRW model with a massive scalar field (the mass parameter m is taken to be 5). Only half of the minisuperspace is shown ($x = a\sinh\phi \geq 0$, $y = a\cosh\phi$). The wave function is exponential for small values of a, near the "light cone" of minisuperspace, but oscillates rapidly for large a.

evaluated numerically. The problem in three dimensions is more formidable but with the coordinates

$$a = \exp(p), \quad b = \exp(q - p), \quad \phi = \phi \tag{3.10}$$

and a convenient choice of operator ordering, the D'Alambertian becomes diagonal. The resulting Wheeler-DeWitt equation is again soluble using a straightforward leapfrog algorithm of fixed grid step size.

The wave function can be plotted for a, b, or ϕ constant slices of minisuperspace. In the a or b constant slices, we observe that the wave function oscillates for

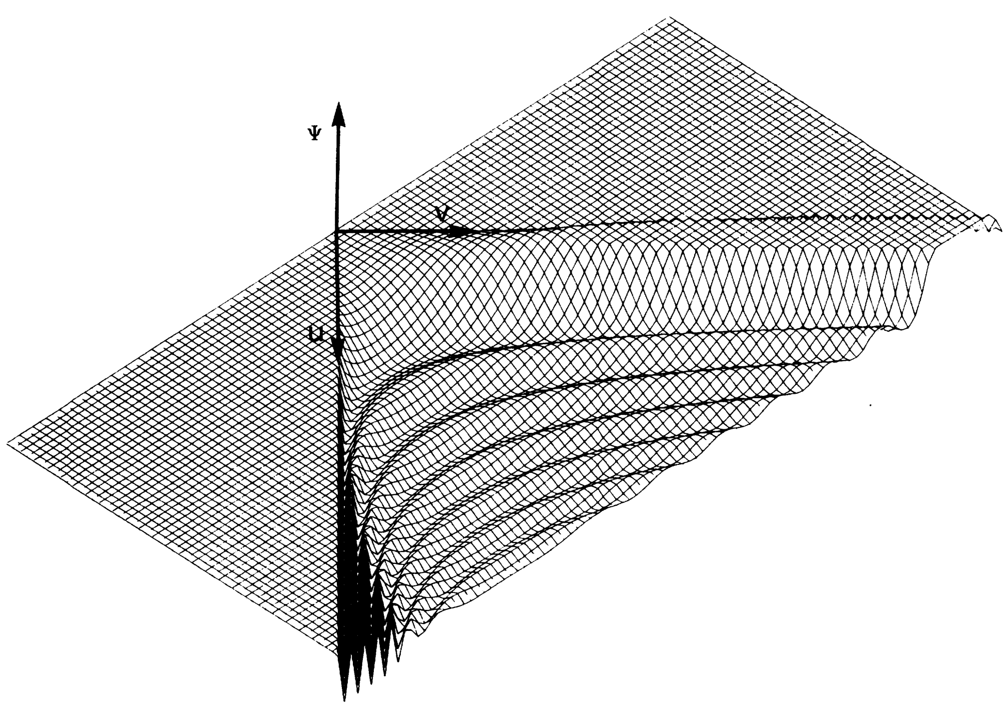

FIG. 2. The wave function for the Kantowski-Sachs model with a massive scalar field in coordinates $u = b, v = a^2 b$. The wave function is exponential for small values of b but oscillates for large b.

large values of the volume ab^2. This confirms the analytic approximations made by Laflamme.[11] These figures are not presented here (see Shellard[12]). It is more interesting to analyze the wave function for slices of constant ϕ. The qualitative features of Ψ can be more easily seen if we take the coordinates

$$u=b, \quad v=a^2 b \ . \tag{3.11}$$

u and v are null coordinates in minisuperspace if the scalar field is constant. This is true for large ϕ and small volume because of the Hartle-Hawking proposal. It is easy to devise an algorithm similar to the one used in the FRW model using $x=u-v$ and $y=u+v$ where only the potential of the Wheeler-DeWitt equation has to be modified. This model in fact reduces to the one studied in Ref. 11, a vacuum KS model with nonzero cosmological constant.

The semiclassical approximation for the wave function reduces to

$$\Psi=C\exp\left[\tfrac{1}{2}(y^2-x^2)^{1/2}\left[1-\frac{m^2\phi^2}{12}(x+y)^2\right]^{1/2}\right] \tag{3.12}$$

for small three-geometries. This implies (taking $C=1$) that the wave function is equal to unity on the boundary of the minisuperspace ($x=\pm y$).

Figure 2 shows the numerical solutions of Ψ for this case. The wave function is exponential for small values of $u=(x+y)/2$ but oscillates for large values. In this region the wave function will represent Lorentzian universes. They will belong to the Schwarzschild—de Sitter family.

IV. THE CLASSICAL SOLUTIONS

Before discussing the classical solutions for these models we must know in which regions of superspace a classical description will be satisfactory. In the oscillatory regions we can write Ψ with the WKB ansatz

$$\Psi=\mathrm{Re}(Ce^{iS}) \ . \tag{4.1}$$

In the case where the prefactor varies slowly in comparison to the phase S, that is,

$$\frac{\nabla^2 C}{C} \ll (\nabla S)^2 \tag{4.2}$$

we obtain, from the Wheeler-DeWitt equation,

$$(\nabla S)^2+V\approx 0 \tag{4.3}$$

and

$$2\nabla S\nabla C+\nabla^2 S=0 \ . \tag{4.4}$$

The first equation is the Hamilton-Jacobi equation for general relativity. The phase S will correspond to a family of solutions of the classical field equations. The gradient of S, ∇S, will indicate the direction of the classical trajectories in superspace. In the models under investigation the Hartle-Hawking proposal picks out one particular Hamilton-Jacobi function $S_{\rm HH}$, the one which is the analytic continuation of the Euclidean action. Because of the regularity conditions imposed on the Euclidean equations of motion it is not surprising that $S_{\rm HH}$ has some very special features.

The second equation is a first-order differential equation for C. Once we know the function S we can calculate C along a particular WKB trajectory. Effectively, the wave function is decomposed into a superposition of WKB wave packets, $\Psi=\mathrm{Re}C_n\exp(iS_n)$, following Hawking and Page.[13] The WKB approximation will be valid along the wave-packet trajectory, provided (4.2) is satisfied.

The classical solutions for the closed FRW model with a scalar field have been examined by Hawking[7] and then in considerably more detail by Page.[14] There is a significant class of solutions which are nonsingular, countably many which are periodic, and perhaps a fractal set of perpetually bouncing aperiodic solutions.

The Lorentzian equations of motion for a and ϕ are (for $N=1$)

$$a\ddot{a}+\frac{\dot{a}^2}{2}+\frac{1}{2}+\frac{3a^2\dot{\phi}^2}{2}-\frac{3a^2m^2\phi^2}{2}=0 \ , \tag{4.5a}$$

$$\ddot{\phi}+3\frac{\dot{a}}{a}\dot{\phi}+m^2\phi=0 \ , \tag{4.5b}$$

with the constraint

$$\dot{a}^2=a^2\dot{\phi}^2+m^2a^2\phi^2-1 \ . \tag{4.5c}$$

There is a degree of redundancy with the field equations written in this form, but this is a positive feature for their numerical solution. The constraint equation provides an independent check on the accuracy of the numerical analysis.

The Lorentzian paths selected by the Hartle-Hawking proposal are those obtained by the analytic continuation of the compact Euclidean paths. The Euclidean solutions for large ϕ are approximately given by

$$a\approx\left[\frac{1}{m^2\phi^2}\right]^{1/2}\sin[(m^2\phi^2)^{1/2}\tau] \ , \tag{4.6}$$

$$\phi\approx\phi_0\ (\text{constant}) \ . \tag{4.7}$$

The analytic continuation is obtained by rotating to imaginary time,

$$\tau=\frac{\pi}{2(m^2\phi^2)^{1/2}}+it \ ,$$

from which we obtain

$$a\approx\left[\frac{1}{m^2\phi^2}\right]^{1/2}\cosh[(m^2\phi^2)^{1/2}t] \ , \tag{4.8}$$

$$\phi\approx\phi_0 \tag{4.9}$$

at small t, $t\approx 0$. This gives the following initial conditions for the Lorentzian field equations

$$a\approx\frac{1}{m|\phi|}, \quad \phi\approx\phi_0, \quad \dot{a}\approx 0, \quad \dot{\phi}\approx 0 \quad \text{at } t=0 \ . \tag{4.10}$$

The equations of motion have been solved numerically using a fourth-order Runge-Kutta algorithm. Variable step length was incorporated to compensate at small volumes and the constraint (4.3) was observed to remain within reasonable limits. Some of the more exceptional classical solutions are illustrated in Fig. 3, plotted in the coordinates defined in (3.6).

If we choose the initial ϕ_0 such that when a reaches a maximum radius ($\dot{a}=0$), we simultaneously have $\phi=0$ or $\dot{\phi}=0$, then it is clear that the evolution must be time symmetric about this point. It will consequently bounce at its minimum radius. Bounces 1 and 3 in Fig. 3 correspond to the former, bounce 2 to the latter condition. Such periodic bouncing solutions (having a local minimum for a, i.e., $\dot{a}=0$) are more the exception than the rule and require very fine tuning of ϕ_0 just to repeat the period twice (about one part in 10^{10}). Indeed this system exhibits classical instability about periodic solutions. The evolution of any wave packet which includes one of these periodic solutions will be irreversible and indeterminate, reminiscent of chaotic behavior.

Typical behavior, however is more like that shown in Fig. 4. After an inflationary epoch with $\dot{\phi}$ remaining comparatively small, ϕ will begin to oscillate, a will expand to a maximum radius and then begin to recollapse. In general, the trajectories become singular ($a \rightarrow 0, \phi \rightarrow \infty$), ending up far from the starting point in configuration space because of the initial inflation. Of all the trajectories surveyed, none were observed to deflate except those fined tuned to be very close to a periodic solution. (Inflation/deflation was checked numerically by observing the ratio of $\dot{a}/a$ and $\dot{\phi}$. It is also evident from the shape of the trajectories.) This does not appear to be in agreement with the conclusion of Ref. 15, that the set of inflating solutions is a set of measure one (though strictly, their analysis is only valid for $k=0,-1$ models and we have only studied values of $\phi_0<2, m=1$). If this were so in the closed model one would expect nondeflating solutions to be rare exceptions. This does not seem to be the case.

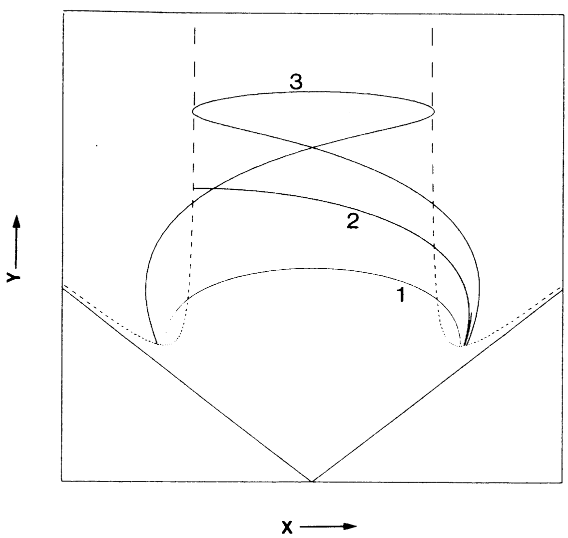

FIG. 3. Classical solutions of the FRW model with the Hartle-Hawking initial conditions. They constitute a family starting at zero potential of the Wheeler-DeWitt equation (the dashed line) and parametrized by the value of ϕ_0. Three periodic bouncing solutions are shown. Bounces 1 and 3 correspond to a solution having $\phi=0$ at the maximum value of a (a_{max}) and bounce 2 to $\dot{\phi}=0$ at a_{max}. Bouncing solutions are more the exception than the rule.

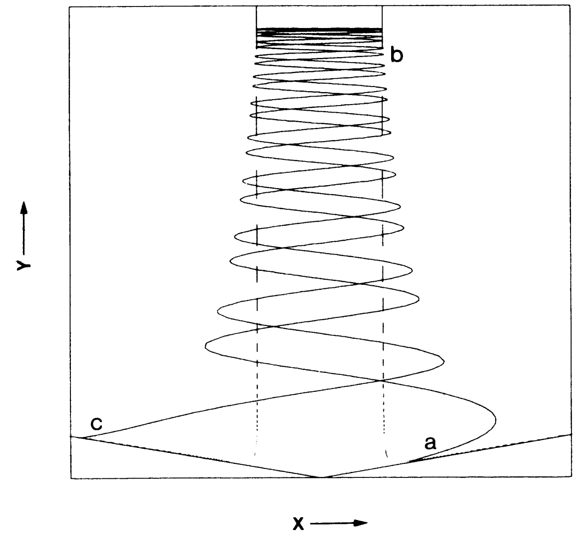

FIG. 4. Typical classical solution of the FRW model with the Hartle-Hawking initial conditions. The universe starts with an inflationary epoch (a), turns into a "dust-filled" model (b), and recollapses to a singularity (c).

In the KS model, the Lorentzian equations of motion for a, b, and ϕ are

$$b\ddot{b}+\frac{\dot{b}^2}{2}+\frac{1}{2}+\frac{b^2\dot{\phi}^2}{2}-\frac{b^2m^2\phi^2}{2}=0 , \tag{4.11a}$$

$$\ddot{a}b+\dot{b}\dot{a}+a\ddot{b}+ab\dot{\phi}^2-abm^2\phi^2=0 , \tag{4.11b}$$

$$\ddot{\phi}+\left[\frac{\dot{a}}{a}+\frac{2\dot{b}}{b}\right]\dot{\phi}+m^2\phi=0 , \tag{4.11c}$$

and the constraint is

$$b\dot{b}\dot{a}+\frac{a\dot{b}^2}{2}+\frac{a}{2}-\frac{ab^2\dot{\phi}^2}{2}-\frac{ab^2m^2\phi^2}{2}=0 . \tag{4.11d}$$

The initial conditions for the Lorentzian trajectories which correspond to the analytic continuation of the Euclidean paths are

$$a\approx 0,\ b\approx\frac{\sqrt{3}}{m\,|\phi|},\ \phi=\phi_0,\ \dot{a}=\dot{a}_0>0 ,$$
$$\dot{b}\approx 0,\ \dot{\phi}\approx 0 . \tag{4.12}$$

The universe will begin in a de Sitter phase and will expand exponentially until the scalar field starts to oscillate. When the oscillations become very rapid the matter acts as dust (the effective pressure is zero). It will then behave like a dust-filled KS universe, somewhat like a FRW solution, until it reaches a maximum three-volume and starts to recollapse. As the three-volume decreases to zero size the universe behaves more and more like the three-geometry of the interior of a black hole. The degree of inflation and the mass parameter of the black hole are a function of the initial value of the scalar field ϕ_0. A typical trajectory is shown in Fig. 5. [Note: this is in u,v coordinates (3.11).]

It is important to note that all classical solutions with the initial conditions (4.12) will become singular. This is more transparent if the equations of motion are written in the following coordinates:

$$a=\exp(p),\quad b=\exp(-p+q)\ ,$$
$$\phi=\phi,\quad N=\exp(-p+2q)\ . \tag{4.13}$$

The equations of motion for p,q,ϕ in Hamiltonian form are

$$p'=\widetilde{\pi}_p\ , \tag{4.14a}$$

$$q'=-\widetilde{\pi}_q\ , \tag{4.14b}$$

$$\phi'=-\widetilde{\pi}_\phi\ , \tag{4.14c}$$

$$\widetilde{\pi}'_p=\exp(-2p+4q)\ , \tag{4.14d}$$

$$\widetilde{\pi}'_q=\exp(2q)-2m^2\phi^2\exp(-2p+4q)\ , \tag{4.14e}$$

$$\widetilde{\pi}'_\phi=-m^2\phi\exp(-2p+4q)\ , \tag{4.14f}$$

where prime denotes $d/N\,dt$. The analytic continuation requires that $p'>0$, so (4.14a) and (4.14d) imply that p must be monotonically increasing, because its first and second derivatives are initially positive. For large ϕ_0, we have $2p'\approx q'$. When the volume of the universe is large $(-p+2q>0)$ and the ϕ field oscillates rapidly, q' will lag behind $2p'$ because of the first term on the right-hand side of (4.14e). This term reverses the sign of π'_q and the universe will recontract. Since p is monotonically increasing, the first term on the right-hand side of (4.14e) will dominate and the universe will then inexorably go to the "cigar" singularity at $q=-\infty,(b=0)$. An extensive numerical search for solutions which bounce at small volume confirmed this conclusion for the initial conditions (4.12).

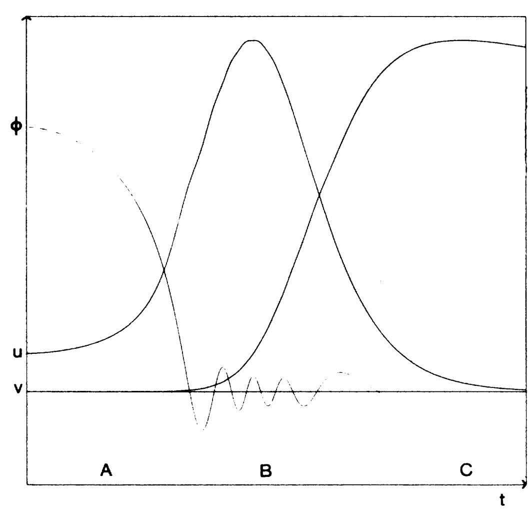

FIG. 5. Generic solution of the classical solution for the Kantowski-Sachs cosmology with the Hartle-Hawking initial conditions. Trajectories of $u=b$, $v=a^2b$, and ϕ in function of time t $(N=ab^2)$ are shown. The universe starts with an inflationary era (A), turns into an isotropic "dust-filled" model (B), and then inexorably recollapses to a singularity (C).

V. SINGULARITIES AND BOUNDARY CONDITIONS

There has been some confusion about the significance of singular Lorentzian trajectories in the Hartle-Hawking formulation. Their inevitability in the KS model contrasts with the FRW model and may help to clarify several ambiguities.

In the first place, there was some doubt whether singular WKB trajectories would contribute to the wave function at all.[7] The set of perpetually bouncing trajectories in the restricted FRW model provided some foundation for this hope. Even in this case, however, Page[16] has argued that singular trajectories will provide the dominant contribution to Ψ because the bouncing solutions are a set of measure zero. In our KS model the veracity of this assertion is not in doubt.

Second, these singular trajectories will traverse into regions of minisuperspace where the wave function is predominantly exponential. The WKB approximation, however, remains valid for wave packets of these trajectories even up to final collapse, so they will contribute to an oscillatory part in the wave function at the boundary and nearby. Previously these regions have been characterized as "Euclidean" or "forbidden."

In order to understand why there can be both oscillatory and exponential contributions to Ψ in certain regions of configuration space, it is more appropriate to consider phase space (the space of positions and momenta). It is only in phase space that certain regions will be strictly classically forbidden. This distinction is easy to demonstrate with the simple quantum mechanical potential shown in Fig. 6. Regions of configuration space forbidden for low-energy particles will be accessible for those which are sufficiently energetic. The same argument applies here, wave packets which recollapse can have sufficient "kinetic" energy from the scalar field to be above the potential barrier.

Regions where the main contribution to the wave function comes from real extrema of the Euclidean action cannot therefore be interpreted as being classically forbidden. Recollapsing universes approaching the final singularity will traverse through them. These regions therefore correspond to a superposition of exponential and rapidly oscillating components.

Third, it should be pointed out that the boundary condition $\Psi=1$ at zero volume for our two minisuperspace models only accounts for contributions from regular Euclidean paths and not the singular Lorentzian trajectories. For this reason, the solutions for Ψ in Sec. III can only be regarded as a first approximation.

Currently we are attempting to estimate these extra contributions close to the boundary by numerically integrating along such singular trajectories. The initial flux of the probability current for a WKB wave packet in the

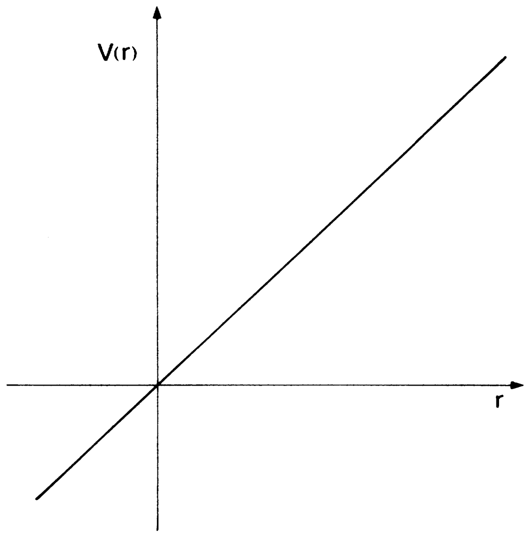

FIG. 6. Simple quantum mechanical potential for a one-dimensional system. Regions of configuration space forbidden for low-energy particles will be accessible for those which are sufficiently energetic.

FRW model,[13] $\delta F = \frac{3}{2}\delta\phi$, will be conserved along its path. Using the WKB approximation, we obtain the relation

$$\delta F = -C^2 a \left[\frac{\partial S}{\partial a}\delta\phi + \frac{1}{a^2}\frac{\partial S}{\partial \phi}\delta a \right] \tag{5.1}$$

in the classical region. The WKB approximation remains valid as the trajectory approaches the boundary. This knowledge about the magnitude of C at small volume, combined with the phase S, will provide a better approximation to Ψ.

Preliminary indications are that this additional contribution is small, the flux is spread out because of the inflationary epoch. However, the program has encountered some difficulties because of the generic occurrence of caustics (trajectory crossing points) on the boundary in the vicinity of the bouncing solutions. In principle, this problem is surmountable with the introduction of a higher order WKB approximation. Alternatively, another minisuperspace model without such caustics may be easier to deal with. The KS model is under investigation.

Finally, we should end this section with a caveat about the existence of singularities in the HH formulation. Perhaps one of the chief motivations for this proposal was to avoid the problem of specifying boundary conditions at the Big Bang singularity. This was achieved in the Euclidean path integral (2.1) by a sum over a class of regular compact geometries. Contributions to the wave function in any part of (mini)superspace come from nonsingular geometries.

The reader may wonder, however, to what extent the semiclassical approximation to (2.1) is valid. It is achieved in an apparently self-consistent manner because the semiclassical approximation is valid in the regime of cosmogenesis. One only requires that $\phi \geq 3m_p$ with $m \approx 10^{-5} m_p$ for a satisfactory inflationary phase. This corresponds to an initial energy density safely below the Planck scale $(m\phi)^2 \ll m_p{}^4$. This will not remain true however for the energy densities associated with the final recollapse of the universe, or in the interior of a black hole. In these situations our confidence in the Einstein action (3.1) cannot be trusted because of higher-order corrections. The Hartle-Hawking formulation will not be complete until it is applied to a quantum gravity theory valid on the smallest scales (see Moss,[17] Hawking[18]).

It it also possible to justify the minisuperspace approach. It will be valid if the transition probabilities between "frozen" modes are small.[19] This is justified in the Hartle-Hawking proposal by the fact that inhomogeneous modes start in their ground states.[4] This approximation will be valid until the inhomogeneous modes reenter the horizon after the de Sitter era.

For these reasons, it might be more appropriate to rename the program quantum cosmogony, at the present stage of development.

VI. THE ANISOTROPY OF THE UNIVERSE

We can characterize the degree of anisotropy in the KS model by rewriting the constraint equation (4.11d) as

$$\frac{1}{6}\frac{\dot{v}^2}{v^2} = \frac{\dot{\phi}^2}{2} + \frac{m^2\phi^2}{2} - \frac{1}{2}\left[\frac{s}{v}\right]^{2/3} + \frac{1}{2}\sigma^2 , \tag{6.1}$$

where $v = ab^2$ is the volume, $s = a/b$ is a measure of the anisotropy, and

$$\sigma = \frac{1}{\sqrt{3}}\left[\frac{\dot{s}}{s}\right] = \frac{1}{\sqrt{3}}\left[\frac{\dot{a}}{a} - \frac{\dot{b}}{b}\right] \tag{6.2}$$

is the magnitude of the shear.[20] The left-hand term in (6.1) corresponds to the expansion energy density, the first two on the right to the matter energy density $\rho_m = T^{00}$, and the latter two to the anisotropy energy density ρ_{anis}.

At the end of the de Sitter phase, during which $\dot{a}/a = \dot{b}/b$ and a and b become very large, it is clear that the shear and the anisotropy energy will be very small. Current observations which place the limit $\rho_{\text{anis}}/\rho_m < 10^{-11}$ (Ref. 21) require that the inflation be driven by an initial $\phi_0 > 3$. A result which is consistent with the requirement that $\phi_0 > 7$ if ρ_m is to be of its present magnitude. These estimates are in good agreement with those of previous perturbative treatments of anisotropy.[4,21]

The significance of this analysis is most evident however when we consider the recollapse of the universe. After the point of maximum expansion, $\dot{b}$ will be negative and the two terms in the shear (6.2) will no longer cancel. The anisotropy energy will inexorably begin to grow and eventually become infinite in the final "cigar" singularity $(a \to \infty, b \to 0)$. From an early stage onward in the recollapse, the energy density will be dominated by the anisotropy energy $\rho_{\text{anis}} > \rho_m$.

The hope that the anisotropy of the universe would necessarily decrease at small volumes in the Hartle-Hawking formulation appears to be unfounded. In general, the reverse will be true for a collapsing universe. Similarly, anisotropy will inevitably grow without limit in the interior of a black hole.

VII. THE ARROW OF TIME

A physical correlate for our subjective experience of directed temporality can be found in the second law of thermodynamics, the observed increase of entropy. Since this correlate apparently cannot be derived from known physical laws, because they are time reversal invariant, some have sought to explain its origin in a low entropy initial state of the universe.[22] In this manner, the second law becomes a selection principle for the boundary conditions of the universe.

Perhaps the most cogent expression of this in the context of classical general relativity has been that of Penrose[5] in his Weyl curvature hypothesis. He proposes that there should be a complete lack of chaos in the initial geometry of the universe: that the Weyl curvature should vanish at any initial singularity and that matter should be in thermal equilibrium. Subsequently, however, during the expansion of the universe, gravitational collapse will take place, black holes will form, and the increased gravitational entropy will manifest itself in a nonzero Weyl tensor.

Taking the term "initial geometry" in a loose sense, it is readily apparent that the KS minisuperspace model discussed here is an explicit example of such behavior. At the earliest stage for which we can define a classical notion of time, the universe is a de Sitter-type, exiting to an FRW-like phase with a vanishing Weyl curvature. After maximum expansion, however, the amplitude of the Weyl curvature grows and eventually diverges as b^{-3}. This is not the case, of course, for the more restricted and unrealistic FRW model because the Weyl tensor is constrained to be zero throughout the evolution.

In the full superspace, the perturbative treatment of Halliwell and Hawking[4] demonstrates that the universe will begin in a de Sitter-type phase with the matter fields as ordered and homogeneous as the Uncertainty Principle will allow. Subsequently, however, matter will clump, black holes will form (as we have seen, irreversibly), and in general upon recollapse the universe will reach a final state of immense complexity with a divergent Weyl tensor. In the Hartle-Hawking formulation, there appear to be no grounds for the assertion that the thermodynamic arrow of time will reverse at the point of maximum expansion,[23] an idea originally due to Gold.[24]

We contend, therefore, that the Hartle-Hawking proposal defines a global and irreversible arrow of time. It naturally incorporates Penrose's Weyl curvature hypothesis, as Vilenkin[25] has already claimed for his own boundary conditions. Once again we see a demonstration of the comprehensive economy latent in a proposal for the boundary conditions of the universe, such as that of Hartle and Hawking.

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