PHYSICAL REVIEW D VOLUME 35, NUMBER 6 15 MARCH 1987

Gauge-invariant objects from Wilson loops

M. Azam
Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400 005, India
(Received 30 September 1986)

We provide the construction of a complete and irreducible set of gauge-invariant objects that can be formed out of Wilson loops passing through a point. Thus we provide a complete solution to the problem of assignment of gauge-invariant variables to loops.

I. INTRODUCTION

Wilson loops[1]

$$W(c,x)=p\exp(\oint_{c,x}A_\mu dx^\mu)$$

are very important objects in a gauge theory. It is possible to describe pure Yang-Mills theory in terms of these objects alone.[2,3] There has been attempts to describe the full QCD in the large-N limit in terms of these objects.[4]

It is obvious that in a gauge theory all observables must be gauge invariant. The set of gauge-invariant objects that can be formed out of Wilson loops in some sense generate all gauge-invariant objects in a gauge theory.[5–7] Moreover, in the description of gauge theories in terms of variables defined on loops, one must have knowledge of the complete and irreducible set of gauge-invariant variables that can be assigned to the loops. The aim of this paper is to provide such a construction. In Sec. II, we provide relevant definitions. In Sec. III, we provide construction of the basic set of gauge-invariant objects on a single loop. In Sec. IV, we provide the same construction for the case of 2 loops passing through a point for gauge theories with structure groups SU(2) and SU(3). In Sec. V, we discuss the multiloop invariants. At first, we construct the complete and irreducible set of gauge-invariant objects for the case of SU(2) and then state the general result for the case of SU(N). At the end, we provide some concluding remarks.

II. LOOPS AND WILSON LOOPS

Let us consider the set of loops $(c_1,c_2,\ldots,c_k,\ldots)$ passing through the point x (Fig. 1). These loops are given directions and the point x on the loops is to play a special role. We can define a rule for the multiplication of these loops. It is just the oriented product of the loops. The inverse of a loop is the loop traversed in the opposite direction. The point x, which we call the 0 loop, plays the role of a unit element under loop multiplication. If we denote the operation of loop multiplication by $\cdot$, then

$$c\cdot c\equiv c^2\,,\qquad c\cdot c^{-1}\equiv\{x\}\,. \tag{2.1}$$

It is clear from the definition that

$$c_1\cdot c_2\neq c_2\cdot c_1\,. \tag{2.2}$$

The set of loops $(c_1,c_2,\ldots,c_n)$ passing through the point x forms a non-Abelian group under loop multiplication with the point x as the unit element. We denote this group by G_L.

Now let us consider a gauge theory with structure group SU(N). Its generators T^a satisfy the commutation relation

$$[T_a,T_b]=2if_{abc}T^c\,, \tag{2.3}$$

where f_{abc}'s are the structure constants of the Lie algebra of the group SU(N). Let $A_\mu=A_\mu^aT^a$ be the gauge potentials. Now for a loop c, we define a path-ordered exponential in the following way. At first, we approximate the loop by a polygon $(x,x_1,\ldots,x_n,x)$ (Fig. 2). The lengths of the sides of the polygon are sufficiently small. Let $(x-x_1)^\mu=a_0^\mu$, $(x_1-x_2)^\mu=a_1^\mu,\ldots,(x_n-x)^\mu=a_n^\mu$. The path-ordered exponential is defined as

$$W(c,x)^\mu\equiv p\exp(\oint_{c,x}A_\mu dx^\mu)\times\lim_{n\to\infty}(e^{A_\mu(x)a_0^\mu}e^{A_\mu(x_1)a_1^\mu}\cdots e^{A_\mu(x_n)a_n^\mu})\,. \tag{2.4}$$

The path-ordered exponential $W(c,x)$ is called the Wilson loop. To every element of the group of loops we now associate a Wilson loop defined as in (2.4). From the definitions, the following properties follow trivially:

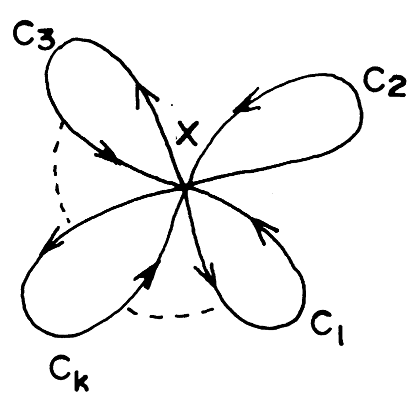

FIG. 1. $(c_1,c_2,\ldots,c_k,\ldots)$ are the loops passing through the point x.

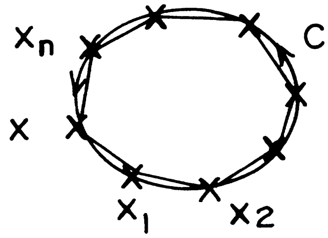

FIG. 2. A loop is approximated by the polygon $(x, x_1, \ldots, x_n, x)$.

$$
\begin{aligned}
W(c_1 \cdot c_2{}^{-1}) &= W(c_1) W(c_2{}^{-1}) \\
&= W(c_1) W^{-1}(c_2) , \\
W\{x\} &= W(c \cdot c^{-1}) = W(c) W^{-1}(c) \\
&= I .
\end{aligned}
\tag{2.5}
$$

Clearly, the set of Wilson loops form a non-Abelian group and is a representation of the group of loops G_L. We denote this group by G_W. Under gauge transformation, the Wilson loops transform as

$$W(c,x) \rightarrow g^{-1}(x) W(c,x) g(x) .$$

The quotient $G_W/\mathrm{SU}(N)$, is the set of gauge-invariant objects constructed out of the Wilson loops. These objects, clearly, are the traces of the product of Wilson loops. We want to construct the minimal set of gauge-invariant monomials, such that any other gauge-invariant object can be expressed as a polynomial in them. The elements of this set we shall call the basic invariant monomials on loops or simply, basic loop invariants.

III. 1-LOOP INVARIANTS

Let us consider the 1 loop c (Fig. 3). The elements of the group of loops G_L for this are $(c, c^2, \ldots, C^n, \ldots)$. The group of Wilson loops G_W consists of the elements

$$
\begin{aligned}
&W(c) , \\
&W(c^2) \equiv W(c \cdot c) = W^2(c) , \\
&\cdots , \\
&W(c^n) \equiv W(\underbrace{c \cdot c \cdot \cdots \cdot c}_{n\,\text{times}}) = W^n(c) .
\end{aligned}
\tag{3.1}
$$

To construct the minimal set of gauge-invariant monomials out of $W(c)$ and its higher powers, we use the theorem of Hamilton and Cayley which states that the characteristic polynomial of any matrix M becomes zero if one replaces the eigenvalue λ by M: i.e.,

$$[\mathrm{Det}(\lambda I - M)]_{\lambda = M} = 0 . \tag{3.2}$$

At first, let us write down the recurrent formula for the Hamilton-Cayley polynomial for W (from now on, we drop the arguments c and x for the convenience of notation):

$$
\begin{aligned}
&p_1 = W - \mathrm{tr} W , \\
&p_n = p_{n-1} W - \frac{1}{n} \mathrm{tr}(p_{n-1} W) .
\end{aligned}
\tag{3.3}
$$

p_n is the polynomial of degree n in the matrix W. According to the theorem of Hamilton and Cayley,

$$p_N = 0 , \tag{3.4}$$

when W is an $N \times N$ matrix. From (3.4) it is clear that $\mathrm{tr}(W^n)$ for $n > N$ can be expressed as polynomial in $(\mathrm{tr} W, \mathrm{tr} W^2, \ldots, \mathrm{tr} W^N)$. Hence, the basic invariant monomials for 1 loop are $(\mathrm{tr} W, \mathrm{tr} W^2, \ldots, \mathrm{tr} W^N)$.

IV. 2-LOOP INVARIANTS

Let us consider the loops c_1 and c_2 passing through the point x (Fig. 4). The elements of the group of loops G_L are of the form

$$c_1^{\,n_1} c_2^{\,n_2} C_1^{\,n_3} \cdots C_2^{\,n_r} \cdots , \tag{4.1}$$

where $n_1, n_2, n_3, \ldots, n_r, \ldots$ are either zero or any arbitrary integer. The elements of the group of Wilson loops are of the form

$$W_1^{n_1} W_2^{n_2} W_2^{n_3} \cdots W_2^{\,n_r} \cdots . \tag{4.2}$$

Here

$$
\begin{aligned}
&W_1 \equiv W(c_1, x) , \\
&W_2 \equiv W(c_2, x) , \\
&W_1{}^n = W(c_1{}^n, x) .
\end{aligned}
\tag{4.3}
$$

At first, we shall construct basic invariant monomials for the Yang-Mills theory with structure group SU(2). In this case the Wilson loops are (2×2) matrices. The Hamilton-Cayley equation for the matrices $W_1, W_2, W_1 W_2$ are

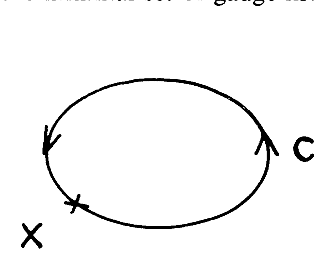

FIG. 3. c is a 1 loop passing through the point x.

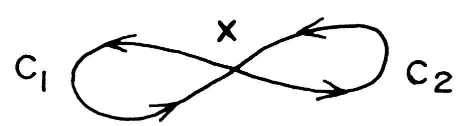

FIG. 4. c_1 and c_2 are the 1 loops passing through the point x.

$$W_1{}^2 - W_1 \mathrm{tr} W_1 + \tfrac{1}{2}(\mathrm{tr} W_1)^2 - \tfrac{1}{2}\mathrm{tr} W_1{}^2 = 0 , \qquad (4.4)$$

$$W_2{}^2 - W_2 \mathrm{tr} W_2 + \tfrac{1}{2}(\mathrm{tr} W_2)^2 - \tfrac{1}{2}\mathrm{tr} W_2{}^2 = 0 , \qquad (4.5)$$

$$(W_1 W_2)^2 - W_1 W_2 \mathrm{tr}(W_1 W_2) + \tfrac{1}{2}[\mathrm{tr}(W_1 W_2)]^2 - \tfrac{1}{2}\mathrm{tr}(W_1 W_2)^2 = 0 . \qquad (4.6)$$

The matrices $(W_1 + W_2)$ and $(W_1 - W_2)$ satisfy the Hamilton-Cayley equation of the type (4.4). Adding up, these two equations, we obtain

$$W_1 W_2 + W_2 W_1 - W_1 \mathrm{tr} W_2 - W_2 \mathrm{tr} W_1 + \mathrm{tr} W_1 \mathrm{tr} W_2 - \mathrm{tr}(W_1 W_2) = 0 . \qquad (4.7)$$

Now let us consider the most general invariant:

$$\mathrm{tr}(W_1^{n_1} W_2^{n_2} \cdots W^{n_r} \cdots) . \qquad (4.8)$$

It is clear from Eqs. (4.4) and (4.5) that if $n_r \geq 2$ for any r, it can be further reduced except for the case of $\mathrm{tr}\, W_1{}^2$, $\mathrm{tr}\, W_2{}^2$. So instead of (4.8) let us consider the invariant

$$\mathrm{tr}(W_1 W_2 W_1 \cdots) \equiv \mathrm{tr}(W_1 W_2)^n . \qquad (4.9)$$

In this equation if $n > 2$, then using (4.6), it can be expressed as a polynomial in $\mathrm{tr}(W_1 W_2)^2$ and $\mathrm{tr}(W_1 W_2)$. Now let us write $\mathrm{tr}(W_1 W_2)^2$ as $\mathrm{tr}[(W_1 W_2)(W_1 W_2)]$ and replace one of the $W_1 W_2$ by using (4.7). We obtain

$$\mathrm{tr}(W_1 W_2)^2 = \mathrm{tr}\{(W_1 W_2)[-W_2 W_1 + W_1 \mathrm{tr} W_2 + W_2 \mathrm{tr} W_1 - \mathrm{tr} W_1 \cdot \mathrm{tr} W_2 \cdot I + \mathrm{tr}(W_1 W_2) I]\}$$
$$\equiv -\mathrm{tr}(W_1 W_2{}^2 W_1) + \mathrm{tr}(W_1 W_2 W_1) \mathrm{tr} W_2 - 2\,\mathrm{tr} W_1 \cdot \mathrm{tr} W_2 + 2\,\mathrm{tr} W_1 W_2 . \qquad (4.10)$$

Using Eqs. (4.4) and (4.5) and the cyclic symmetry of the trace (4.10) can be expressed as a polynomial in $\mathrm{tr}\, W_1$, $\mathrm{tr}\, W_2$, $\mathrm{tr}\, W_1{}^2$, $\mathrm{tr}\, W_2{}^2$, and $\mathrm{tr}(W_1 W_2)$. So, the 2-loop basic invariant monomials for Yang-Mills theory with structure group SU(2) are

$$\mathrm{tr} W_1, \mathrm{tr} W_2, \mathrm{tr} W_1{}^2, \mathrm{tr} W_2{}^2, \mathrm{tr}(W_1 W_2) . \qquad (4.11)$$

Now let us consider the 2-loop invariants for the Yang-Mills theory with structure group SU(3). The elements of the group of loops G_L are given in (4.1). To each element of the group we associate a Wilson loop defined as in (2.4). The elements of the group of Wilson loops are of the form given in (4.2). Now, the Wilson loops are (3×3) matrices. The Hamilton-Cayley equation for a (3×3) matrix M is

$$P_3(M) \equiv M^3 - M^2 \mathrm{tr} M + \tfrac{1}{2} M\, \mathrm{tr} M^2 - \tfrac{1}{2} M (\mathrm{tr} M)^2 - \frac{1}{3!}\mathrm{tr} M^3 + \frac{1}{3!}\mathrm{tr} M^2 \cdot \mathrm{tr} M + \frac{1}{3!}(\mathrm{tr} M)^3 = 0 . \qquad (4.12)$$

Now the Hamilton-Cayley equations for $W_1, W_2, W_1 W_2, [W_1, W_2], W_1 + W_2, W_1 - W_2$ are

$$\begin{aligned} &P_3(W_1) = 0 ,\\ &P_3(W_2) = 0 ,\\ &P_3(W_1, W_2) = 0 ,\\ &P_3[W_1, W_2] = 0 ,\\ &P_3(W_1 + W_2) = 0 ,\\ &P_3(W_1 - W_2) = 0 . \end{aligned} \qquad (4.13)$$

Using the Hamilton-Cayley equation for $W_1, W_2, (W_1 + W_2)$ and $(W_1 - W_2)$, we obtain the equations

$$W_1 W_2{}^2 + W_1 W_2 W_1 + W_2{}^2 W_2 = 2 W_2 \mathrm{tr}(W_1 W_2) + W_1 \mathrm{tr} W_2{}^2 + \mathrm{tr}(W_1 W_2{}^2) , \qquad (4.14)$$

$$W_2 W_1{}^2 + W_2 W_1 W_2 + W_1{}^2 W_2 = 2 W_1 \mathrm{tr}(W_2 W_1) + W_2 \mathrm{tr} W_1{}^2 + \mathrm{tr}(W_2 W_1{}^2) . \qquad (4.15)$$

From Eq. (4.13), it is clear that any invariant of form

$$\mathrm{tr}(W_1^{r_1} W_2^{r_2} \cdots W_2^{r_n} \cdots) ,$$

where some r_i greater than 2 can be expressed as polynomials in $\mathrm{tr}(W_1^{r_1} W_2^{r_2} \cdots W_2^{r_n} \cdots)$ where $r_i \leq 2$ for any i. Hence, we shall consider only these invariants and show that they can be further reduced. If $r_i = 1$ for any i, then the invariant must have one of the following forms:

$$\mathrm{tr}(W_1 W_2)^n,\quad \mathrm{tr}[(W_1 W_2)^n W_1],\quad \mathrm{tr}[W_2 (W_1 W_2)^n] . \qquad (4.16)$$

Using the Hamilton-Cayley equations (4.13), it is easy to show that $\mathrm{tr}(W_1, W_2)^n$ for $n \geq 4$, and $\mathrm{tr}[(W_1 W_2)^n W_1]$ and $\mathrm{tr}[W_2 (W_1 W_2)^n]$ for $n \geq 3$ can be expressed as polynomials in $\mathrm{tr}\, W_1$, $\mathrm{tr}\, W_1{}^2$, $\mathrm{tr}\, W_1{}^3$, $\mathrm{tr}\, W_2$, $\mathrm{tr}\, W_2{}^2$, $\mathrm{tr}\, W_2{}^3$, $\mathrm{tr}(W_1 W_2)$, $\mathrm{tr}(W_1 W_2)^2$, and $\mathrm{tr}(W_1 W_2)^3$. Now let us consider the invariant $\mathrm{tr}(W_1{}^2 W_2{}^2 W_1 W_2{}^2)$. Using (4.15), we obtain

$$\begin{aligned} \mathrm{tr}(W_1{}^2 W_2{}^2 W_1 W_2{}^2) &= -\mathrm{tr}[W_1{}^2 (W_2 W_1 W_2 + W_1 W_2{}^2) W_2{}^2] \\ &\quad + (\text{sums of products of traces of lower order}) \\ &= -\mathrm{tr} W_1{}^2 W_2 W_1 W_2{}^3 - \mathrm{tr} W_1{}^3 W_2{}^4 + \cdots . \end{aligned} \qquad (4.17)$$

Now using Hamilton-Cayley equations for the matrices, it can be expressed as polynomials in trace monomials of degree less than or equal to 6. These are precisely the ones, mentioned above. With the help of the procedure described above, any monomial of the form $\mathrm{tr}(W_1^{r_1}W_2^{r_2}\cdots W_2^{r_n}\cdots)$ where $r_i\le 2$ for any i, can be expressed as polynomials in

$$\begin{aligned}&\mathrm{tr}W_1,\ \mathrm{tr}W_1{}^2,\ \mathrm{tr}W_1{}^3\ ,\\&\mathrm{tr}W_2,\ \mathrm{tr}W_2{}^2,\ \mathrm{tr}W_2{}^3\ ,\\&\mathrm{tr}(W_1W_2),\ \mathrm{tr}(W_1W_2)^2,\ \mathrm{tr}(W_1W_2)^3\ .\end{aligned} \tag{4.18}$$

They cannot be further reduced (see the next section). Hence these are the 2-loop basic invariant monomials for the Yang-Mills theory with structure group SU(3).

V. MULTILOOP INVARIANTS

Let us consider the set of loops $(c_1,c_2,\ldots,c_k,\ldots)$ passing through the point x (Fig. 1). The elements of the group of loops G_L are of the form

$$(c_{i_1}^{r_1}c_{i_2}^{r_2}\cdots c_{i_k}^{r_k})\ , \tag{5.1}$$

where the indices $(i_1,i_2,\ldots,i_k)$ numbering the loops are either the same or different and $r_k=0,1,2,\ldots,$ for any k.

Let us, at first, consider the case of the Yang-Mills theory with structure group SU(2). To every element of the group of loops G_L, we associate a Wilson loop as defined in (2.4). Now the Wilson loops are (2×2) matrices. Let $W_i\equiv W(c_i,x)$ for any i. Let σ_i, $i=1,2,3$ be the Pauli matrices:

$$\{\sigma_i,\sigma_j\}=2\delta_{ij}\ , \tag{5.2}$$

$$[\sigma_i,\sigma_j]=2i\epsilon_{ijk}\ , \tag{5.3}$$

where ϵ_{ijk} is the totally antisymmetric tensor:

$$\epsilon_{ijk}\epsilon_{kmn}=\delta_{im}\delta_{jn}-\delta_{in}\delta_{jm}\ . \tag{5.4}$$

We can take $\{I,\sigma_i\}$ as the basis of (2×2) matrices. In this basis

$$\begin{aligned}&W_i=W_{i0}\cdot I+W_{ij}\sigma_j\ ,\\&W_{i0}=\tfrac{1}{2}\mathrm{tr}W_i\ .\end{aligned} \tag{5.5}$$

It is easy to see that

$$\mathrm{tr}W_iW_jW_k=i\epsilon_{ijk}W_iW_jW_k\ . \tag{5.6}$$

Now using (5.2)—(5.5) it is easy to show that $\mathrm{tr}(W_{i_1}^{r_1}W_{i_2}^{r_2}\cdots W_{i_k}^{r_k})$ for any set of indices $(i_1,i_2,\ldots,i_k)$ and integer powers $(r_1,r_2,\ldots,r_k)$ is a polynomial in the monomials of the form

$$\mathrm{tr}W_i,\ \mathrm{tr}(W_iW_j),\ \mathrm{tr}(W_iW_jW_k) \tag{5.7}$$

for any i,j,k. Hence, these are the basic multiloop invariants for the Yang-Mills theory with structure group SU(2).

Let us now consider the most general case of the Yang-Mills theory with the structure group SU(N). To every element of the form (5.1) of the group of loops G_L, we now associate a Wilson loop as defined in (2.4). Clearly, the Wilson loops are $(N\times N)$ matrices. Let $W_i\equiv W(c_i,x)$. The problem of construction of multiloop basic invariant polynomial reduces to the construction of basic invariant monomials out of matrices W_{i_k}'s where $k=1,\ldots,n$. For this construction we use a theorem due to Processi.[6,8] The proof of the theorem is based on the Hamilton-Cayley theorem. However, we shall not provide the proof. The statement of the theorem is every invariant polynomial in F_i, F_i's being $(N\times N)$ matrices, $1\le i,\le n$ and transforming adjointly under the action of the group SU(N), i.e., $F_i\to g^{-1}F_ig$, $g\in\mathrm{SU}(N)$, is a polynomial in a small set of invariant polynomials consisting of $\mathrm{tr}(F_{i_1},F_{i_2},\ldots,F_{i_k})=S_{i_1i_2\cdots i_k}$, where $i_1,i_2,\ldots,i_k$ are either the same or different indices and k obeys the bound $k\le 2^N-1$ for $N\neq 3$, independent of n. For $N=3$, the bound is $k\le 2^3-2=6$ and is also independent of n.

Now using Processi's theorem we have the following results. The multiloop basic invariant monomials for the Yang-Mills theory with structure group SU(N) are of the form

$$\mathrm{tr}(W_{i_1}^{r_1}W_{i_2}^{r_2}\cdots W_{i_k}^{r_k})\ ,$$

where $i_1,i_2,\ldots,i_k$ are either the same or different indices and $(r_1+r_2+\cdots+r_k)\le 2^N-1$ for $N\neq 3$ and $(r_1+r_2+\cdots+r_k)\le 6$ for $N=3$.

VI. CONCLUSION

It is to be noted that the study of the loop invariants and the basic loop-invariant monomials was started by Christos.[7] However, he considers only the symmetric product of loops and, hence, the group of loops he considers is an Abelian group. We have considered here the most general case where a group of loops is non-Abelian and thus we provide complete solution to the problem of loop invariants. Our result is applicable for the case of lattice gauge theory. In particular, it might be useful for the construction of the most general Symanzik-type Lagrangian for the lattice gauge theory.

ACKNOWLEDGMENT

I am grateful to the members of theoretical physics group of the Tata Institute of Fundamental Research for many fruitful discussions.

[1]K. G. Wilson, Phys. Rev. D 7, 2971 (1973); 10, 2245 (1974).
[2]A. M. Polyakov, Nucl. Phys. **B164**, 171 (1980).
[3]A. Jevicki and B. Sakita, Nucl. Phys. **B185**, 89 (1981).
[4]Yu. M. Makeenko, and A. A. Migdal, Nucl. Phys. **B188**, 269 (1981).
[5]J. Frohlich, Lecture at Colloquium on random fields, Esztergam, Hungary, 1979 (unpublished).
[6]B. Durhuus, Lett. Math. Phys. **4**, 515 (1980).
[7]G. A. Christos, Phys. Lett. **110B**, 471 (1982).
[8]M. Azam and P. P. Divakaran, Phys. Lett. **125B**, 207 (1983).