Solving momentum-space integral equations for quarkonia spectra with confining potentials. II

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Singular integral equations for quarkonia ($q\bar{q}$) spectra are solved in momentum space for nonrelativistic and relativistic Coulomb plus confinement potentials. The confinement potential in momentum space is defined using an analytical regularization scheme. Further manipulations give rise to integro-differential equations and we obtain analytical expressions for the remaining singular integrals. The procedure is tested on previously solved relativistic and nonrelativistic cases. The energies of the first few eigenstates are obtained accurately to six significant figures. The method works in all partial waves. Straightforward extensions are sufficiently general to treat nonlocal potentials and combinations of singular potentials.

I. INTRODUCTION

As interest has grown in developing a covariant description of the mass spectra of elementary particles based on QCD, there has emerged a need for stable and accurate methods to solve the Bethe-Salpeter[1] integral equation in momentum space with nonlocal and singular kernels. We have adopted the quarkonia problem to illustrate a method for treating singular potentials.

We begin by showing how the potential $V^i(r)=\alpha_i r^i$ can be written in manageable form in the momentum representation. We next show how a suitable basis for the solution of integral equations containing such potentials can be chosen. We then indicate how the appropriate integrals may be evaluated accurately and efficiently. When this combination of techniques is applied to several problems we obtain results with a small basis which agree quite well with exact results or with precision numerical results from other techniques. In an earlier treatment of such problems,[2] which we refer to as I, confinement potentials were treated by introducing a numerical cutoff and then correcting for the effects of the cutoff by using perturbation theory. In this treatment we introduce techniques to evaluate analytically the limit in which the cutoff goes to zero. In addition, we present here results for the Bethe-Salpeter equation in the instantaneous approximation. Thus, the current effort improves the the results of I in several ways and retains the advantage that general nonlocal potentials can be treated in addition to these singular potentials without requiring new techniques.

II. TREATMENT OF SINGULARITIES

It can be shown[2] that the momentum-space partial-wave projection of a power-law potential, $V^i(r)=\alpha_i r^i$, can be expressed in terms of Legendre functions of the second kind as

$$V^i(q,q')=\frac{2\alpha_i}{\pi}\lim_{\mu\to 0}\left[\frac{-\partial}{\partial\mu}\right]^{i+1}\frac{Q_L(Z)}{2qq'} \tag{1}$$

with the definitions

$$Q_0(Z)=\tfrac{1}{2}\ln\left[\frac{Z+1}{Z-1}\right], \tag{2}$$

$$Q_{L+1}(Z)=\left[\frac{2L+1}{L+1}\right]ZQ_L(Z)-\left[\frac{L}{L+1}\right]Q_{L-1}(Z)+\delta_{L,0}, \tag{3}$$

$$Z=\left[\frac{q'^2+q^2+\mu^2}{2qq'}\right]. \tag{4}$$

Because of their singularities when $q=q'$ and $\mu=0$, these expressions require care in their interpretation.

As an example of our procedure for treatment of the singularities due to confining potentials,[3] we consider the nonrelativistic, spin-zero case of two particles, with a linear confinement potential $V(r)=\alpha_1 r$. We choose a wave function in the Lth partial wave which has a convenient explicit dependence as $q\to 0$. That is, we choose

$$\Psi_L(\mathbf{q})=\frac{R_L(q)}{q}Y_{LM}(\hat{\mathbf{q}}), \tag{5}$$

$$R_L(q)=q^L\chi_L(q), \tag{6}$$

whereby the momentum representation for the Schrödinger equation is

$$\epsilon\chi_L(q)=\frac{q^2}{2m}\chi_L(q)+\left[\frac{\alpha_1}{\pi q^L}\right]\lim_{\mu\to 0}\int_0^\infty dq'\frac{\partial^2}{\partial\mu^2}Q_L(Z)q'^L\chi_L(q'). \tag{7}$$

In this case we use integration by parts and the boundary conditions to reason as

$$-\lim_{\mu\to 0}\frac{\partial^2}{\partial\mu^2}Q_0(Z)=\lim_{\mu\to 0}\left[\frac{1}{(q'-q)^2+\mu^2}-\frac{1}{(q'+q)^2+\mu^2}\right] \tag{8}$$

$$=\lim_{\mu\to 0}\frac{1}{2i\mu}\left[\left[\frac{1}{(q'-q)-i\mu}-\frac{1}{(q'-q)+i\mu}\right]-\left[\frac{1}{(q'+q)-i\mu}-\frac{1}{(q'+q)+i\mu}\right]\right] \tag{9}$$

$$=-\frac{d}{dq'}\left[P\left[\frac{1}{q'-q}\right]-P\left[\frac{1}{q'+q}\right]\right] \tag{10}$$

$$=\left[P\left[\frac{1}{q'-q}\right]-P\left[\frac{1}{q'+q}\right]\right]\frac{d}{dq'}\,, \tag{11}$$

where P denotes the Cauchy principal value.

For $L>1$ we use the recursion relation and definition of Z to obtain

$$\lim_{\mu\to 0}\frac{\partial^2}{\partial\mu^2}Q_{L+1}(Z)$$
$$=\lim_{\mu\to 0}\left[\left[\frac{2L+1}{L+1}\right]\left[\frac{1}{qq'}Q_L(Z)+Z\frac{\partial^2}{\partial\mu^2}Q_L(Z)\right]-\left[\frac{L}{L+1}\right]\frac{\partial^2}{\partial\mu^2}Q_{L-1}(Z)\right]. \tag{12}$$

With these manipulations and with the spline basis introduced below, the integrals occurring in Eq. (7) for any partial wave may be performed analytically. This represents the most important practical achievement of our method.

III. CHOICE OF BASIS

The wave equations were solved by approximating

$$\chi_L(q)\approx\sum_{\nu=1}^{N}\beta_\nu B_\nu(q)$$

with the functions $B_\nu(q)$ being cubic B splines having two continuous derivates. The coefficients β_ν were then determined by either the colocation method or the Galerkin method.[4] The functions $B_\nu(q)$ are then defined by a recursion relation[5] in terms of $N+4$ (distinct) knots $\{\tau_j\}$. For $j>4$ these were chosen[4] to be the images of the zeros of a Chebyshev polynomial $\{x_j\}$,

$$x_j=-\cos\left[\left[\frac{2j-1}{2N}\right]\pi\right], \tag{13}$$

under a mapping so that

$$\tau_{j+4}=\bar{q}\left[\frac{1+x_j}{1-x_j}\right]^{\gamma}+\delta\,,\quad j=1,\ldots,N\,, \tag{14}$$

and for $j\le 4$, $\tau_4=0$ and the remaining are chosen symmetrically so that

$$\tau_{4-i}=-\tau_{i+4}\,,\quad i=1,2,3\,. \tag{15}$$

Satisfactory choices for the remaining free parameters in these calculations were $\bar{q}=1.0$, $\gamma=\frac{1}{2}$, $\delta=0$, for the colocation method, and $\bar{q}=1.0$, $\gamma=1$, $\delta=10^{-3}$ for the Galerkin method. For the colocation method, one colocation point was placed at the origin and used to enforce the condition $\chi_L(0)=0$, one point was placed at $\tau_5/2$ and the remainder at the knots τ_j; $j=5,\ldots,N+2$. We wish to emphasize this attractive feature that a single set of the parameters governing the knots for each method worked for all our potentials and in all partial waves we tested. These parameters can be chosen in other applications by simply requiring that eigenvalues are independent of the number of splines.

IV. EVALUATION OF INTEGRALS

For the colocation calculations it was necessary to evaluate integrals of the general form

$$I_L^i(\nu)=\int_0^\infty V_L^i(q,q')q'^L B_\nu(q')dq'\,, \tag{16}$$

where we found greatest stability by analytically evaluating integrals of the form

$${}^jI_M^i=\int_{\tau_j}^{\tau_{j+1}}V_L^i(q,q')q'^M dq' \tag{17}$$

and substituting the resulting expressions into the recursion relation defining $B_\nu(q')$. Since the resulting functions of q were reasonably smooth, we elected to perform numerically the second integrations required by the Galerkin calculations with Gauss quadrature over the subintervals (τ_j,τ_{j+1}). This choice to evaluate the second integration numerically was made solely to avoid the extra labor of developing a computer code based on the analytic expressions for these simple test cases.

V. RESULTS

We have tested this combination of procedures with the nonrelativistic (NR) Schrödinger equation for Coulomb,

TABLE I. Bound states of the NR Coulomb potential, $L=0$, $V^i=\alpha_i r^i (i=-1)$. The energies are in units of $\alpha_i^{2/(2+i)}m^{-i/(2+i)}$ with $\hbar=c=1$.

n	Exact	Galerkin	Colocation
1	−0.250 000	−0.249 999	−0.250 190
2	−0.062 500	−0.062 500	−0.062 526
3	−0.027 777	−0.027 777	−0.027 796
4	−0.015 625	−0.015 625	−0.015 750
5	−0.010 000	−0.010 010	−0.015 675

TABLE II. Bound states for the NR linear confining potential ($i=1$) for $L=0$. The units are the same as those quoted in Table I.

n	Exact	Galerkin	Colocation
1	2.338 107	2.338 110	2.338 24
2	4.087 957	4.087 957	4.095 18
3	5.520 560	5.520 597	5.536 47
4	6.786 708	6.786 847	6.810 32
5	7.944 134	7.944 787	7.971 78

linear confinement, and Coulomb plus linear confinement (charmonium) potentials. In addition, we have tested it for the relativistic case of the instantaneous approximation to the Bethe-Salpeter (BS) equation. In the relativistic case we reproduced results to the accuracy available[6] (four significant figures) for the Cutkosky model.[7] This was done in the strong-coupling limit with coupling constant equal to 1.0 in the $l=0$ and 1 partial waves and with equal-mass particles exchanging a zero-mass meson. For details of the model, the eigenvalues, and another method of solution not easily applicable to the confinement case, see Ref. 7.

The results for the exactly soluble NR pure Coulomb and s-wave pure linear confinement are illustrated in Tables I and II using $N=31$ splines. They indicate that the Galerkin method yields approximately twice as many significant figures as the colocation method. This seems reasonable as the Galerkin method is a variational method while the colocation method is not. The cost of this extra accuracy is the increased computer time to do the second integration numerically. As might be expected the accuracy of the computed eigenvalues decreases with increasing energy. We note again that the same set of knots was used for all of these Galerkin calculations, and a different, but fixed, set of knots was used for all of the colocation calculations.

Our results for the energies for the lowest states for the first few partial waves in a NR model for charmonium are given in the third column of Table III. Note that they are identical with precision numerical results[8,9] to the accuracy available. Presented for comparison in the second column are the results obtained using the numerical regularization technique of I. In the fourth column of this table we present our results for a corresponding relativistic problem obtained by adding a confinement term to the kernal of the BS equation for the Cutkosky model.

TABLE III. Eigenenergies of charmonium in the lowest state of each partial wave in GeV units obtained with our method for the Schrödinger (NR) and Bethe-Salpeter (BS) equations. The Coulomb-plus-linear confining potential $V(r)=ar^{-1}+br$ ($a=0.49, b=0.176$ GeV^2, $m=1.35$ GeV) is taken from Ref. 9 where it was obtained by fitting the spin-averaged data. The NR results with a regularization method are those of Ref. 2 and the exact results are given in Ref. 8.

	NR (Exact)	NR (Regularization)	NR	BS
$1S$	0.364	0.368	0.364	0.375
$1P$	0.772	0.775	0.772	0.787
$1D$	1.060	1.062	1.060	1.037

VI. CONCLUSIONS

For Coulomb and confining potentials the procedures outlined here seem to satisfy the criteria of stability and accuracy with a small spline basis, and because one integration was done analytically, the amount of computer time needed was quite small. Since the results obtained are in excellent agreement with exact results and with other precision numerical studies we have chosen not to carry the comparisons with other methods beyond the eigenvalues. Our main goal is now to apply these techniques to relativistic two-body and relativistic three-body problems with spin.

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