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Potential and sum-rule approach in QCD

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In a model in which the gluon condensate is simulated by a stochastic background field we can evaluate the shift of the energy levels of heavy quarkonia due to the gluon condensate as a function of the correlation time of the background field. By introducing a correlation time for the quark system it is possible to decide whether the description by Shifman-Vainshtein-Zakharov sum rules or potential models is more appropriate.

I. INTRODUCTION

Both Shifman-Vainshtein-Zakharov (SVZ) sum rules[1] as well as potential models (see, e.g., Ref. 2) make most clear-cut statements and have had their greatest successes in heavy-quarkonium spectroscopy. There are however serious theoretical contradictions between the two approaches. Voloshin[3] and Leutwyler[4] have shown that the presence of a gluon condensate excludes the existence of local potentials even in the limit of an infinite quark mass. Eichten and Feinberg,[5] on the other hand, give, starting from the Wilson loop, expressions for a local potential, which are supposed to be exact in the limit of infinite quark mass. Bell and Bertlmann[6] have derived, from SVZ sum rules, "equivalent" potentials which are strongly flavor dependent and thus in striking contrast to the phenomenological potentials. In the following we give a model, motivated from QCD, which allows the investigation of the above contradictions and leads at least to a partial clarification. In Sec. II we will briefly discuss the model, in Sec. III give a motivation for it from QCD, and in Sec. IV evaluate it. In Sec. V we summarize our results and discuss the contradictions raised above.

II. DISCUSSION OF THE MODEL

As a model we consider a pair of nonrelativistic quarks, bound together by a Coulomb potential in an external stochastic color-electric field in *Euclidean* space-time. The latter represents the nonperturbative QCD vacuum. It is essential to consider a model in Euclidean rather than Minkowski space-time: If we had a color-electric field fluctuating in real time, the time variation of it would lead to an energy uncertainty even of the ground state. Since the typical fluctuation times are supposed to be of the order of the hadronic scale, the energy smearing would thus be of similar magnitude.

For simplicity we confine ourselves to the dipole interaction of the quarks with the external field, since this is the leading interaction in the large-quark-mass limit. The model Lagrangian thus has the following form:

$$L(\mathbf{x},\mathscr{E})=\frac{\mu}{2}\dot{\mathbf{x}}^2+(\lambda_{\text{total}}{}^2-\lambda_q{}^2-\lambda_{\bar q}{}^2)\frac{g_s{}^2}{32\pi|\mathbf{x}|}+i\frac{g_s}{4}(\lambda_q^a-\lambda_{\bar q}^a)\mathbf{x}\cdot\mathscr{E}^a(t) \ . \tag{2.1}$$

Here μ is the reduced mass of the quark pair, $\mathbf{x}$ their relative distance, g_s the strong coupling constant renormalized at some suitable point, $\mathscr{E}^a$ the external color-electric field with color index a, and λ_q^a, $\lambda_{\bar q}^a$ are the Gell-Mann matrices, acting on the color spinor of the quark and antiquark, respectively.

We now consider the Schwinger function (i.e., Green's function continued to Euclidean time):

$$G(\mathbf{x}_f,\mathbf{x}_i;t)=\int[d\mathbf{x}]\exp\left[-\int_0^t L(\mathbf{x},\mathscr{E})d\tau\right]\mu(\mathscr{E})[d\mathscr{E}] \ . \tag{2.2}$$

The quantum-mechanical path integration $[d\mathbf{x}]$ is over all paths with $\mathbf{x}(0)=\mathbf{x}_i$, $\mathbf{x}(t)=\mathbf{x}_f$. The integration over the external fields with the measure $\mu(\mathscr{E})[d\mathscr{E}]$ is due to the stochastic nature of the color-electric field. The Schwinger function (2.2) is the average (over $\mathscr{E}$) of Schwinger functions for a Schrödinger equation in an external field $\mathscr{E}$. The ground-state energy is obtained by the Feynman-Kac formula:

$$E_1=-\lim_{t\to\infty}\frac{1}{t}\ln G(0,0;t) \ . \tag{2.3}$$

III. MOTIVATION OF THE MODEL

The field-theoretical analog of the Schwinger function (2.2) in QCD is the four-point function

$$\begin{aligned}G^F(x,0)&=\langle\bar\psi(x)\mathscr{O}\psi(x)\bar\psi(0)\mathscr{O}\psi(0)\rangle\\&=\int[d\psi][d\bar\psi][d\mathscr{B}^G]\bar\psi(x)\mathscr{O}\psi(x)\bar\psi(0)\mathscr{O}\psi(0)\exp\left[-\int L^{\text{QCD}}(\psi,\mathscr{B})d^4x\right] \ ,\end{aligned} \tag{3.1}$$

where x is the four-vector $(\mathbf{x},t)$ and $\psi(x)$ is the quark field. (We have omitted all color, Dirac, and Lorentz indices.) The measure $[d\mathscr{B}^G]$ indicates the integration over the color potentials in a fixed gauge.

In order to handle this expression, we make the following approximations: We neglect internal fermion loops (quenched approximation); then the Grassmann integrals over the fermion fields reduce to sums over the quark paths (see, e.g., Ref. 7). Furthermore we restrict ourselves to nonrelativistic kinematics. These approximations are justified in the large-quark-mass limit. We thus arrive at the following expression for $G^F(x,0)$:

$$G^F(x,0)=\int [d\mathbf{x}]\exp\left[-\int L^{\text{quark}}(\mathbf{x},\mathscr{B})d\tau\right]\times\exp\left[-\int L^{\text{YM}}(\mathscr{B})d^4x\right][d\mathscr{B}^G] \quad (3.2)$$

with a nonrelativistic Lagrange function $L^{\text{quark}}(\mathbf{x},\mathscr{B})$ of a heavy quark in an external color potential $\mathscr{B}$ and the pure gauge Lagrangian density $L^{\text{YM}}(\mathscr{B})$.

We now split the gluon fields into two parts:[8–10] $\mathscr{B}=\mathscr{B}^f\cup\mathscr{B}^s$, a rapidly varying part $\mathscr{B}^f$ with a Gaussian distribution and a slowly varying part $\mathscr{B}^s$. The integration over the rapidly varying part can be performed and leads in our approximation to a Coulomb potential between the quarks. We furthermore neglect the interaction of the rapidly varying with the slowly varying gluon field and thus arrive at the following expression for $G^F(x,0)$:

$$G^F(x,0)=\int [d\mathbf{x}]\exp\left[-\int L^{Q}(\mathbf{x})d\tau+\int L(\mathbf{x},\mathscr{B}^s)d^4x\right]\times\mu(\mathscr{B}^s)[d\mathscr{B}^s]\;. \quad (3.3)$$

With

$$L^{Q}(\mathbf{x})=\frac{\mu}{2}\dot{\mathbf{x}}^2+(\lambda_{\text{total}}{}^2-\lambda_q{}^2-\lambda_{\bar q}{}^2)\frac{g_s{}^2}{32\pi|\mathbf{x}|}\;, \quad (3.4)$$

$L(\mathbf{x},\mathscr{B}^s)$ gives the interaction between the quarks and the slowly varying gluon fields, $\mu(\mathscr{B}^s)$ is the distribution of the slowly varying gluon fields after the integration of the rapidly varying ones. For the slowly varying fields we choose the gauge proposed by Balitzky:[9]

$$\mathbf{x}\cdot\mathscr{B}^a(x)=0,\quad \mathscr{B}_0^a(0,t)=0\;,\quad \forall t\;. \quad (3.5)$$

In that gauge we obtain

$$\mathscr{B}_0^a(\mathbf{x},t)=\mathbf{x}\cdot\mathscr{E}^a(0,t)+\cdots\;. \quad (3.6)$$

In our model we retain only this lowest term leading to the dipole interaction and thus to our Lagrange function (2.1).

It should be noted that some of the approximations made here are crucial, some are technical. The most crucial assumption is the splitting of the gluon fields in a rapidly varying one, which can be treated perturbatively and a slowly varying "background field." This assumption is crucial in the philosophy of SVZ (Ref. 11) sum rules in order to justify their approach to QCD sum rules. The restriction imposed by the use of nonrelativistic kinematics can at least partially be loosened by a $1/m_q$ expansion.

IV. THE EVALUATION OF THE MODEL

The Schwinger function (2.2) is the expectation value of the exponential $\exp[ig_s\xi^a\int \mathbf{x}(\tau)\cdot\mathscr{E}^a(\tau)d\tau]$ with respect to the paths $\mathbf{x}$ with the measure $\exp[-\int L^{Q}(\mathbf{x})d\tau][d\mathbf{x}]$ and the field $\mathscr{E}$ with the measure $\mu(\mathscr{E})[d\mathscr{E}]$ where we have used the abbreviation $\xi^a=\frac{1}{4}(\lambda_q^a-\lambda_{\bar q}^a)$. We introduce the notation

$$\langle O\rangle_{\mathscr{E}}=\int O\mu(\mathscr{E})[d\mathscr{E}] \quad (4.1)$$

and

$$\langle O\rangle_x=\int O\exp\left[-\int L^{Q}(\mathbf{x})d\tau\right][d\mathbf{x}]\;. \quad (4.2)$$

The evaluation of the expectation value is most conveniently done in the cluster approximation:[12]

$$G(0,0,t)=\left\langle \exp\left[ig_s\xi^a\int_0^t\mathbf{x}(\tau)\cdot\mathscr{E}^a(\tau)d\tau\right]\right\rangle_{x,\mathscr{E}}
=\exp\left[\sum_{m=1}^{\infty}\frac{(ig_s)^m}{m!}\int_0^t\cdots\int_0^t\langle\langle\mathbf{x}(\tau_1)\cdot\mathscr{E}(\tau_1)\cdots\mathbf{x}(\tau_m)\cdot\mathscr{E}(\tau_m)\rangle\rangle_{x,\mathscr{E}}d\tau_1\cdots d\tau_m\right]\;, \quad (4.3)$$

where the cumulants $\langle\langle\;\rangle\rangle$ are the connected parts of the expectation values. Since we confine ourselves to the dipole approximation, there is no use of going beyond a cumulant with more than two entries:

$$G(0,0,t)=\exp\left[ig_s\int_0^t\langle\langle\mathbf{x}(\tau)\cdot\mathscr{E}(\tau)\rangle\rangle_{x,\mathscr{E}}d\tau-\frac{g_s{}^2}{2}\int_0^t\int_0^t\langle\langle\mathbf{x}(\tau_1)\cdot\mathscr{E}(\tau_1)\mathbf{x}(\tau_2)\cdot\mathscr{E}(\tau_2)\rangle\rangle_{x,\mathscr{E}}d\tau_1d\tau_2+O(g_s{}^3)\right]\;. \quad (4.4)$$

The integrations over $[d\mathbf{x}]$ and $[d\mathscr{E}]$ are independent of each other; furthermore it follows from the properties of the measure (invariance of the QCD vacuum),

$$\langle x_j\rangle_x=0,\quad \langle\mathscr{E}_j\rangle_{\mathscr{E}}=0\;,\quad \langle\mathscr{E}_j^a(\tau_1)\mathscr{E}_l^b(\tau_2)\rangle_{\mathscr{E}}=\tfrac{1}{24}\delta^{ab}\delta_{jl}R(\tau_1-\tau_2)\;. \quad (4.5)$$

We consider only Schwinger functions leading from color-singlet to color-singlet states and obtain, using

$$\langle\text{singlet}|\xi^a\xi^b|\text{singlet}\rangle=\tfrac{1}{6}\delta^{ab}\;, \quad (4.6)$$

$$G(0,0,t)=\exp\left[-\frac{g_s{}^2}{36}\int_0^t\int_0^t\langle\mathbf{x}(\tau_1)\cdot\mathbf{x}(\tau_2)\rangle_x R(\tau_1-\tau_2)d\tau_1d\tau_2+O(g_s{}^4)\right]\;. \quad (4.7)$$

This result is exact, if $\mathscr{E}$ has a Gaussian distribution. In order to have a stochastic process which guarantees for the Schwinger function the wanted Markov property, $R(\tau_1-\tau_2)$ must be of the form[12]

$$R(\tau_1-\tau_2)=\langle \mathscr{E}^2\rangle e^{-|\tau_1-\tau_2|/T^{\mathscr{E}}}\ , \tag{4.8}$$

$T^{\mathscr{E}}$ being the correlation time of the background field.[13] In the evaluation of the path integral there occurs another time scale, namely, that of the quark correlation $\langle \mathbf{x}(\tau_1)\cdot\mathbf{x}(\tau_2)\rangle_x$, which we shall call T^Q. Two cases can easily be discussed.

$$T^{\mathscr{E}} \gg T^Q$$

In this case we neglect the time dependence of $\langle \mathscr{E}_j^a(\tau_1)\mathscr{E}_l^b(\tau_2)\rangle$ and obtain

$$\begin{aligned}G(\mathbf{0},\mathbf{0},t)&\approx \exp\left[-\frac{g_s^2}{36}\langle \mathscr{E}^2\rangle \int_0^t\int_0^t \langle \mathbf{x}(\tau_1)\cdot\mathbf{x}(\tau_2)\rangle_x d\tau_1 d\tau_2\right]\\ &=\int [d\mathbf{x}]\exp\left[-\int_0^t\left[\frac{\mu}{2}\dot{\mathbf{x}}^2+(\lambda_{\text{total}}{}^2-\lambda_q{}^2-\lambda_{\bar q}{}^2)\frac{g_s^2}{32\pi|\mathbf{x}|}\right]d\tau\right]\exp\left[-\frac{g_s^2}{36}\langle \mathscr{E}^2\rangle\int_0^t\int_0^t \mathbf{x}(\tau_1)\cdot\mathbf{x}(\tau_2)d\tau_1 d\tau_2\right].\end{aligned} \tag{4.9}$$

We see directly that this expression is not compatible with an instantaneous potential.[3,4] Under certain circumstances the ground-state energy can be evaluated in that case by SVZ techniques. The short-time expansion of $G(\mathbf{0},\mathbf{0},t)$ corresponds to the asymptotic expansion of the Borel-improved operator-product expansion of the field-theoretical Schwinger function.[6] If the quark correlation time is sufficiently short then the first few terms of the short-time expansion of $G(\mathbf{0},\mathbf{0},t)$ are sufficient in order to extract the ground-state energy from the Feynman-Kac formula (2.3) (Ref. 10). Normally the minimum of the right-hand side of Eq. (2.3) is then identified with the ground-state energy. The "equivalent" potential of Bell and Bertlmann[6] is constructed in order to reproduce the short-time behavior of the Schwinger function $G(\mathbf{0},\mathbf{0},t)$.

$$T^Q \gg T^{\mathscr{E}}$$

Here we can treat the fluctuation of $\mathscr{E}$ like a white noise, i.e., we approximate[12]

$$R(\tau_1-\tau_2)=2T^{\mathscr{E}}\langle \mathscr{E}^2\rangle\delta(\tau_1-\tau_2) \tag{4.10}$$

and hence obtain[8]

$$G(\mathbf{0},\mathbf{0},t)=\int [d\mathbf{x}]\exp\left[-\int_0^t\left[\frac{\mu}{2}\dot{\mathbf{x}}^2+(\lambda_{\text{total}}{}^2-\lambda_q{}^2-\lambda_{\bar q}{}^2)\frac{g_s^2}{32\pi|\mathbf{x}|}+\frac{g_s^2}{18}T^{\mathscr{E}}\langle \mathscr{E}^2\rangle\mathbf{x}^2(\tau)\right]d\tau\right]. \tag{4.11}$$

Thus a rapidly varying background field leads to an additional local potential

$$V(\mathbf{x})=\tfrac{g_s^2}{18}T^{\mathscr{E}}\langle \mathscr{E}^2\rangle\mathbf{x}^2 \tag{4.12}$$

proportional to the product of the field correlation time $T^{\mathscr{E}}$ and the expectation value of the external field $\langle \mathscr{E}^2\rangle$. The quadratic space dependence is a consequence of the dipole approximation.

In order to get an expression for the quark correlation time T^Q, we must specify which angular and radial excitation of the quark system we consider. For that purpose we form the matrix element $\langle nl\,|\,G\,|\,nl\rangle$ of the Schwinger function and make use of the Feynman-Kac formula in order to get the energy shift due to the gluon condensate. For the level with principal quantum number n and angular momentum l we obtain the shift

$$M_{nl}(T^{\mathscr{E}})=\lim_{t\to\infty}\frac{g_s^2}{t}\int_0^t\int_0^t \langle nl\,|\,\xi^a x_j(\tau_1)\xi^b x_l(\tau_2)\,|\,nl\rangle\langle \mathscr{E}_j^a(\tau_1)\mathscr{E}_l^b(\tau_2)\rangle d\tau_1 d\tau_2\ . \tag{4.13}$$

With the ansatz (4.5) and (4.8) for the gluon correlation function one obtains, after performing the time integrations,

$$M_{nl}(T^{\mathscr{E}})=\frac{g_s^2}{18}\langle \mathscr{E}^2\rangle\sum_{|m\rangle}\frac{|\langle nl^{(0)}\,|\,\mathbf{x}\,|\,m^{(8)}\rangle|^2}{E_m^{(8)}-E_{nl}^{(0)}+\dfrac{1}{T^{\mathscr{E}}}}\ . \tag{4.14}$$

Note that the summation involves states from the octet sector only, because the color matrix ξ^a just links singlet states to octet states, denoted by superscripts (0) and (8), respectively. In the limit $T^{\mathscr{E}}\to\infty$ this expression goes over into the formula derived by Leutwyler:[4]

$$\begin{aligned}M_{nl}(\infty)&=\frac{g_s^2}{18}\langle \mathscr{E}^2\rangle\sum_{|m\rangle}\frac{|\langle nl^{(0)}\,|\,\mathbf{x}\,|\,m^{(8)}\rangle|^2}{E_m^{(8)}-E_{nl}^{(0)}}\\ &=n^6\epsilon_{nl}\frac{g_s^2\langle \mathscr{E}^2\rangle a_0^3}{8\beta}\end{aligned} \tag{4.15}$$

with

$$a_0=\frac{1}{m\beta},\quad \beta=\tfrac{4}{3}\alpha_s,\quad \alpha_s=\frac{g_s^2}{4\pi}\ ,$$

and the reduced level shift ϵ_{nl}

$$\epsilon_{nl}=\frac{2}{9}\frac{1}{n^3(2l+1)}\{(l+1)[F(n,l)-F(-n,l)]+l[F(n,-l-1)-F(-n,-l-1)]\}\ ,$$
$$F(n,l)=2n[n^2-(l+1)^2]+(n+1+2)(n+l+1)\left[\frac{(n-l)(n+l+3)}{9n+16}+\frac{4(2n-l)^2}{9n+8}\right]\ . \tag{4.16}$$

On the other hand, let us assume that the quark system in Euclidean space-time can—at least approximately—be described by a stationary Markov process with a correlation time T^Q_{nl} depending on the angular and radial excitation n,l of the quark system:

$$\langle nl\,|\,x_j(\tau_1)x_l(\tau_2)\,|\,nl\rangle=\tfrac{1}{3}\delta_{jl}\langle \mathbf{x}^2\rangle_{nl}e^{-(|\tau_1-\tau_2|)/T^Q_{nl}}\ . \tag{4.17}$$

Inserting this ansatz into (4.13) yields

$$\widetilde{M}_{nl}(T^{\mathscr{E}})=n^6\widetilde{\epsilon}_{nl}(T^{\mathscr{E}})\frac{g_s{}^2\langle\mathscr{E}^2\rangle a_0{}^3}{8\beta}$$
$$=\frac{g_s{}^2}{18}\langle\mathscr{E}^2\rangle\frac{\langle nl\,|\,\mathbf{x}^2\,|\,nl\rangle}{\dfrac{1}{T^Q_{nl}}+\dfrac{1}{T^{\mathscr{E}}}} \tag{4.18}$$

or

$$\widetilde{M}_{nl}(\infty)=\frac{g_s{}^2}{18}\langle\mathscr{E}^2\rangle\langle\mathbf{x}^2\rangle_{nl}T^Q_{nl}\ . \tag{4.19}$$

Thus we obtain the following expression for the quark correlation time:

$$T^Q_{nl}=\tfrac{27}{16}n^6\frac{a_0{}^3}{\alpha_s}\frac{\epsilon_{nl}}{\langle\mathbf{x}^2\rangle_{nl}} \tag{4.20}$$

with

$$\langle\mathbf{x}^2\rangle_{nl}=\frac{n^2}{2}[5n^2+1-3l(l+1)]a_0{}^2\ . \tag{4.21}$$

In Fig. 1 the n dependence of T^Q_{nl} for s states is shown for $c\bar{c}$, $b\bar{b}$, and $t\bar{t}$ in units of $T^{\mathscr{E}}\approx 1/\Lambda_{\rm QCD}\approx 1/(150\ \text{MeV})$.

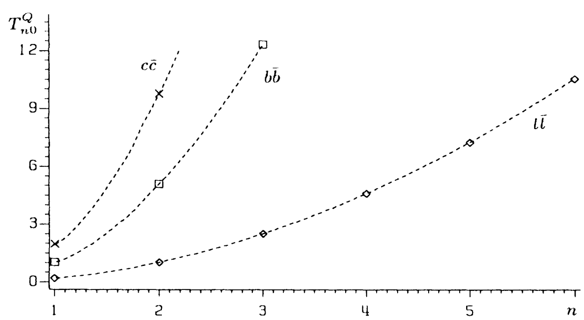

FIG. 1. The quark correlation time T^Q_{n0} [see (4.17)] of the $c\bar{c}$ system (×) ($m_c=1.5$ GeV, $\alpha_s=0.25$), $b\bar{b}$ system (□) ($m_b=4.5$ GeV, $\alpha_s=0.20$), $t\bar{t}$ system (◇) $m_t=45$ GeV, $\alpha_s=0.14$), as a function of the principal quantum number n with $l=0$. Note that the dashed lines are just displayed in order to guide the eye.

Now it is also possible to calculate the shift of the Schrödinger level $|\,nl\rangle$ due to the gluon condensate as a function of the vacuum correlation time $T^{\mathscr{E}}$:

$$\widetilde{\epsilon}_{nl}(T^{\mathscr{E}})=\frac{U_{nl}}{\dfrac{U_{nl}}{\epsilon_{nl}}+\dfrac{a_0}{\alpha_s T^{\mathscr{E}}}}\ , \tag{4.22}$$

where we have set

$$U_{nl}=\frac{16}{27}\frac{\langle\mathbf{x}^2\rangle_{nl}}{n^6a_0{}^2}\ . \tag{4.23}$$

We have also calculated $M_{nl}(T^{\mathscr{E}})$ numerically,[10] treating Eq. (4.14) by the method of Delgarno and Lewis.[14] In Figs. 2–4 we compare the numerical results with the values of Eq. (4.22). The good agreement shows that our ansatz describing quantum-mechanical systems in Euclidean space-time by a stationary Markov process is well justified.

The same calculations can be done in the case of static quarks.[9] We have to omit the kinetic energy of the quarks in Eq. (4.14):

$$M^{\rm static}_{nl}(T^{\mathscr{E}})=\frac{g_s{}^2}{18}\langle\mathscr{E}^2\rangle\left\langle nl\left|\frac{\mathbf{x}^2}{3\alpha_s/2\,|\,\mathbf{x}\,|+1/T^{\mathscr{E}}}\right|nl\right\rangle\ . \tag{4.24}$$

Thus we obtain the correlation time

$$T^{Q,\rm static}_{nl}=\frac{2}{3\alpha_s}\frac{\langle\,|\,\mathbf{x}\,|^3\rangle_{nl}}{\langle\mathbf{x}^2\rangle_{nl}} \tag{4.25}$$

with

$$\langle\,|\,\mathbf{x}\,|^3\rangle_{nl}=\tfrac{7}{8}n^2a_0{}^3[5n^4+n^2-3l(l+1)n^2]$$
$$-\tfrac{3}{32}n^2a_0{}^3[(2l+1)^2-9][3n^2-l(l+1)]\ . \tag{4.26}$$

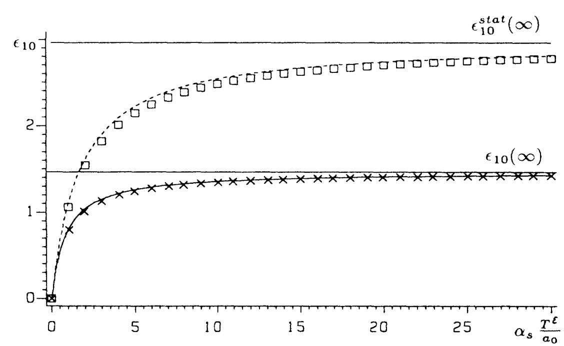

FIG. 2. Reduced level shifts [see Eq. (4.18)] $\widetilde{\epsilon}_{10}(T^{\mathscr{E}})$ (solid line), $\widetilde{\epsilon}^{\rm static}_{10}(T^{\mathscr{E}})$ (dashed line) as calculated according to Eqs. (4.22) and (4.28), respectively, in comparison to the numerical results $\epsilon_{10}(T^{\mathscr{E}})$ (×) and $\epsilon^{\rm static}_{10}(T^{\mathscr{E}})$ (□). The horizontal lines denote $\epsilon_{10}(\infty)$ and $\epsilon^{\rm static}_{10}(\infty)$, respectively.

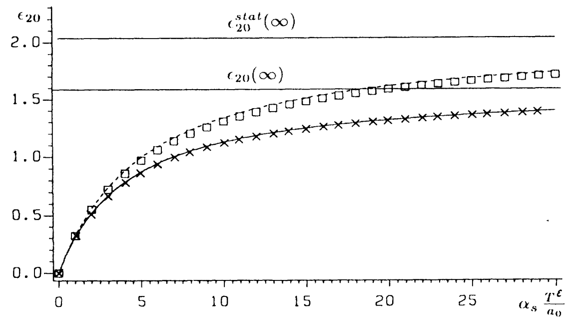

FIG. 3. As Fig. 1, but for the Schrödinger level $n=2, l=0$.

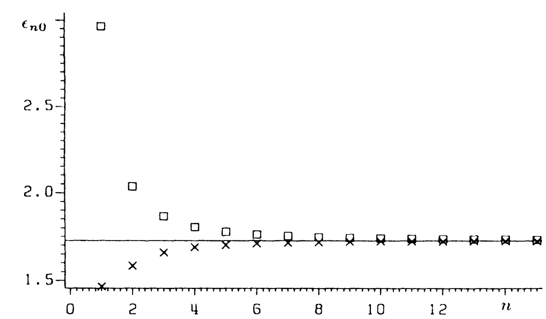

FIG. 5. ϵ_{n0} ($\times$) and $\epsilon_{n0}^{\text{static}}$ ($\square$) as function of the principal quantum number n. The horizontal line denotes their common limit.

The reduced level shift for $T^{\mathscr{E}} \to \infty$ is

$$\epsilon_{nl}^{\text{static}} \equiv \epsilon_{nl}^{\text{static}}(\infty) = \frac{32}{81}\frac{\langle |\mathbf{x}|^3 \rangle_{nl}}{n^6 a_0{}^3} \tag{4.27}$$

and the reduced level shift for static quarks as function of $T^{\mathscr{E}}$ is

$$\tilde{\epsilon}_{nl}^{\text{static}}(T^{\mathscr{E}}) = \frac{U_{nl}}{\dfrac{U_{nl}}{\epsilon_{nl}^{\text{static}}} + \dfrac{a_0}{\alpha_s T^{\mathscr{E}}}} \tag{4.28}$$

with the same U_{nl} as in Eq. (4.23). Again Eq. (4.24) can be integrated numerically and those numerical results as well as the results of Eq. (4.28) are also shown in Figs. 2–4. Now it is easy to prove that for large n the reduced level shifts for dynamic quarks and static quarks will be the same[9] since though both $\tilde{\epsilon}_{nl}(T^{\mathscr{E}})$ and $\tilde{\epsilon}_{nl}^{\text{static}}(T^{\mathscr{E}})$ depend on one different parameter $\tilde{\epsilon}_{nl}(\infty)$ and $\tilde{\epsilon}_{nl}^{\text{static}}(\infty)$, they become equal in the limit $n \to \infty, l$ fixed. This is a simple consequence of the relations

$$\lim_{n\to\infty} \tilde{\epsilon}_{nl}(\infty) = \lim_{n\to\infty} \tilde{\epsilon}_{nl}^{\text{static}}(\infty) = \tfrac{140}{81} \; . \tag{4.29}$$

The n dependence of the reduced level shifts is shown in Fig. 5.

V. DISCUSSION

We have presented a realistic model for a system of two heavy quarks in QCD: they interact via Coulomb forces and the effects of a gluon condensate are described by an external stochastic color-electric field. In this model the treatment by SVZ sum rules or by potentials is appropriate in two limiting cases.

If the Euclidean correlation time of the quark system is short as compared to the one of the external field, potential models are inadequate (4.9), but under certain circumstances SVZ sum rules are appropriate in order to obtain the ground-state energy. If however the correlation time of the external field is short as compared to that of the quark system, the effects of the stochastic field can be well approximated by a flavor-independent local potential [(4.12), (4.24)]. We have given a good analytical approximation for the correlation time of the quark system (4.20). It is displayed in Fig. 1 for different quark masses as a function of the principal quantum number of the two-quark system. We see that for highly excited states the correlation time increases, so that the potential picture will be appropriate. On the other hand, the static approximation of Ref. 4 is not justified, if m stays fixed and n increases.[13] The obvious discrepancy between the results of Voloshin[3] and Eichten and Feinberg[5] is thus solved: If we stay at fixed distances (as we do in evaluating the Wilson loop) and let the quark mass increase, we go to highly excited states and hence the vacuum structure can be approximately described by flavor-independent local potentials. If we however confine ourselves to the lowest-lying states and very high quark masses, the external field varies slowly in comparison to the quark system and cannot be approximated by a flavor-independent local potential. In that case an equivalent potential,[6] which reproduces the Green's function in perturbation theory for *short times* has no connection to the phenomenological potentials of the potential models. This situation is a consequence of the Coulomb potential; the Bohr radius of the ground state decreases as m_q increases, and therefore the correlation time of the quark system decreases with increasing m_q. If we assume the inverse of the correlation time of the vacuum fluctuations to be of the order of the hadronic scale (~ 150 MeV), we see that for the ground state of charmonium both correlation times are approximately equal, whereas for the higher excited states the potential approximation works well. For the lower-lying states of the heavier quarkonia, the sum-rule approach is

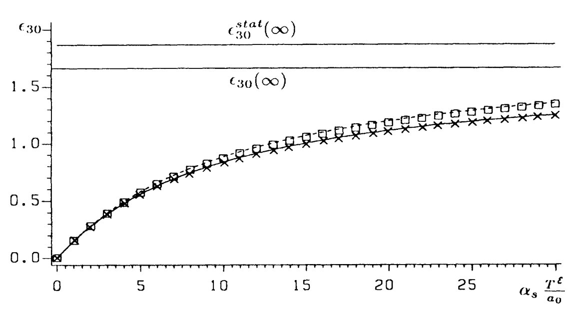

FIG. 4. As Fig. 1, but for the Schrödinger level $n=3, l=0$.

definitely more appropriate. However, by the same reason, by which the correlation time is short (namely, the shrinking radius) also the interaction of the quarks with the gluon condensate is reduced.

Note added. After the submission of our paper we learned of the work of Campostrini, Di Giacomo, and Olejnik[15] which also includes nonrelativistic corrections and as do Refs. 8 and 13 stresses the importance of the correlation length of the gluon condensate. They advocate very short correlation times, corresponding to $(T^{\mathscr{G}})^4\ \langle(\alpha_s/\pi)G^a_{\mu\nu}G^a_{\mu\nu}\rangle \ll 1$. On the other hand, for small values of x the expression

$$\left\langle \frac{\alpha_s}{\pi} G^a_{\mu\nu}(x)\mathbb{P}\exp\left[ig_s \int_0^x \mathbb{A}_\mu dx^\mu\right] G^b_{\mu\nu}(0)\right\rangle \equiv G_2(x)\ ,$$

where the path-ordered integral is taken along a straight line connecting x and 0, can be obtained through Taylor expansion:

$$G_2(x) = \left\langle \frac{\alpha_s}{\pi} G^a_{\mu\nu}(0)G^a_{\mu\nu}(0)\right\rangle + \tfrac{1}{2}x^2\left\langle \frac{\alpha_s}{\pi} D^2 G^a_{\mu\nu}(0)G^a_{\mu\nu}(0)\right\rangle,$$

where D is the covariant derivative. Standard estimates of the six-dimensional condensate[1] rather suggest a correlation time satisfying

$$(T^{\mathscr{G}})^4\left\langle \frac{\alpha_s}{\pi} G^a_{\mu\nu}G^a_{\mu\nu}\right\rangle \simeq 1\ .$$

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