Mass splitting of hadron ground-state multiplets

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We use certain qualitative features of quantum chromodynamics in the framework of a potential model in order to obtain a number of inequalities and equalities among the masses of ground-state hadrons. Of those relations which have been tested experimentally, every one is in agreement with the data within the experimental errors.

I. INTRODUCTION

Many authors have obtained various relations among the masses of hadrons within the framework of the constituent-quark model.[1–8] In this paper, we shall consider this problem once again, relaxing some of the assumptions made in previous works. We shall be guided by the theory of quantum chromodynamics (QCD), but shall avoid as much as we can predictions which depend on the details of specific models. We shall also make use of the experimental values of masses in cases in which we can relate those masses to certain mass splittings.

Our approach limits us in the number of things we can say about hadron masses, but it is gratifying that none of the inequalities and equalities that we have been able to obtain within our framework is in disagreement with experiment. Furthermore, future experiments can provide further tests of our predictions.

We include in our discussion hadrons which contain the quarks of the standard model with three generations, including the t quark. We confine ourselves to those ground-state mesons and baryons which in the simple quark model have no orbital angular momentum: namely, the vector and pseudoscalar mesons and the baryons of spin $\frac{1}{2}$ and $\frac{3}{2}$. Furthermore, we omit those mesons which have only hidden flavor (self-conjugate mesons of isospin zero) because of complications arising from annihilation and mixing effects. We also omit the ρ and π mesons because relativistic effects are strongest for these mesons.

We frame our discussion in terms of a potential model with constituent-quark masses, and consider a Hamiltonian containing an unperturbed part H_0 and a perturbation H'. The unperturbed Hamiltonian includes the rest energy of the quarks, their relativistic kinetic energy, and a spin-independent potential V, which consists of a term arising from one-gluon exchange and a confinement term, which presumably arises from multigluon exchange.

We assume that the spin-independent potential is flavor independent. By this we mean that V does not explicitly contain terms which depend on the quark masses and that any parameters in V are the same for all quarks. The variation of the running coupling constant α_s with quark masses, which one might expect to be a problem, is not. This is because we can incorporate asymptotic freedom into the potential by letting α_s depend on the separation r between quarks rather than on the quark masses.

The perturbation H' includes a Coulomb term V_C, a magnetic term V_A, and Fermi-Breit corrections to the one-gluon-exchange potential and to the confining potential. These corrections include a hyperfine or color-magnetic term V_R, the existence of which is well established empirically.

Because we are restricting ourselves to those hadron ground states which (to a good approximation) have zero orbital angular momentum, we can omit the spin-orbit term and the tensor term of the Fermi-Breit interaction. The spin-independent term in the Fermi-Breit interaction depends strongly on whether the confining potential transforms like a Lorentz scalar or like the time component of a Lorentz four-vector.[9] Unfortunately, the transformation properties of the confining potential are not established empirically, and so this term is highly model dependent. For this reason we shall omit the spin-independent term is so far as it depends explicitly on quark masses. Any part which does not contain the quark masses can be lumped into the potential V.

An eigenvalue of the unperturbed Hamiltonian for a hadron is given in our formulation as a sum of the constituent-quark masses plus an energy E, which includes the effect of the potential and the relativistic kinetic energy. We note that E can be either positive or negative, depending on whether the state in question (neglecting the perturbations) has a mass greater or less than the sum of the masses of its constituent quarks. Because the constituent-quark masses are not known precisely, different models can yield opposite signs for E. Fortunately, this sign ambiguity is not troublesome because we are concerned, not with the value of E, but only with how E changes when a quark of one flavor is replaced by a quark of another flavor. As we shall see later, if the new quark has a larger mass than the old one, the change in E is negative.

The perturbative terms V_C, V_A, and V_R are operators whose expectation values are of known sign in any meson, although their magnitudes depend on a specific model. In baryons the perturbative terms are sums of two-body operators. The expectation value of each of these two-body operators is of known sign in any baryon, although the sign of the sum may be model dependent.

We are concerned first with the mass difference between two ground-state hadrons which contain quarks of the same flavor but which differ either in their total spin or in their spin configuration. These mass differences arise primarily from the color-magnetic (or hyperfine) term V_R. The magnetic term also contributes, but is much smaller and may be neglected. We are also concerned with the mass splitting of isospin multiplets containing the same quark flavors except that one or more u quarks may be replaced by d quarks, and in this case we must keep all terms.

The isospin mass differences are often called electromagnetic. There is an electromagnetic contribution to these mass differences, but the mass difference between the u and d quark also plays an essential role. At present we do not have a good explanation for why any of the quarks have their observed masses and cannot make a good case that the $u-d$ mass difference is electromagnetic. Once the u and d have different masses, then the strong interaction will lead to additional mass-dependent effects. In particular, a change in mass will change both the binding energy and the color-magnetic energy.

In Sec. II we formulate the problem. In Sec. III we obtain some inequalities among meson mass differences, and in Sec. IV we obtain both equalities and inequalities among baryon mass differences. Then in Sec. V we briefly present our conclusions.

II. GENERAL FORMULATION

According to the discussion in the previous section, the mass of a hadron containing quarks with masses m_i can be written as

$$M=\sum_i m_i+E+E_C+E_A+E_R \ , \tag{2.1}$$

where $\sum_i m_i+E$ is the eigenvalue of H_0, and E_C, E_A, and E_R are, respectively, the expectation values of the Coulomb, magnetic, and color-magnetic energies. These energy terms depend on the quark content of the hadron M, and we shall sometimes make this explicit by using appropriate subscripts.

The quantities E_C and E_A depend on the charges of the quarks, and E_A and E_R depend on the quark spins. We can factor out this known charge and spin dependence as follows[2] (we introduce a factor 3 into certain definitions for later convenience):

$$E_C=3\sum_{i<j} Q_iQ_jC_{ij} \ , \tag{2.2}$$

$$E_A=-3\sum_{i<j} Q_iQ_j\sigma_i\cdot\sigma_jA_{ij} \ , \tag{2.3}$$

$$E_R=\sum_{i<j} \sigma_i\cdot\sigma_jR_{ij} \ , \tag{2.4}$$

where

$$C_{ij}=\alpha\langle 1/r_{ij}\rangle/3 \ , \tag{2.5}$$

$$A_{ij}=2\pi\alpha\langle\delta(r_{ij})\rangle/(9m_im_j) \ , \tag{2.6}$$

$$R_{ij}=\langle\nabla^2V_v(r_{ij})\rangle/(6m_im_j) \ . \tag{2.7}$$

In the expression for R_{ij}, $V_v(r_{ij})$ is that part of the potential V which acts between quarks i and j and which transforms like the time component of a Lorentz four-vector.[9] In order to obtain Eq. (2.7), we have had to assume that the vector part of V is a sum of two-body potentials. The quark masses appearing in Eqs. (2.6) and (2.7) must be constituent masses for these equations to be even approximately correct for the light quarks.

A simplification arises if the spin-spin interaction arises entirely from one-gluon exchange and if the variation of α_s with r is neglected.[2,6] In this case the color-magnetic energy has the same form as the magnetic energy except for color factors and the replacement of the fine-structure constant α by the strong-interaction coupling strength α_s. Then

$$R_{ij}=3F\alpha_sA_{ij}/\alpha \ , \tag{2.8}$$

where F is $\frac{4}{3}$ for mesons and $\frac{2}{3}$ for baryons. However, it turns out that we do not need to make this assumption.

We can see from Eqs. (2.5), (2.6), and (2.8) that C_{ij}, A_{ij}, and R_{ij} are positive-definite quantities. In a wide variety of potential models, including models with potentials of the form

$$V_v(r)=\sum_n a_nr^{z_n}, \;\; a_nz_n>0, \;\; z_n\geq -1 \ , \tag{2.9}$$

with a_n and z_n constants, the R_{ij} are positive definite even if the more general expression given in Eq. (2.7) is used. Because the expression for V_v is quite flexible, we believe the R_{ij} are positive for any realistic potential.

Although C_{ij}, A_{ij}, and R_{ij} are the expectation values of two-quark operators, for baryons these quantities depend in general on the third quark through their dependence on the baryon wave functions.[3] Despite this additional complication for baryons, our framework allows us to obtain more predictions concerning baryons than about mesons. The basic reason for this is that there are more ground-state baryons than mesons with no orbital angular momentum, and furthermore, there exist baryon isospin multiplets with three or four different masses, whereas the meson isospin multiplets composed of a single quark-antiquark pair have at most two distinct masses.

In the following sections, we denote the mass of a hadron by its symbol as given by the Particle Data Group.[10] We use an asterisk on the symbol to denote a meson of spin 1 or a baryon of spin $\frac{3}{2}$. Otherwise (except for the Δ baryon) mesons have spin 0 and baryons have spin $\frac{1}{2}$.

III. MESONS

Let us consider meson states first, as they are composed of only two particles and are therefore easier to handle, although, as we have remarked, our method does not allow us to say very much about their mass splittings. We first briefly mention the difference in mass of a vector and pseudoscalar meson containing the same quarks. As has been pointed out many times previously, the sign of the color-magnetic term is such as to make the vector meson heavier than the pseudoscalar in all cases.

The question of the mass dependence of the magnitude of the splitting requires some discussion. It is true that the color-magnetic term R_{ij} contains the quark masses in the denominator. However, for some potentials the magnitude of the matrix element $\langle\nabla^2V_v(r_{ij})\rangle$ appearing in the numerator also increases with increasing quark masses. In these cases, whether R_{ij} increases or decreases with increasing mass depends on the form of the potential V.

We have examined the behavior of R_{ij} using a number of different potentials motivated from QCD. All the potentials we have considered have the property that for a meson containing a light quark (u,d,s), $\langle \nabla^2 V_v(r_{ij})\rangle$ increases as any quark mass increases, but the magnitude of the increase is not sufficient to overcome the effect of $m_i m_j$ in the denominator. Therefore, R_{ij} decreases as the quark masses increase. If we take this to be a general result and if we use approximate values of the constituent-quark masses as input, we obtain the inequalities (we omit the symbol for the meson charge in cases where the inequality holds independently of the charge):

$$K^*-K > D^*-D > D_s^*-D_s > B^*-B > B_s^*-B_s > T^*-T \ , \tag{3.1}$$

$$T^{*+}-T^+ > T_s^{*+}-T_s^+ \ , \tag{3.2}$$

Of these inequalities, the ones involving the B_s, T, and T_s have not been tested, but the others hold experimentally, as can be seen from Table I. Mesons composed only of heavy quarks may not obey the pattern of (3.1) because the part of the potential which goes like $1/r$ becomes more and more important as the quark masses increase. Eventually, for sufficiently heavy quark masses, the matrix element can increase faster than $m_i m_j$ and so can overcome the effect of this factor in the denominator.

We now turn to consideration of the splitting of meson isospin doublets, each of which contains one u or d quark and one other quark h, which may be s, c, b, t, or another heavy quark. There are contributions to the mass splitting both from electromagnetic effects and from the $d-u$ mass difference.

The mass difference of the members of an isospin doublet can be written

TABLE I. Experimental values of hadron mass splittings (excluding isospin splittings) which are relevant to the considerations of this paper. The data are from the Particle Data Group (Ref. 10) except for the $\Sigma_c-\Lambda_c$ splitting, which is from Macfarlane (Ref. 16).

Hadrons	Mass difference (MeV)
K^*-K	397
D^*-D	142
$D_s^*-D_s$	139
B^*-B	52
$\Sigma-N$	253
$\Xi-\Sigma$	191
$\Sigma-\Lambda$	77
$\Sigma^*-\Sigma$	191
$\Xi^*-\Xi$	216
$\Sigma^*-\Delta$	151
$\Sigma_c-\Lambda_c$	168

$$M_{dh}-M_{uh}=\epsilon+E_{dh}-E_{uh}+C_Q C_{qh}+A_{QS}A_{qh} +R_S(R_{dh}-R_{uh}) \ , \tag{3.3}$$

where $\epsilon=m_d-m_u$ and C_Q, A_{QS}, and R_S are factors which depend on charge Q and/or spin S and can be calculated by making use of Eqs. (2.2)–(2.4). For example, the spin factor R_S $(S=1,0)$ is given by $R_1=1$, $R_0=-3$. We need not distinguish between the masses of the u and d in C_{qh} and A_{qh} because these quantities are already small, being proportional to the fine-structure constant α. On the other hand, the differences in binding energy and color-magnetic energy vanish if we neglect ϵ. However, because ϵ is small, we can write

$$E_{dh}-E_{uh}=\epsilon E'_{qh}, \quad R_{dh}-R_{uh}=\epsilon R'_{qh} \ , \tag{3.4}$$

where E'_{qh} and R'_{qh} are the derivatives of E_{qh} and R_{qh}, respectively, with respect to the average light-quark mass. Then we get

$$M_{dh}-M_{uh}=\epsilon(1+E'_{qh}+R_S R'_{qh})+C_Q C_{qh}+A_{QS}A_{qh} \ . \tag{3.5}$$

We are assuming that the constituent mass difference ϵ can be treated like a unique quantity which is independent of the hadron in which the u or d is bound. This assumption is not true in all models. However, even if ϵ is a single constant, we see from Eq. (3.5) that the meson isospin mass splitting depends on electromagnetic effects and on an *effective* $d-u$ mass difference ϵ_{hS}, which is given by

$$\epsilon_{hS}=\epsilon(1+E'_{qh}+R_S R'_{qh}) \ . \tag{3.6}$$

This effective mass difference depends explicitly on the binding energy and color-magnetic energy of the meson, and so is not the same for all mesons. If we make use of Eq. (3.6) and evaluate the Coulomb and magnetic factors, we obtain the following expressions for the mass splittings of K, K^*, D, and D^* isospin doublets:

$$K^0-K^+=\epsilon_{s0}-C_{qs}-3A_{qs} \ , \tag{3.7}$$

$$K^{*0}-K^{*+}=\epsilon_{s1}-C_{qs}+A_{qs} \ , \tag{3.8}$$

$$D^+-D^0=\epsilon_{c0}+2C_{qc}+6A_{qc} \ , \tag{3.9}$$

$$D^{*+}-D^{*0}=\epsilon_{c1}+2C_{qc}-2A_{qc} \ . \tag{3.10}$$

Expressions for the mass differences B^0-B^+ and $B^{*0}-B^{*+}$ can be obtained by replacing s by b in the right-hand side of Eqs. (3.7) and (3.8). In the model, analogous expression hold for the mass splitting of any meson isospin doublet containing a heavy quark of charge $-\frac{1}{3}$. Likewise, expressions analogous to those of Eqs. (3.9) and (3.10) hold for any meson isodoublet containing a heavy quark of charge $+\frac{2}{3}$. Expressions similar to those given in Eqs. (3.7)–(3.10), have been written down some time ago,[5] except that it was assumed in Ref. 5 that ϵ_{hS} did not depend on mass or spin. Chan[6] has also written similar expressions, neglecting the variation of matrix elements with h.

What can we say about the meson isospin mass splittings within this rather general framework? First, in nearly any model, C_{qh} will increase as the mass of h increases because the size of the wave function decreases with increasing mass, thereby causing the expectation value $\langle 1/r \rangle$ to increase. However, the magnitude of the variation of the expectation value with m_h depends upon the form of the potential V. Second, just as R_{qh} decreases with increasing m_h, so also will A_{qh} and for the same reason.

The behavior of E_{qh} with mass depends on the potential V. According to QCD, V goes like α_s/r at small distances; if we want to include the effect of asymptotic freedom, we can let α_s vary approximately logarithmically with r. Also, there is evidence from lattice QCD that the potential rises linearly at larger distances. A potential of this form is consistent with what we know about the mass spectra of quarkonium states.

However, we do not need to specify the form of V precisely. It is sufficient for our present considerations to assume that V is a local potential which rises monotonically as r increases and also that V does not depend on the quark masses. Then it is a rather general property of both relativistic and nonrelativistic two-body quantum-mechanical wave equations that the energy eigenstates E_{qh} (excluding the rest energy) decrease as the mass of either particle increases. The reason for this behavior is that as the mass increases, the kinetic energy decreases. It follows that E'_{qh} is negative.

For some potentials, it is possible that $E'_{qh} < -1$, so that a positive value of ϵ_{hS}, which is required phenomenologically, might require a negative value of ϵ. (It is conceivable that the constituent mass of the u quark might be larger than that of the d quark even if the d has the larger current mass.) The possibility that ϵ might be negative has previously been considered.[11] However, it has not been shown that potentials which require ϵ to be negative are compatible with QCD or that they can be used successfully in calculating quarkonia mass spectra. For the successful quarkonium potentials of which we are aware, whether phenomenological or motivated by QCD, we have

$$-1 < E'_{qh} < 0 . \tag{3.11}$$

It is plausible that this inequality holds for all viable models, a result which, when combined with our expressions for the meson mass differences and with the experimental values of those differences, requires ϵ to be positive.

A loophole in our argument is that the spin-independent part of the Breit-Fermi correction to V, which we have omitted, contains the quark masses explicitly, and therefore contradicts our assumption that V is flavor independent. Nevertheless, we believe our result is correct, because it is unlikely that the mass-dependent contribution to V, for which there is little if any empirical evidence, can be large enough to overturn our conclusion.

We next turn to the quantity R'_{qh}. Although R_{qh} is positive definite, in order to ascertain whether R'_{qh} is positive or negative, we need to compare R_{dh} with R_{uh}, and this depends on whether $\langle \delta(r_{dh}) \rangle / m_d$ is larger or smaller than $\langle \delta(r_{uh}) \rangle / m_u$. Although both the magnitude and sign of R'_{ij} are model dependent, we shall give a plausibility argument that R'_{qh} is negative and small in magnitude compared to unity. Empirically, as noted previously,[12] there is an approximate relation between the mass M^* of the ground-state vector meson and the mass M of the ground-state pseudoscalar meson containing the same quarks:

$$M_{ij}^{*2} - M_{ij}{}^2 = 0.56\ \text{GeV}^2 = \kappa . \tag{3.12}$$

This relation holds quite well, except for self-conjugate mesons with zero isospin (which, as we have already remarked, are subject to complications from annihilation and mixing effects). If we neglect electromagnetic effects, in our picture we have

$$R_{ij} = (M^*_{ij} - M_{ij})/4 = \kappa/[4(M^*_{ij} + M_{ij})] . \tag{3.13}$$

If we make the approximation

$$M^*_{ij} + M_{ij} = 2(m_i + m_j) , \tag{3.14}$$

we obtain

$$R_{ij} = \kappa/[8(m_i + m_j)] . \tag{3.15}$$

Then, taking the derivative, we obtain

$$R'_{ij} = -\kappa/[8(m_i + m_j)^2] . \tag{3.16}$$

We can improve this expression by once again using Eq. (3.14) in order to obtain a formula involving known meson masses. We obtain

$$R'_{ij} = -\kappa/[2(M^*_{ij} + M_{ij})^2] . \tag{3.17}$$

If we substitute into this expression the masses of the strange and charmed mesons, we find

$$R'_{qs} = -0.15, \quad R'_{qc} = -0.02 , \tag{3.18}$$

and R'_{qb} is negligibly small. We see from Eqs. (3.5) and (3.18) that even in the case of the s quark, R' is considerably too small in magnitude to change the sign of ϵ relative to that of ϵ_{s1}.

In view of Eqs. (3.7)—(3.10), our argument that R' is negative and small has as a consequence the following inequalities:

$$D^+ - D^0 > D^{*+} - D^{*0}, \quad D^+ - D^0 > 0 , \tag{3.19}$$

both of which are satisfied by experiment, as can be seen from Table II. Somewhat stronger relations hold for B and T mesons because the magnetic and color-magnetic contributions are negligible. We obtain

$$B^0 - B^+ = B^{*0} - B^{*+} , \tag{3.20}$$

$$T^+ - T^0 = T^{*+} - T^{*0}, \quad T^+ - T^0 > 0 . \tag{3.21}$$

The above equalities should hold considerably better than 1 MeV. The first is consistent with the data, but the experimental uncertainty is rather large. The second is untested. We cannot say whether the T^{*0} or T^+ has a larger mass because in this case the effects of the $d-u$ mass difference and the Coulomb energy may be larger than the color-magnetic energy.

TABLE II. Experimental values of isospin mass splittings which are relevant to this paper. Data are from the Particle Data Group (Ref. 10) except for the $\Sigma_c^0-\Sigma_c^{++}$ mass difference, which is from Macfarlane (Ref. 16).

Hadrons	Mass difference (MeV)
K^0-K^+	4.05 ± 0.07
D^+-D^0	4.7 ± 0.3
B^0-B^+	4.0 ± 3.4
$K^{*0}-K^{*+}$	4.4 ± 0.5
$D^{*+}-D^{*0}$	3 ± 2
$n-p$	1.29
$\Sigma^0-\Sigma^+$	3.09 ± 0.07
$\Sigma^--\Sigma^+$	7.97 ± 0.07
$\Xi^--\Xi^0$	6.4 ± 0.6
$\Delta^0-\Delta^{++}$	2.7 ± 0.3
$\Sigma^{*0}-\Sigma^{*+}$	1 ± 1
$\Sigma^{*-}-\Sigma^{*+}$	4.4 ± 0.7
$\Xi^{*-}-\Xi^{*0}$	3.2 ± 0.6
$\Sigma_c^0-\Sigma_c^{++}$	-2.5 ± 1

IV. BARYONS

We shall restrict ourselves to baryons containing at most one heavy quark, as measurement of the masses of other baryons seems more remote. However, it is straightforward to obtain formulas and inequalities for baryons containing two or more heavy quarks which are analogous to the ones we derive here.

The mass of a baryon is given by Eq. (2.1), with E_C, E_A, and E_R defined by Eqs. (2.2)–(2.6) and (2.7) or (2.8). The charge matrix elements are simple to evaluate in terms of the constants C_{ij} and A_{ij}. However, in order to evaluate spin matrix elements, we need to specify the baryon spin wave functions. We use wave functions of Franklin, Lichtenberg, Namgung, and Carydas[13] which are either symmetric or antisymmetric under the interchange of the spin coordinates of the first two quarks.

In order to make the above wave functions unique, we need to specify which quarks in the baryon are the first two. We follow the prescription of Franklin, Lichtenberg, Namgung, and Carydas.[13] If a baryon contains two identical quarks, these are the first two, and the spin wave function is symmetric under their interchange. If all three quarks are different, the two lightest quarks are the first two, and their spin wave function may be either symmetric or antisymmetric. It was shown that with this choice, mixing effects in eigenstates of the mass operator are small, and so we shall neglect them. The spin wave function which is antisymmetric in the spins of the first two quarks is the same as the spin wave function of the Λ baryon, and so we shall use a subscript Λ on the symbol for any baryon which has this wave function.

The energy eigenvalue of H_0 for a baryon depends on all three quarks it contains, and therefore we shall indicate all three quarks by subscripts, i.e., E_{ijk}. Because H_0 contains no spin-dependent terms and because the potential is independent of flavor, the eigenvalue E_{ijk} depends on the quark masses only through the kinetic energy operator, which is a sum of one-body terms. It follows that E_{ijk} is symmetric under the interchange of any pair of its indices.

As we have remarked previously, matrix elements of two-quark operators depend on the presence of the third, or spectator, quark through its influence on the baryon wave function. We take note of this explicitly by indicating the spectator quark as a third subscript on the matrix element. Thus, we make Eq. (2.5) more explicit by writing

$$C_{ij}\to C_{ijk}=\alpha\langle ijk\,|\,1/r_{ij}\,|\,ijk\rangle/3\ . \tag{4.1}$$

Similarly $A_{ij}\to A_{ijk}$ and $R_{ij}\to R_{ijk}$. The C_{ijk}, A_{ijk}, and R_{ijk} are symmetric under the interchange of their first two indices.

We next take up the difficult question of the variation of E_{ijk}, C_{ijk}, A_{ijk}, and R_{ijk} with quark masses. For a general three-body potential we can say little. However, there is good evidence that the potential between quarks in a baryon is well approximated by a sum of two-body potentials of strength half as great as the quark-antiquark potential in mesons. This evidence is partly from the successful phenomenological treatment of baryons with such a potential[14] and partly from lattice gauge theory.[15]

With this approximate potential, increasing the mass of any quark lowers its kinetic energy and does not increase its potential energy, so that the eigenvalue E_{ijk} will decrease. Furthermore, the heavier the quarks i and j, the more their wave function will be concentrated at small distances and the larger will be the expectation value of $1/r_{ij}$. If the mass of the third quark k increases, the wave function will also shrink, but will affect the expectation value of $1/r_{ij}$ less than the expectation value of $1/r_{ik}$ or $1/r_{jk}$. Therefore, the following inequalities hold among the C_{ijk}:

$$C_{ijk}<C_{ijl},\quad C_{ikj}<C_{ilj}\ ,\quad C_{ikl}<C_{ilk},\quad m_k<m_l\ . \tag{4.2}$$

Similarly, the expectation value of $\delta(\mathbf{r}_{ij})$ will increase as the mass of any quark increases. However, this does not imply that A_{ijk} will increase because the expression for A_{ijk} contains the quark masses m_i and m_j in the denominator. We have found that for a number of models with realistic potentials and baryons containing no more than one heavy quark, A_{ijk} in fact decreases with increasing m_i or m_j. Because m_k does not appear in the denominator, A_{ijk} will increase with increasing m_k.

The above considerations lead us to believe that the A_{ijk} satisfy the inequalities

$$A_{ijk}<A_{ijl},\quad A_{ilj}<A_{ikj},\quad A_{ilk}<A_{ikl},\quad m_k<m_l\ . \tag{4.3}$$

Furthermore, in our picture the R_{ijk} have structure which is similar to the A_{ijk}, and so satisfy inequalities analogous to those of Eq. (4.3). In applying these inequalities, we do not distinguish between the u and d quarks in C_{ijk} and A_{ijk}.

We now make the further approximation that replacing a u spectator quark by a d spectator has a negligible effect on the average distance between the nonspectator quarks. Then, even in R_{ijk}, we need not distinguish between u and d spectator quarks, and can write $R_{iju}=R_{ijd}=R_{ijq}$.

The sign of the color-magnetic term is such as to make any baryon of spin $\frac{3}{2}$ heavier than its spin-$\frac{1}{2}$ counterpart (containing the same flavors). Aside from this well-known result, we can use the inequalities satisfied by the R_{ijk} to obtain the following inequalities:

$$\Xi^* - \Xi > \Sigma^* - \Sigma \ , \tag{4.4}$$

$$\Sigma^* - \Delta > (3\Lambda - \Sigma - 2N)/2 \ , \tag{4.5}$$

$$\Sigma - N > \Xi - \Sigma \ , \tag{4.6}$$

$$\Sigma_c - \Lambda_c > \Sigma - \Lambda > 0 \ , \tag{4.7}$$

$$\Xi_t - \Xi_{t\Lambda} > \Xi_b - \Xi_{b\Lambda} > \Xi_c - \Xi_{c\Lambda} > 0 \ , \tag{4.8}$$

$$\Sigma^* - \Sigma > \Sigma_c^* - \Sigma_c > \Sigma_b^* - \Sigma_b > \Sigma_t^* - \Sigma_t \ , \tag{4.9}$$

$$\Sigma_t - \Lambda_t > \Sigma_b - \Lambda_b > \Sigma_c - \Lambda_c \ , \tag{4.10}$$

$$\Xi_c^* - \Xi_c > \Xi_b^* - \Xi_b > \Xi_t^* - \Xi_t \ , \tag{4.11}$$

$$\Xi_c^* - \Xi_{c\Lambda} > \Sigma_c^* - \Sigma_c > \Xi_c^* - \Xi_c \ . \tag{4.12}$$

In obtaining the first of the inequalities given in (4.12), we have had to use a rough estimate of the constituent-quark masses as well as the inequalities satisfied by the R_{ijk}. Inequalities analogous to those of (4.12) hold if the c quark is replaced by a b or t.

The inequalities (4.4)—(4.7) are in agreement with experiment, as can be seen from Table I (the remaining ones are untested), and so give us further confidence in our model. In particular, many models predict that (4.4) and (4.5) should be equalities. However, experimentally, the $\Xi^* - \Xi$ splitting is about 25 MeV greater than the $\Sigma^* - \Sigma$ splitting, and the $\Sigma^* - \Lambda$ splitting is about 13 MeV greater than the combination $(3\Lambda - \Sigma - 2N)/3$, as can be seen from Table I. The reason we get an inequality for (4.4) is that in the case of the Ξ^* and Ξ, the relevant matrix element is R_{qss}, while in the case Σ^* and Σ, it is R_{qsq}. The latter matrix element is smaller because of a lighter spectator quark. Similarly, on the left-hand side of (4.5), the relevant matrix element is R_{qqs}, while on the right-hand side it is the smaller R_{qqq}. Again, the spectator quark influences the matrix element through its effect on the baryon wave function.[3]

We next define derivatives analogous to those of Sec. III as

$$E_{djk} - E_{ujk} = \epsilon E'_{qjk}, \quad R_{djk} - R_{ujk} = \epsilon R'_{qjk} \ . \tag{4.13}$$

With these results we can write down formulas for the baryon isospin mass splittings in terms of E'_{ijk}, R'_{ijk}, C_{ijk}, A_{ijk}, and ϵ. There are quite a few such formulas and so we relegate them to an Appendix. Making use of these formulas, we obtain the following relations, which have been previously obtained many times under a variety of more restrictive assumptions:

$$n - p = \Delta^0 - \Delta^+ = (\Delta^- - \Delta^{++})/3 \ , \tag{4.14}$$

$$\Sigma^- - 2\Sigma^0 + \Sigma^+ = \Sigma^{*-} - 2\Sigma^{*0} + \Sigma^{*+} \ , \tag{4.15}$$

$$\Sigma_c^0 - 2\Sigma_c^+ + \Sigma_c^{++} = \Sigma_c^{*0} - 2\Sigma_c^{*+} + \Sigma_c^{*++} \ . \tag{4.16}$$

The Δ mass splittings are not known well enough from experiment to test Eq. (4.14). Equation (4.15) is satisfied within the experimental error (see Table II) and Eq. (4.16) is not yet tested. We obtain an equation similar to Eq. (4.15) if the s quark is replaced by the b quark or by any heavy quark of charge $-\frac{1}{3}$, and an equation similar to Eq. (4.16) if the c quark is replaced by the t or any heavy quark of charge $\frac{2}{3}$.

We gave arguments in Sec. III on mesons why the quantities E'_{qh} and R'_{qh} should be negative and smaller than unity in magnitude. It is plausible that analogous results also hold for baryons. We assume that this is the case: namely, that E'_{ijk} and R'_{ijk} are negative and that their magnitudes are sufficiently less than unity so that the effective values of ϵ_B, where the subscript denotes any baryon, are positive. (These statements are true for models we have investigated.) Then we obtain

$$\Sigma^- - \Sigma^+ > \Sigma^- - \Sigma^0, \quad \Sigma^- - \Sigma^+ > 0 \ , \tag{4.17}$$

$$\Sigma^- - \Sigma^+ > \Sigma^{*-} - \Sigma^{*+}, \quad \Sigma^- - \Sigma^0 > \Sigma^{*-} - \Sigma^{*0} \ , \tag{4.18}$$

$$\Xi^- - \Xi^0 > \Xi^{*-} - \Xi^{*0}, \quad \Xi^- - \Xi^0 > 0 \ , \tag{4.19}$$

$$\Xi_{c\Lambda}^0 - \Xi_{c\Lambda}^+ > \Xi_c^0 - \Xi_c^+ \ , \tag{4.20}$$

as well as

$$\Xi_{c\Lambda}^0 - \Xi_{c\Lambda}^+ > \Xi_c^{*0} - \Xi_c^{*+} \ , \tag{4.21}$$

$$\Xi_{b\Lambda}^- - \Xi_{b\Lambda}^0 > \Xi_b^- - \Xi_b^0 > 0 \ . \tag{4.22}$$

The inequalities (4.17)—(4.19) are in agreement with the data, but (4.20)—(4.22) have not been tested.

The inequalities (4.8) and (4.20) tell us that, of the two sets of Ξ_c isospin doublets, the doublet of smaller mean mass has the larger isospin mass splitting. Our formulas do not enable us to say whether the $\Xi_c - \Xi_{c\Lambda}$ splitting is greater or less than that of the $\Sigma - \Lambda$, and so we guess that the splittings are comparable. The significance of this is that the Ξ_c should decay electromagnetically to $\Xi_{c\Lambda}$, being stable against strong decay. If so, the Ξ_c should be quite narrow, and in principle the mass difference $\Xi_c^0 - \Xi_c^+$ should be measurable quite accurately.

An inequality analogous to (4.20) should hold if the c quark is replaced by a t, but it should be very nearly an equality because the magnetic and color-magnetic matrix elements which violate the equality are small. We also get two other inequalities, which, because of the smallness of the relevant magnetic and color-magnetic matrix elements, are approximate equalities. Although the prospects for measuring the relevant masses in the near term are not good, we exhibit these equalities here:

$$\Xi_b^- - \Xi_b^0 = \Xi_b^{*-} - \Xi_b^{*0} \ , \tag{4.23}$$

$$\Xi_t^0 - \Xi_t^+ = \Xi_t^{*0} - \Xi_t^{*+} \ . \tag{4.24}$$

It can be seen from the Appendix that

$$\Sigma^- - 2\Sigma^0 + \Sigma^+ = 3(C_{qqs} - A_{qqs}) \ . \tag{4.25}$$

We see from experiment that the left-hand side of Eq. (4.25) is positive, and so can be satisfied only if $C_{qqs} > A_{qqs}$. It is plausible that the same inequality holds independently of the flavor of the spectator quark. Then the following inequalities hold:

$$\Sigma^{*-}-\Sigma^{*0}>0\ , \tag{4.26}$$

$$\Delta^{-}-\Delta^{0}>\Delta^{0}-\Delta^{+}>\Delta^{+}-\Delta^{++}\ , \tag{4.27}$$

$$\Sigma_c^0-\Sigma_c^+>\Sigma_c^+-\Sigma_c^{++}\ , \tag{4.28}$$

as well as analogous inequalities involving the Σ_b and Σ_t. So far, except for (4.26), these inequalities have not been tested.

V. CONCLUSIONS

In this work we have obtained a number of equalities and inequalities among the masses of hadrons, all of which are in agreement with the present data.[10,16] Our arguments have depended principally on general features of QCD which are believed to be qualitatively correct, on the approximate validity of the constituent-quark picture to describe hadron bound states, and on the existence of ordinary electric and magnetic interactions among bound quarks.

We have avoided, on the one hand, making quantitative calculations which depend on the details of particular models, and, on the other hand, assuming that energy eigenvalues and certain two-body matrix elements are independent of quark masses. Both of these latter approaches lead to additional results, but some of them are in disagreement with experiment. In particular, our taking into account the effect of the spectator quark on the matrix elements of two-body operators has led to two inequalities (4.4) and (4.5) which agree with experiment, whereas the neglect of the influence of the spectator quark leads to predictions in disagreement with experiment.

Because we have not made predictions which depend on the details of particular models, we expect that our inequalities will hold without exception. Owing to our neglect of certain small terms and our use of perturbation theory, our equalities cannot be exactly right, but we expect them to be good to better than 1 MeV. The complete agreement with experiments done so far gives us confidence that the interactions of quarks inside hadrons are qualitatively understood. Additional measurements of hadron mass differences can provide further tests of our general picture.

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APPENDIX

The considerations of Secs. II and IV lead to the following expressions for the mass splittings of a number of baryon isospin multiplets:

$$n-p=\epsilon_N-C_{qqq}+A_{qqq}\ , \tag{A1}$$

$$\Sigma^0-\Sigma^+=\epsilon_\Sigma-2C_{qqs}+C_{qsq}+2A_{qqs}+2A_{qsq}\ , \tag{A2}$$

$$\Sigma^--\Sigma^+=2\epsilon_\Sigma-C_{qqs}+2C_{qsq}+A_{qqs}+4A_{qsq}\ , \tag{A3}$$

$$\Xi^--\Xi^0=\epsilon_\Xi+2C_{qss}+4A_{qss}\ , \tag{A4}$$

$$\Delta^+-\Delta^{++}=\epsilon_N-4C_{qqq}+4A_{qqq}\ , \tag{A5}$$

$$\Delta^0-\Delta^+=\epsilon_N-C_{qqq}+A_{qqq}\ , \tag{A6}$$

$$\Delta^--\Delta^{++}=3\epsilon_N-3C_{qqq}+3A_{qqq}\ , \tag{A7}$$

$$\Sigma^{*0}-\Sigma^{*+}=\epsilon_{\Sigma^*}-2C_{qqs}+C_{qsq}+2A_{qqs}-A_{qsq}\ , \tag{A8}$$

$$\Sigma^{*-}-\Sigma^{*+}=2\epsilon_{\Sigma^*}-C_{qqs}+2C_{qsq}+A_{qqs}-2A_{qsq}\ , \tag{A9}$$

$$\Xi^{*-}-\Xi^{*0}=\epsilon_{\Xi^*}+2C_{qss}-2A_{qss}\ , \tag{A10}$$

$$\Sigma_c^+-\Sigma_c^{++}=\epsilon_{\Sigma_c}-2C_{qqc}-2C_{qcq}+2A_{qqc}-4A_{qcq}\ , \tag{A11}$$

$$\Sigma_c^0-\Sigma_c^{++}=2\epsilon_{\Sigma_c}-C_{qqc}-4C_{qcq}+A_{qqc}-8A_{qcq}\ , \tag{A12}$$

$$\Xi_{c\Lambda}^0-\Xi_{c\Lambda}^+=\epsilon_{\Xi_{c\Lambda}}+C_{qsc}-2C_{qcs}+3A_{qsc}\ , \tag{A13}$$

$$\Xi_c^0-\Xi_c^+=\epsilon_{\Xi_c}+C_{qsc}-2C_{qcs}-A_{qsc}-4A_{qcs}\ , \tag{A14}$$

$$\Sigma_c^{*+}-\Sigma_c^{*++}=\epsilon_{\Sigma_c^*}-2C_{qqc}-2C_{qcq}+2A_{qqc}+2A_{qcq}\ , \tag{A15}$$

$$\Sigma_c^{*0}-\Sigma_c^{*++}=2\epsilon_{\Sigma_c^*}-C_{qqc}-4C_{qcq}+A_{qqc}+4A_{qcq}\ , \tag{A16}$$

$$\Xi_c^{*0}-\Xi_c^{*+}=\epsilon_{\Xi_c^*}+C_{qsc}-2C_{qcs}-A_{qsc}+2A_{qcs}\ , \tag{A17}$$

where

$$\epsilon_N=\epsilon(1+E'_{qqq}+2R'_{qqq})\ ,$$

$$\epsilon_\Sigma=\epsilon(1+E'_{qqs}+R'_{qqs}-2R'_{qsq})\ ,$$

$$\epsilon_\Xi=\epsilon(1+E'_{qss}-4R'_{qss})\ ,$$

$$\epsilon_{\Sigma^*}=\epsilon(1+E'_{qqs}+R'_{qqs}+R'_{qsq})\ ,$$

$$\epsilon_{\Xi^*}=\epsilon(1+E'_{qss}+2R'_{qss})\ ,$$

$$\epsilon_{\Sigma_c}=\epsilon(1+E'_{qqc}+R'_{qqc}-2R'_{qcq})\ ,$$

$$\epsilon_{\Xi_{c\Lambda}}=\epsilon(1+E'_{qsc}-3R'_{qsc})\ ,$$

$$\epsilon_{\Xi_c}=\epsilon(1+E'_{qsc}+R'_{qsc}-2R'_{qcs})\ ,$$

$$\epsilon_{\Sigma_c^*}=\epsilon(1+E'_{qqc}+R'_{qqc}+R'_{qcq})\ ,$$

$$\epsilon_{\Xi_c^*}=\epsilon(1+E'_{qsc}+R'_{qsc}+R'_{qcs})\ .$$

We obtain equations for $\Sigma_b^0-\Sigma_b^+$ and $\Sigma_b^--\Sigma_b^+$ by replacing s by b in Eqs. (A2) and (A3). Likewise, we obtain equations for $\Sigma_b^{*0}-\Sigma_b^{*+}$ and $\Sigma_b^{*-}-\Sigma_b^{*+}$ by making this replacement in Eqs. (A8) and (A9). It is straightforward to obtain expressions for the mass splittings of other baryon multiplets.

*On leave from Indiana University.

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