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Torsion as a source of expansion in a Bianchi type-I universe in the self-consistent Einstein-Cartan theory of a perfect fluid with spin density

James C. Bradas
U.S. Army Missile Command, Research Development and Engineering Center, AMSMI-RD-GC-T, Redstone Arsenal, Alabama 35898 and Department of Physics, University of Alabama in Huntsville, Huntsville, Alabama 35899

Alphonsus J. Fennelly
Teledyne Brown Engineering, Mail Stop MS-47 Cummings Research Park, Huntsville, Alabama 35805 and Department of Physics, University of Alabama in Huntsville, Huntsville, Alabama 35899

Larry L. Smalley
ES65/Space Science Laboratory, NASA Marshall Space Flight Center, Alabama 35812 and Department of Physics, University of Alabama in Huntsville, Huntsville, Alabama 35899

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We show that a generalized (or "power law") inflationary phase arises naturally and inevitably in a simple (Bianchi type-I) anisotropic cosmological model in the self-consistent Einstein-Cartan gravitation theory with the improved stress-energy-momentum tensor with the spin density of Ray and Smalley. This is made explicit by an analytical solution of the field equations of motion of the fluid variables. The inflation is caused by the angular kinetic energy density due to spin. The model further elucidates the relationship between fluid vorticity, the angular velocity of the inertially dragged tetrads, and the precession of the principal axes of the shear ellipsoid. Shear is not effective in damping the inflation.

I. INTRODUCTION

The inflationary universe model, in which expansion is accelerated ($\ddot{R}>0$), is expected to occur in the early Universe containing matter in the form of bare quantum fields, the expansion being exponential $[R\sim\exp(kt)]$ (Ref. 1). Power-law inflation, in which the expansion scale factor obeys a power-law relation with the time, is also a possible result of those physical processes in the early Universe. It is important because it too will solve such cosmological problems as horizons, homogeneity, and flatness.[2]

Inflation is important further because it is thought that it can solve the problem of the apparent large-scale isotropy of the Universe.[3] This is so because its presence mimics the acceleration produced by the existence of a cosmological constant, which isotropizes the Universe.[4] There is, however, the model in the Einstein-Cartan gravitational theory with the Ray-Smalley improved energy-momentum tensor with spin. Gasperini has proven that for the case of an rms spin density.[5] We will, however, show there that the inflationary epoch occurs because of the density of spin angular kinetic energy, which is a local quantity dependent on the spin, and so it is not necessary to resort to an rms expectation value of the spin-density operator to generate the spin terms necessary to induce inflation. Our model is a simple anisotropic (Bianchi type-I) cosmological model with shear, but vanishing spatial curvature (Euclidean model). Formal solutions of the Einstein-Cartan equations and of the fluid equations of motion are exhibited and shown to lead to conditions producing an inflationary epoch in the very early Universe. The inflation is due to the angular kinetic energy density of the spin. The shear is not effective in preventing or damping the inflation in the models. The model further brings out the relationship between the fluid vorticity, the angular velocity of observers' inertially dragged and Fermi-transported reference tetrads, the precession of the principal axes of the ellipsoid of the shear rate, and the torsion. We will conclude with comments of further work and suggestions for new investigations of these and related models.

In Sec. II we give the basic equations of the model following the pattern of Bianchi type-I spacetime and show its behavior in a Riemann-Cartan spacetime in Sec. III. We present our conclusion in Sec. IV.

II. FORM OF THE MODEL

In this paper, we choose the metric to have the following form, as used by Misner:[6]

$$ds^2=-dt^2+e^{2\alpha}e^{2\beta}{}_{ij}dx^i dx^j\ , \tag{1}$$

where α is a scalar function of time and $\exp(\beta)$ a traceless, 3×3 matrix, also a function of time. Following the method of differential forms, we write Cartan's first equation, connecting the basis forms with torsion $S_{\mu\nu}{}^{\alpha}$ as

$$d\omega^{\alpha}+\omega^{\alpha}{}_{\nu}\wedge\omega^{\nu}=\tfrac{1}{2}S_{\mu\nu}{}^{\alpha}\omega^{\mu}\wedge\omega^{\nu}\ , \tag{2}$$

where we define the torsion to be

$$S_{\mu\nu}{}^{\alpha}=\Gamma^{\alpha}_{[\mu\nu]}-\tfrac{1}{2}C_{\mu\nu}{}^{\alpha}\ . \tag{3}$$

The torsion here is defined to be the true antisymmetric portion of the affine connection (nonzero in a holonomic frame), with the C's the antisymmetric portion (if any) due to the choice of tetrads. Choosing a basis one-form set which diagonalizes the above metric and puts its into Minkowski form gives

$$\omega^0 = dt \; , \tag{4a}$$

$$\omega^i = e^{\alpha} e^{\beta i}{}_j dx^j \; . \tag{4b}$$

In a Bianchi type-I cosmology, all the spatial structure constants are zero. Thus,

$$C^i{}_{jk} = 0 \; . \tag{5}$$

The tetrads have the following properties. We use capital latin indices to refer to anholonomic coordinates, and greek indices refer to holonomic coordinates. Thus

$$E_{A\mu} E_B{}^{\mu} = g_{AB} \; , \tag{6a}$$

$$E^A{}_{\mu} E_{A\nu} = \eta_{\mu\nu} \; . \tag{6b}$$

Following Ray and Smalley,[7] we write an expression giving the orientation of the spin density $s_{\mu\nu}$, angular momentum $w_{\mu\nu}$ of the tetrads, and the improved stress-energy-momentum tensor (SEMT):

$$s_{\mu\nu} = k(x)(E^1{}_{\mu} E^2{}_{\nu} - E^1{}_{\nu} E^2{}_{\mu}) \; , \tag{7a}$$

$$w_{\mu\nu} = \tfrac{1}{2}[D(E^A{}_{\mu}) E_{A\nu} - D(E^A{}_{\nu}) E_{A\mu}] \; , \tag{7b}$$

$$\begin{aligned} T^{\alpha\beta}_{\text{Ray-Smalley}} &= \rho(1+\epsilon+P/\rho) u^{\alpha} u^{\beta} + P g^{\alpha\beta} \\ &\quad + 2\rho D(u_{\nu}) u^{(\alpha} s^{\beta)\nu} + \nabla^*_{\nu}(\rho u^{(\alpha} s^{\beta)\nu}) \\ &\quad - \rho w_{\nu}^{(\alpha} s^{\beta)\nu} \; . \end{aligned} \tag{7c}$$

Using tetrads consistent with the choice of a Bianchi type-I (see Ref. 8, p. 110) structure in Eq. (7a), the nonzero components of $s_{\mu\nu}$ are

$$s_{12} = k(x) = -s_{21} \; . \tag{8}$$

Again, following Ray and Smalley,[7] the trace-free (proper) torsion $\hat{S}_{\alpha\beta}{}^{\mu}$, which is defined as

$$\hat{S}_{\mu\nu}{}^{\alpha} = S_{\mu\nu}{}^{\alpha} + \tfrac{2}{3} \delta^{\alpha}_{[\mu} S_{\nu]\beta}{}^{\beta} \tag{9}$$

is related to the spin density $s_{\mu\nu}$ by the relationship

$$\hat{S}_{\mu\nu}{}^{\alpha} = \tfrac{1}{2} \kappa \rho s_{\mu\nu} u^{\alpha} \; , \tag{10}$$

where $\kappa = 8\pi G$, and G is the gravitational constant. For the purposes of this paper, we choose to work in a comoving frame ($u^{\alpha} = \delta^{\alpha}_0$) with normalized four-velocity ($u_{\alpha} u^{\alpha} = -1$). Thus, the nonzero components of trace-free torsion are

$$\hat{S}_{12}{}^0 = \tfrac{1}{2} \kappa \rho s_{12} u^0 = -\hat{S}_{21}{}^0 \tag{11}$$

which gives a relationship between the torsion and proper torsion:

$$S_{12}{}^0 = \hat{S}_{12}{}^0 = \tfrac{1}{2} \kappa \rho k(x) u^0 = -S_{21}{}^0 \; . \tag{12}$$

These values of torsion will be used in the model under consideration.

Proceeding with the calculation, we calculate the connection two-forms using Cartan's first equation. In this paper we use the following notation for derivatives. The overdot indicates partial differentiation with respect to time, as in

$$\dot{A}_{ij\ldots} = \frac{\partial}{\partial t} A_{ij\ldots} \; , \tag{13a}$$

while the directional covariant derivative along the four-velocity is indicated by

$$D(A_{ij\ldots}) = A_{ij\ldots;a} u^a \; . \tag{13b}$$

The quantity

$$\dot{e}^{\beta}{}_{ij} e^{-\beta j}{}_k \; , \tag{14}$$

which occurs when doing the computations involved in Cartan's first equation, is split into a symmetric and antisymmetric part, the former related to the shear, the latter related to the twist of the congruence of the normals to the homogeneous hypersurfaces. These are written as

$$\tilde{\sigma}_{ik} = \dot{e}^{\beta}{}_{(i|j|} e^{-\beta j}{}_{k)} \; , \tag{15a}$$

$$\tilde{\tau}_{ik} = \dot{e}^{\beta}{}_{[i|j|} e^{-\beta j}{}_{k]} \; . \tag{15b}$$

The connection two-forms are summarized in Table I.

Using the connection two-forms summarized above, we compute Cartan's second equation, which is

$$\begin{aligned} \bar{\theta}_{\beta}{}^{\alpha} &= d\bar{\omega}_{\beta}{}^{\alpha} + \bar{\omega}_{\nu}{}^{\alpha} \wedge \bar{\omega}_{\beta}{}^{\nu} \\ &= \tfrac{1}{2} R^{\alpha}{}_{\beta\mu\nu} \omega^{\mu} \wedge \omega^{\nu} \; . \end{aligned} \tag{16}$$

The computations involved in Eq. (16) are somewhat tedious, and we shall state the results only. The nonzero components of the Riemann tensor are ($\tilde{k}_{lj} = \tilde{\sigma}_{lj} + \tilde{\tau}_{lj}$)

$$\begin{aligned} \bar{R}^i{}_{00j} &= -\bar{R}^i{}_{0j0} \\ &= \ddot{\alpha}\delta_{ij} + \dot{\tilde{\sigma}}_{ij} + (\dot{\alpha} + \tilde{\sigma})_{ij}{}^2 + (\tilde{\sigma} \cdot \tilde{\tau})_{ij} \; , \end{aligned} \tag{17a}$$

$$\bar{R}^i{}_{000} = \bar{R}^i{}_{0jk} = \bar{R}^0{}_{000} = 0 \; , \tag{17b}$$

$$\begin{aligned} \bar{R}^0{}_{i0j} &= -\bar{R}^0{}_{ijo} \\ &= \ddot{\alpha}\delta_{ij} + \dot{\tilde{\sigma}}_{ij} + (\dot{\alpha} + \tilde{\sigma})_{ij}{}^2 + (\tilde{\sigma} \cdot \tilde{\tau})_{ij} \\ &\quad + \tfrac{1}{2}(\dot{S}_{ij}{}^0 + \dot{\alpha} S_{ij}{}^0 + \tilde{k}_{lj} S_i{}^{l0} - \tilde{\tau}_{li} S_j{}^{l0}) \; , \end{aligned} \tag{17c}$$

$$\begin{aligned} \bar{R}^i{}_{jkl} &= (\dot{\alpha}\delta^i{}_k + \tilde{\sigma}^i{}_k)(\dot{\alpha}\delta_{jl} + \tilde{\sigma}_{jl}) \\ &\quad - (\dot{\alpha}\delta^i{}_l + \tilde{\sigma}^i{}_l)(\dot{\alpha}\delta_{jk} + \tilde{\sigma}_{jk}) \\ &\quad + \tfrac{1}{2}[S_{jl}{}^0(\dot{\alpha}\delta^i{}_k + \tilde{\sigma}^i{}_k) - S_{jk}{}^0(\dot{\alpha}\delta^i{}_l + \tilde{\sigma}^i{}_l)] \; . \end{aligned} \tag{17d}$$

Using the above Riemann tensor components, we next calculate the Ricci tensor components $R_{\mu\nu}$ by contracting on the first and third indices of the Riemann tensor. The nonzero components of the Ricci tensor are

TABLE I. Connection two-forms.

$\bar{\omega}^0{}_j$	$(\dot{\alpha}\delta_{jk} + \tilde{\sigma}_{jk} + \tfrac{1}{2} S_{jk}{}^0)\omega^k$
$\bar{\omega}^j{}_0$	$(\dot{\alpha}\delta_{jk} + \tilde{\sigma}_{jk})\omega^k$
$\bar{\omega}^j{}_k$	$-\tilde{\tau}_{jk} dt$
$\bar{\omega}^0{}_0$	0
$\bar{\omega}^i{}_i$	0

TABLE II. The components of the Einstein tensor.

μ	ν	$\overline{G}_{\mu\nu}$ (unprimed G's are standard results from GR)
0	0	G_{00}
0	i	$0 \quad (i=1,2,3)$
i	0	$0 \quad (i=1,2,3)$
1	1	G_{11}
1	2	$G_{12}+\frac{1}{2}(\dot{S}_{12}{}^{0}+3\dot{\alpha}S_{12}{}^{0})$
1	3	$G_{13}+\frac{1}{2}\widetilde{\tau}_{23}S_{12}{}^{0}$
2	1	$G_{21}+\frac{1}{2}(\dot{S}_{21}{}^{0}+3\dot{\alpha}S_{21}{}^{0})$
2	2	G_{22}
2	3	$G_{23}+\frac{1}{2}\widetilde{\tau}_{13}S_{21}{}^{0}$
3	1	$G_{31}-\frac{1}{2}\widetilde{\tau}_{23}S_{12}{}^{0}$
3	2	$G_{32}-\frac{1}{2}\widetilde{\tau}_{13}S_{21}{}^{0}$
3	3	G_{33}

$$\overline{R}_{0i}=\overline{R}_{i0}=0\ ,$$

$$\overline{R}\,{}^{0}{}_{0}=3\ddot{\alpha}+3(\dot{\alpha})^2+\widetilde{\sigma}_{ij}\widetilde{\sigma}^{ij}\ , \tag{18a}$$

$$\overline{R}_{ij}=(\ddot{\alpha}\delta^{0}_{ij}+\dot{\widetilde{\sigma}}_{ij})+3\dot{\alpha}(\dot{\alpha}\delta_{ij}+\widetilde{\sigma}_{ij})+(\widetilde{\sigma}\cdot\widetilde{\tau})_{ij}$$
$$+\tfrac{1}{2}(\dot{S}_{ij}{}^{0}+3\dot{\alpha}S_{ij}{}^{0}-2\widetilde{\tau}_{k[i}S_{j]}{}^{k0})\ . \tag{18b}$$

The curvature scalar R is

$$\overline{R}=6\ddot{\alpha}+12\dot{\alpha}^2+\widetilde{\sigma}_{ij}\widetilde{\sigma}^{ij}\ . \tag{19}$$

The 16 components of the Einstein tensor, defined by

$$\overline{G}_{\mu\nu}=\overline{R}_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}\overline{R}\ , \tag{20}$$

are summarized in Table II. For brevity's sake, we have written explicitly only the additional terms due to torsion. The Einstein tensor which one obtains in the regular general-relativistic (GR) theory is listed only symbolically (see Misner[6] for regular GR components).

To calculate the Ray-Smalley tensor components in our model we will need the affine connections, which are contained implicitly in Table I. Using the relationship that $\omega^{\alpha}{}_{\gamma}=\Gamma^{\alpha}{}_{\beta\gamma}\omega^{\beta}$ one determines

$$\Gamma^{0}_{ij}=\dot{\alpha}\delta_{ij}+\widetilde{\sigma}_{ij}-\tfrac{1}{2}S_{ij}{}^{0}\ , \tag{21a}$$

$$\Gamma^{i}_{j0}=\dot{\alpha}\delta^{i}{}_{j}+\widetilde{\sigma}^{i}{}_{j}\ , \tag{21b}$$

$$\Gamma^{i}_{0j}=-\widetilde{\tau}^{i}{}_{j}\ , \tag{21c}$$

$$\Gamma^{i}_{jk}=0\ , \tag{21d}$$

$$\Gamma^{0}_{00}=\Gamma^{0}_{0i}=0\ , \tag{21e}$$

$$\Gamma^{i}_{i0}=\Gamma^{i}_{00}=0\ . \tag{21f}$$

TABLE III. Field equations.

α	β	$\overline{G}^{\alpha\beta}-\nabla^{*}_{\mu}(Q^{\alpha\beta\mu}-Q^{\beta\mu\alpha}+Q^{\mu\alpha\beta})=\kappa T^{\alpha\beta}_{\text{Ray-Smalley}}$ Field equations
0	0	$3\dot{\alpha}^2-\frac{1}{2}\widetilde{\sigma}_{ij}\widetilde{\sigma}^{ij}=\kappa\rho(1+\epsilon)$
0	i	$0=0, \quad i=1,2,3$
i	0	$0=0, \quad i=1,2,3$
1	1	$-2\ddot{\alpha}-3(\dot{\alpha})^2-\frac{1}{2}\widetilde{\sigma}_{ij}\widetilde{\sigma}^{ij}+\dot{\widetilde{\sigma}}_{11}+[\widetilde{\sigma},\widetilde{\tau}]_{11}=\kappa P+\kappa\rho\widetilde{\tau}_{21}s^{12}$
1	2	$\dot{\widetilde{\sigma}}_{12}+3\widetilde{\sigma}_{12}\dot{\alpha}+[\widetilde{\sigma},\widetilde{\tau}]_{12}-\frac{1}{2}(\dot{S}_{12}{}^{0}+3\dot{\alpha}S_{12}{}^{0})=0$
1	3	$\dot{\alpha}_{12}+3\widetilde{\sigma}_{13}\dot{\alpha}+[\widetilde{\sigma},\widetilde{\tau}]_{13}-\frac{1}{2}S_{12}{}^{0}\widetilde{\tau}_{23}=\frac{1}{2}\kappa\rho\widetilde{\tau}_{23}s^{12}$
2	1	$\dot{\widetilde{\sigma}}_{21}+3\widetilde{\sigma}_{21}\dot{\alpha}+[\widetilde{\sigma},\widetilde{\tau}]_{21}-\frac{1}{2}(\dot{S}_{21}{}^{0}+3\dot{\alpha}S_{21}{}^{0})=0$
2	2	$-2\ddot{\alpha}-3(\dot{\alpha})^2-\frac{1}{2}\widetilde{\sigma}_{ij}\widetilde{\sigma}^{ij}+\dot{\widetilde{\sigma}}_{22}+[\widetilde{\sigma},\widetilde{\tau}]_{22}=\kappa P+\kappa\rho\widetilde{\tau}_{12}s^{21}$
2	3	$\dot{\widetilde{\sigma}}_{23}+3\widetilde{\sigma}_{23}\dot{\alpha}+[\widetilde{\sigma},\widetilde{\tau}]_{23}-\frac{1}{2}S_{21}{}^{0}\widetilde{\tau}_{13}=\frac{1}{2}\kappa\rho\widetilde{\tau}_{13}s^{21}$
3	1	$\dot{\widetilde{\sigma}}_{31}+3\widetilde{\sigma}_{31}\dot{\alpha}+[\widetilde{\sigma},\widetilde{\tau}]_{31}-\frac{1}{2}S_{12}{}^{0}\widetilde{\tau}_{23}=\frac{1}{2}\kappa\rho\widetilde{\tau}_{23}s^{12}$
3	2	$\dot{\widetilde{\sigma}}_{32}+3\widetilde{\sigma}_{32}\dot{\alpha}+[\widetilde{\sigma},\widetilde{\tau}]_{32}-\frac{1}{2}S_{21}{}^{0}\widetilde{\tau}_{13}=\frac{1}{2}\kappa\rho\widetilde{\tau}_{13}s^{21}$
3	3	$-2\ddot{\alpha}-3(\dot{\alpha})^2-\frac{1}{2}\widetilde{\sigma}_{ij}\widetilde{\sigma}^{ij}+\dot{\widetilde{\sigma}}_{33}+[\widetilde{\sigma},\widetilde{\tau}]_{33}=\kappa P$

The coupling of the Einstein tensor in the Einstein-Cartan theory with the Ray-Smalley improved SEMT involves writing the field equations in the so-called self-consistent form:

$$\bar{G}^{\alpha\beta} - \nabla^*_\mu (Q^{\alpha\beta\mu} - Q^{\beta\mu\alpha} + Q^{\mu\alpha\beta}) = \kappa T^{\alpha\beta}_{\text{Ray-Smalley}} , \quad (22)$$

where

$$\nabla^*_\nu(\;) = \nabla_\nu(\;) + 2S_{\nu\alpha}{}^{\alpha}(\;) \quad (23a)$$

and $Q_{\alpha\beta}{}^{\mu}$ is the modified torsion, defined as

$$Q_{\alpha\beta}{}^{\mu} = S_{\alpha\beta}{}^{\mu} + 2\delta^{\mu}_{[\alpha}S_{\beta]\nu}{}^{\nu} . \quad (23b)$$

Because we are considering the mass-conserving case, the form of the torsion in our model becomes

$$Q_{\alpha\beta}{}^{\mu} = S_{\alpha\beta}{}^{\mu} = \hat{S}_{\alpha\beta}{}^{\mu} \quad (23c)$$

and $\nabla^*_\nu(\;) = \nabla_\nu(\;)$.

Summarizing the field equations generated in the model, we summarize below versus the values of α and β in Table III.

One note should be made on the appearance of the field equations. The $(\alpha=0,\beta=0)$ equation, at first appearance, seems to contain no spin energy terms. The energy term, however, is there by virtue of the ϵ term on the right-hand side of the equation. Recalling the thermodynamic laws of the fluid, as presented by Ray and Smalley,[7] the differential of energy $d\epsilon$ is given by

$$d\epsilon = T\,ds - P\,d(1/\rho) + \tfrac{1}{2} w_{\alpha\beta} ds^{\alpha\beta} . \quad (24)$$

Although the specific form of each of the thermodynamic variables for the fluid is unknown in our model, one may say that the integrated energy ϵ represents a correction to the standard energy term usually written in the standard theory of perfect fluids. The usual term for the T_{00} energy-density component of the SEMT is simply $\kappa\rho$. In this model, we assume that the correction term ϵ is small compared to the total energy due to fluid density. Thus, in the scaling laws developed in the next section, we let $\rho' = \rho(1+\epsilon)$ have the same scaling behavior as ρ.

III. BEHAVIOR OF THE MODEL WITH TORSION

From the field equations contained in Table III, we may make the following observations. The $(\alpha=0,\beta=0)$, $(\alpha=0,\beta=i)$, and $(\alpha=i,\beta=0)$ components are identical to their GR counterparts. Torsion appears in all the other field equations. In order to determine the effect of torsion on the solutions to the field equations, we must demand self-consistency. To elucidate, let us consider the $(\alpha=1,\beta=2)$ and $(\alpha=2,\beta=1)$ equations. These two are related since the GR terms involve shear terms which are symmetric, the expansion which is a scalar, and the antisymmetric product of shear and vorticity. Let us examine the $(\alpha=1,\beta=2)$ and $(\alpha=2,\beta=1)$ equations in more detail. Writing them out, we have

$$(\alpha=1,\beta=2):\; \dot{\tilde{\sigma}}_{12} + 3\tilde{\sigma}_{12}\dot{\alpha} + [\tilde{\sigma},\tilde{\tau}]_{12} - \tfrac{1}{2}(\dot{S}_{12}{}^{0} + 3\dot{\alpha}S_{12}{}^{0}) = 0 , \quad (25a)$$

$$(\alpha=2,\beta=1):\; \dot{\tilde{\sigma}} + 3\tilde{\sigma}_{21}\dot{\alpha} + [\tilde{\sigma},\tilde{\tau}]_{21} - \tfrac{1}{2}(\dot{S}_{21}{}^{0} + 3\dot{\alpha}S_{21}{}^{0}) = 0 . \quad (25b)$$

Reversing the indices in Eq. (25b), we obtain an equation identical to Eq. (25a) except for the terms involving torsion. (It can be shown that $[\tilde{\sigma},\tilde{\tau}]_{ij} = [\tilde{\sigma},\tilde{\tau}]_{ji}$.) Doing this operation, Eq. (25b) becomes

$$(\alpha=2,\beta=1):\; \dot{\tilde{\sigma}}_{12} + 3\tilde{\sigma}_{12}\dot{\alpha} + [\tilde{\sigma},\tilde{\tau}]_{12} + \tfrac{1}{2}(\dot{S}_{12}{}^{0} + 3\dot{\alpha}S_{12}{}^{0}) = 0 . \quad (26)$$

Comparing Eq. (26) with (25a), we immediately see that the term involving torsion (fourth term) must be zero. Thus, the fourth term must satisfy the identity

$$(\dot{S}_{12}{}^{0} + 3\dot{\alpha}S_{12}{}^{0}) = 0 \quad (27)$$

which allows us to solve for the torsion as a function of time:

$$S_{12}{}^{0} = S_{12}{}^{0}(0)e^{-3\alpha} . \quad (28)$$

Examining the $(\alpha=1,\beta=3)$ field equation, we can make an identity immediately by recalling that

$$S_{12}{}^{0} = \tfrac{1}{2}\kappa\rho s_{12}u^{0} = \tfrac{1}{2}\kappa\rho s^{12}u^{0} = \tfrac{1}{2}\kappa\rho s^{12} \quad (29)$$

since $u^0 = 1$. Thus, substituting Eq. (29) into the right-hand side of the $(\alpha=1,\beta=3)$ field equation and subtracting from both sides, we can write

$$\dot{\tilde{\sigma}}_{13} + 3\tilde{\sigma}_{13}\dot{\alpha} + [\tilde{\sigma},\tilde{\tau}]_{13} - \tfrac{3}{2}S_{12}{}^{0}\tilde{\tau}_{23} = 0 \quad (30a)$$

while the $(\alpha=3,\beta=1)$ equation, using the same identity, can be written

$$\dot{\tilde{\sigma}}_{31} + 3\tilde{\sigma}_{31}\dot{\alpha} + [\tilde{\sigma},\tilde{\tau}]_{31} - \tfrac{1}{2}S_{12}{}^{0}\tilde{\tau}_{23} = 0 . \quad (30b)$$

Since the shear and shear-vorticity commutator terms are symmetric in their indices, one can transform Eq. (30b) into the following, by switching indices:

$$\dot{\tilde{\sigma}}_{13} + 3\tilde{\sigma}_{13}\dot{\alpha} + [\tilde{\sigma},\tilde{\tau}]_{13} - \tfrac{1}{2}S_{12}{}^{0}\tilde{\tau}_{23} = 0 . \quad (30c)$$

To have consistency between Eqs. (30a) and (30c), we set $\tilde{\tau}_{23} = 0$. Similar reasoning using the $(\alpha=2,\beta=3)$ and $(\alpha=3,\beta=2)$ field equations yields $\tilde{\tau}_{13} = 0$.

Adding the $(\alpha=1,\beta=1)$, $(\alpha=2,\beta=2)$, and $(\alpha=3,\beta=3)$ equations together, and using the fact that $\text{Tr}[\tilde{\sigma}] = 0$ and $[\tilde{\sigma},\tilde{\tau}]_{ii} = 0$. The resulting equation can be written

$$-6\ddot{\alpha} - 9(\dot{\alpha})^2 - \tfrac{3}{2}\tilde{\sigma}^2 = 3\kappa P + 2\kappa\rho\tilde{\tau}_{12}s^{21} . \quad (31)$$

Writing the $(\alpha=0,\beta=0)$ field equation

$$3(\dot{\alpha})^2 - \tfrac{1}{2}\tilde{\sigma}_{ij}\tilde{\sigma}^{ij} = \kappa\rho(1+\epsilon) , \quad (32)$$

where, recalling from previous argument $[\rho \approx \rho' = \rho(1+\epsilon)]$, we simply replace $\rho(1+\epsilon)$ by ρ in the following equations. Thus, one can solve for the time rate of change of the expansion to be

$$(\dot{\alpha})^2 = \tfrac{1}{6}\tilde{\sigma}^2 + \tfrac{1}{3}\kappa\rho . \quad (33)$$

Equation (33) can be substituted into Eq. (31) to yield

$$\ddot{\alpha}=-\kappa\tilde{\sigma}^2-\tfrac{1}{2}\kappa(\rho+P)+(\kappa/3)\tau_{12}s^{12} . \quad (34)$$

The expansion term α can be written in terms of an apparent rate of change of radius of the Universe (Hubble expansion) as

$$\dot{\alpha}=\dot{R}/R , \quad (35a)$$

$$\ddot{\alpha}=\ddot{R}/R-(\dot{R}/R)^2 . \quad (35b)$$

Substituting the above plus writing the shear scalar as $\rho_\sigma=\frac{1}{2}\tilde{\sigma}^2$, Eq. (34) can be written as (let $\kappa=1$)

$$\ddot{R}/R=-\tfrac{2}{3}\rho_\sigma-\rho/6-P/2+\tfrac{1}{3}\rho\tau_{12}s^{12} . \quad (36)$$

To arrive at the appropriate power laws for each of the terms in Eq. (36), it is necessary to make use of several identities. Using the contracted Bianchi identities,[9] it is possible to arrive at the first-order differential equation relating density and pressure (letting $P=\rho\gamma$); then,

$$\dot{\rho}=-3\dot{\alpha}(\rho+P) \quad (37a)$$

$$=-3\dot{\alpha}(1+\gamma)p , \quad (37b)$$

gives

$$\rho=\rho_0 e^{-3\alpha(1+\gamma)} . \quad (38)$$

From Eq. (35a), one can write $\exp(\alpha)=R$ which we can substitute in Eq. (38) to find

$$\rho=\rho_0 R^{-3(1+\gamma)} . \quad (39)$$

Because of the relationship between ρ and P, we can immediately write a power law for pressure P as

$$P=\tfrac{1}{2}\gamma\rho_0 R^{-3(1+\gamma)} . \quad (40)$$

The scaling for the shear scalar term, $\rho_\sigma=\frac{1}{2}\tilde{\sigma}_{ij}\tilde{\sigma}^{ij}$, can be determined by the following analysis. From the form of the torsion chosen (spin vector aligned along the z axis), we express the nonzero components of the coordinate shear in terms of a "shear ellipsoid" with the tetrads aligned such that[9]

$$\tilde{\sigma}_{11}=\Sigma_{11}^0(1+\cos\phi)+\Sigma_{22}^0(1-\cos\phi)+2\Sigma_{12}^0\sin\phi , \quad (41a)$$

$$\tilde{\sigma}_{22}=\Sigma_{22}^0(1+\cos\phi)+\Sigma_{11}^0(1-\cos\phi)-2\Sigma_{12}^0\sin\phi , \quad (41b)$$

$$\tilde{\sigma}_{33}=\Sigma_{33}^0 , \quad (41c)$$

$$\tilde{\sigma}_{12}=2\Sigma_{12}^0\cos\phi+(\Sigma_{11}^0-\Sigma_{22}^0)\sin\phi , \quad (41d)$$

where

$$\Sigma_{ij}^0=\tfrac{1}{2}\tilde{\sigma}_{ij}^0 e^{-3\alpha} \quad (41e)$$

and

$$\phi=2\int\tilde{\tau}_{12}dt . \quad (41f)$$

The shear scalar can be represented as (sum of nonzero terms)

$$\tilde{\sigma}^2=\tilde{\sigma}_{11}{}^2+\tilde{\sigma}_{22}{}^2+\tilde{\sigma}_{33}{}^2+2\tilde{\sigma}_{12}{}^2 =2(\tilde{\sigma}_{11}{}^2+\tilde{\sigma}_{22}{}^2+\tilde{\sigma}_{12}{}^2+\tilde{\sigma}_{11}\tilde{\sigma}_{22}) . \quad (42)$$

Performing the indicated operations, we obtain

$$\rho_\sigma=4[(\Sigma_{11}^0)^2+(\Sigma_{22}^0)^2+(\Sigma_{12}^0)^2+\Sigma_{11}^0\Sigma_{22}^0+\Sigma_{12}^0(\Sigma_{11}^0-\Sigma_{22}^0)\sin2\phi] . \quad (43)$$

In Eq. (43), we replace all Σ_{ij}^0 by their corresponding $\tilde{\sigma}_{ij}$ terms:

$$\rho_\sigma=[(\tilde{\sigma}_{11}^0)^2+(\tilde{\sigma}_{22}^2)^2+(\tilde{\sigma}_{12}^0)^2+\tilde{\sigma}_{11}^0\tilde{\sigma}_{22}^0+\tilde{\sigma}_{12}^0(\tilde{\sigma}_{11}^0-\tilde{\sigma}_{22}^0)\sin2\phi]e^{-6\alpha} . \quad (44)$$

Thus, the term ρ_σ scales as R^{-6}.

To determine the scaling laws for the shear term τ_{12} involved in Eq. (36), we use the result of Ref. 9 which shows by dimensional analysis that the shear evolution is related to the proper time T by the relationship

$$\tilde{\tau}_{ij}=\tau_{ij}^0 T^{-1} . \quad (45)$$

To get a scaling relationship between R and time T, we proceed as follows. From the ($\alpha=0,\beta=0$) field equation, one can write

$$(\dot{R}/R)^2=\tfrac{1}{3}\rho_\sigma^0 R^{-6}+\rho_0 R^{-3(1+\gamma)} . \quad (46)$$

For $\gamma=0$ (dust), the last term goes as R^{-3}. For $\gamma=\frac{1}{3}$ (radiation), the last term goes as R^{-4}. For $\gamma=1$ (stiff matter), the last term goes as R^{-6}. Depending on the region of interest ($R>$ or <1) and the value of γ, one or the other term in Eq. (46) will dominate. As an example, for dust ($\gamma=0$) and $R>1$, the second term will dominate. Thus, we approximate the differential equation in Eq. (46) and write

TABLE IV. Scaling relationships.

Type matter	Dominant term	R values	Scaling law
Dust: $\gamma=0$	$\frac{1}{3}\rho_\sigma^0 R^{-6}$	$R<1$	$R\sim T^{1/3}$
	$\rho^0 R^{-3}$	$R>1$	$R\sim T^{2/3}$
Radiation: $\gamma=\frac{1}{3}$	$\frac{1}{3}\rho_\sigma^0 R^{-6}$	$R<1$	$R\sim T^{1/3}$
	$\rho^0 R^{-4}$	$R>1$	$R\sim T^{1/2}$
Stiff matter: $\gamma=1$	$\frac{1}{3}(\rho_\sigma^0+\rho_0)R^{-6}$	$R<1$	$R\sim T^{1/3}$
	$\frac{1}{3}(\rho_\sigma^0+\rho_0)R^{-6}$	$R>1$	$R\sim T^{1/3}$

$$(\dot{R}/R)^2=\rho_0 R^{-3} . \quad (47)$$

Solving for R, we obtain

$$R^{3/2}=\tfrac{3}{2}(p_0)^{1/2}(T-T_0) , \quad (48a)$$

which indicates that R scales with respect to time as

$$R \sim T^{2/3} . \quad (48b)$$

Proceeding similarly for the other values of γ both for $R>1$ and $R<1$, we derive similar scaling relationships between R and T which are summarized in Table IV. As can be seen, for all matter types and $R<1$, $R\sim T^{1/3}$. Thus, T goes as R^3 and we write the scaling law for shear as

$$\tilde{\tau}_{12}=\tau^0_{12}R^{-3} . \quad (49)$$

To complete the scaling laws for all the terms involved in Eq. (36), we need to postulate the scaling properties for the spin density s^{12}. In order to do this, we write the first-order differential equation relating the time evolution of spin density to tetrad rotation rate and as spin density:[7]

$$D(s_{ij})=w_{li}s^l{}_j+w_{jl}s^l{}_i \quad (50)$$

which shows that the time derivative of spin density is proportional to terms such as $w_{il}s^l{}_j$. Thus, we may write a differential equation relating the time rate of spin and spin and tetrad rotation as

$$\dot{s}=ws , \quad (51a)$$

which gives ($w=-\tau=\tau^0/T$)

$$ds/s=-\tau^0 dT/T . \quad (51b)$$

Solving for s gives

$$s_{12}\sim 1/T \sim R^{-3} . \quad (51c)$$

Combining the scaling laws for density, shear, and spin, we obtain

$$\tfrac{1}{3}\rho\tau_{12}s^{12}=\tfrac{1}{3}\rho_0\tau^0_{12}s^0_{12}R^{-3(3+\gamma)} . \quad (51d)$$

Combining Eqs. (39), (40), (44), and (51d) in Eq. (36), we arrive at

$$\ddot{R}/R=-(\gamma/2+\tfrac{1}{6})\rho_0 R^{-3(1+\gamma)}-\tfrac{2}{3}\rho^0_\sigma R^{-6} + \tfrac{1}{3}\rho_0\tau^0_{12}s^0_{12}R^{-3(3+\gamma)} . \quad (52)$$

The usual relationships between matter density and pressure used in cosmological models are noninteracting dust ($\gamma=0$), radiation ($\gamma=\frac{1}{3}$), and so-called "stiff" matter ($\gamma=1$). One immediately concludes that for dust, radiation, and stiff matter, the term containing shear and spin density scales as R^{-9}, R^{-10}, and R^{-12}, respectively. Putting in the respective values for γ into Eq. (52), we obtain, for dust ($\gamma=0$),

$$\ddot{R}=-\tfrac{1}{3}\rho_0 R^{-2}-\tfrac{2}{3}\rho^0_\sigma R^{-5} + \tfrac{1}{3}\rho_0\tau^0_{12}s^0_{12}R^{-8} , \quad (53a)$$

for radiation ($\gamma=\frac{1}{3}$)

$$\ddot{R}=-\tfrac{1}{3}\rho_0 R^{-3}-\tfrac{2}{3}\rho^0_\sigma R^{-5} + \tfrac{1}{3}\rho_0\tau^0_{12}s^0_{12}R^{-9} , \quad (53b)$$

and for stiff matter ($\gamma=1$)

$$\ddot{R}=-\tfrac{2}{3}(\rho_0-\rho^0_\sigma)R^{-5}+\tfrac{1}{3}\rho_0\tau^0_{12}s^0_{12}R^{-11} . \quad (53c)$$

It can be seen from Eqs. (53a)–(53c), that for small values of R (less than 1 in normalized coordinates), the spin density term will dominate as a large positive term, thus providing the source for increasing expansion; shear does not effectively damp the expansion. According to our model, a radiation-dominated, early epoch cosmology will have its expansion strongly driven by the spin kinetic energy of the fluid.

III. CONCLUSIONS

We have shown by exact solution that expansion, in the early, radiation-dominated phases of a Bianchi type-I Einstein-Cartan cosmology, is driven positively by the spin kinetic energy of a perfect fluid using the improved energy-momentum tensor of Ray and Smalley. It is immediately obvious that $R=0$ is not a solution to either Eq. (53a), (53b), or (53c). A solution set to these equations can be found, but their exact form depend on the boundary conditions. We conclude that even in this simple model, the spin-energy not only drives the expansion, but prevents the occurrence of the singularity and causes the apparent radius of the Universe to "bounce."

Gasperini[5] has demonstrated a spin-driven inflation using a time-averaging and scaling analysis of the Einstein-Cartan equations. Kopczyński[10] and Trautman[11] have obtained minimal radius solutions for torsion cosmological models containing polarized dust.

It is apparent from this simple model that the properties of the standard Bianchi type-I cosmology are drastically changed when one goes to an Einstein-Cartan cosmology with spin density using the Ray-Smalley improved energy-momentum tensor of spinning fluids. One could thus argue that astronomical observations which lead one to classify behavior as a Bianchi type of higher number in standard general-relativistic theory, could, in reality, be torsion "masquerading" itself as some sort of pseudocurvature in a universe which obeys the Einstein-Cartan formalism. Much work remains to be done in this area, including a systematic reexamination of cosmological properties of all Bianchi-type structures within the framework of the correct description of spinning fluids in the torsion theory of gravitation.

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