Bifurcation of the quark self-energy: Infrared and ultraviolet cutoffs

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The quark self-energy in massless QCD is studied in the approximation that both the quark-gluon vertex and the gluon propagator remain bare. It is shown that chiral invariance is not spontaneously broken at a critical coupling $\lambda_c > 0$, unless both infrared and ultraviolet cutoffs are introduced.

I. INTRODUCTION

The possibility that quarks obtain their effective ("constituent") masses by dynamical breaking of chiral symmetry is appealing. The Dyson-Schwinger equation for the quark mass operator in chirally symmetric QCD (vanishing quark bare mass) always has the trivial vanishing solution: the question arises as to whether it can also have a nontrivial solution that violates chiral symmetry.

In this paper we consider the simple approximation in which the full vertex function and the full gluon propagator are replaced by their bare values. We show in Sec. II that, in the absence of cutoffs, there is no critical coupling $\lambda_c > 0$ such that, for $\lambda < \lambda_c$, there is only the trivial solution, while a nontrivial solution bifurcates away from it at $\lambda = \lambda_c$. In fact, there are nontrivial solutions for all positive values of λ, as was shown earlier[1] by numerical means. In Sec. III we impose an infrared cutoff by introducing an effective gluon mass: although this reduces the dimension of the manifold of the solutions, there still is no critical coupling $\lambda_c > 0$. Finally, in Sec. IV we employ both an infrared cutoff and a Pauli-Villars ultraviolet cutoff Λ. The situation is dramatically different, for now the linearized form of the equation (the functional differential evaluated at the trivial solution) is a homogeneous Fredholm equation. The spectrum is discrete and the smallest value of λ for which the linearized equation has a nontrivial solution is the first bifurcation point, λ_c, of the full nonlinear equation away from the trivial solution. The theory of Fredholm equations guarantees that $\lambda_c > 0$.

The basic lesson of this analysis, which reproduces in a simpler way certain earlier results,[2] is that *both* an infrared and an ultraviolet cutoff are necessary, if a bifurcation point, $\lambda_c > 0$, is to exist. The gluon mass is supposed to be caused by gluon self-interactions, and is intuitively appealing. The Pauli-Villars ultraviolet cutoff is not so well motivated, although we show, also in Sec. III, that the equation remains well behaved in the limit $\Lambda \to \infty$. Moreover, in this limit, λ_c does not tend to zero, as one would naively expect, but to a definite positive value. Nevertheless, a better-motivated ultraviolet regularization would be preferable; and this can be provided by the logarithmic decrease of the running coupling constant in QCD (Refs. 3 and 4). The present model has the character of a didactic example in which certain basic issues are clarified.

II. NO CUTOFF

With the vertex function and the gluon propagator replaced by bare values, the inverse quark propagator satisfies the approximate Dyson-Schwinger equation

$$S'_F{}^{-1}(p) = \not p - \frac{i\lambda}{(2\pi)^4} \int d^4p' \gamma_\mu S'_F(p') \gamma_\nu D_F^{\mu\nu}(p'-p) \ . \tag{2.1}$$

In the Landau gauge, viz.,

$$D_F^{\mu\nu}(k) = \frac{-g^{\mu\nu} + k^\mu k^\nu / k^2}{k^2 + i\epsilon} \ , \tag{2.2}$$

one finds

$$S'_F{}^{-1}(p) = \alpha(-p^2) + \not p \ , \tag{2.3}$$

where α is a solution of the integral equation

$$\alpha(x) = \frac{3\lambda}{16\pi^2} \int_0^\infty \frac{dy}{x_{\max}} \frac{y\alpha(y)}{y + \alpha^2(y)} \ , \tag{2.4}$$

with the notation $x_{\max} = \max(x,y)$.

Equation (2.4) always has the trivial solution $\alpha(x) \equiv 0$. A bifurcation of this solution into a nontrivial solution occurs at the smallest value of the coupling parameter λ for which the linearized equation ($f = \delta\alpha$)

$$f(x) = \frac{3\lambda}{16\pi^2} \int_0^\infty \frac{dy}{x_{\max}} f(y) \tag{2.5}$$

has a nontrivial solution.

This integral equation can be converted into the differential equation

$$[x^2 f'(x)]' + \frac{3\lambda}{16\pi^2} f(x) = 0 \ , \tag{2.6}$$

which has the general solution

$$f(x) = Ax^{\gamma_+} + Bx^{\gamma_-} \ , \tag{2.7}$$

with

$$\gamma_\pm = -\frac{1}{2} \pm \left[\frac{1}{4} - \frac{3\lambda}{16\pi^2} \right]^{1/2} \ . \tag{2.8}$$

The function (2.7) does indeed solve the integral equation (2.5), with arbitrary constants A and B, *for any real* $\lambda > 0$.

Thus we expect a nontrivial solution of the nonlinear equation (2.4) for any positive λ; and this has been confirmed by numerical methods.[1] There is no critical point, $\lambda_c > 0$, such that only the trivial solution exists for $0 < \lambda < \lambda_c$.

III. INFRARED CUTOFF

A simple way to introduce an infrared cutoff is to posit the existence of an effective gluon mass, m, presumably induced by gluon self-interaction.[5] Inserting a massive gluon propagator in (2.1) results in a relatively complicated integration over the direction of the loop momentum p'. It is sufficient for our purposes simply to replace the factor $x_{\max}$ in Eq. (2.4) by $x_{\max} + m^2$. Such a replacement is accurate unless p'^2 and p^2 are comparable in magnitude, and even then the relative error in the kernel is uniformly less than one. As shown in Ref. 2 the character of solutions is not affected by this simplification. In our approximation, Eq. (2.4) is modified to

$$\alpha(x) = \frac{3\lambda}{16\pi^2} \int_0^\infty \frac{dy}{x_{\max} + m^2} \frac{y\alpha(y)}{y + \alpha^2(y)} . \tag{3.1}$$

Its functional derivative with respect to α at $\alpha \equiv 0$ is

$$f(x) = \frac{3\lambda}{16\pi^2} \int_0^\infty \frac{dy}{x_{\max} + m^2} f(y) . \tag{3.2}$$

The differential equation is modified to

$$[(x + m^2) f'(x)]' + \frac{3\lambda}{16\pi^2} f(x) = 0 , \tag{3.3}$$

the general solution of which is

$$f(x) = A(x + m^2)^{\gamma_+} + B(x + m^2)^{\gamma_-} , \tag{3.4}$$

with $\gamma_\pm$ as in (2.8).

However, the solution to the integral equation (3.2) must also satisfy the boundary condition

$$f'(0) = 0 , \tag{3.5}$$

so that the integration constants A and B are not independent. The general solution to (3.2), apart from a normalization constant, is

$$f(x) = \gamma_- \left[\frac{x + m^2}{m^2} \right]^{\gamma_+} - \gamma_+ \left[\frac{x + m^2}{m^2} \right]^{\gamma_-} . \tag{3.6}$$

It is interesting that the two-parameter solution of Sec. II has been reduced to a solution with one normalization parameter by the introduction of the infrared cutoff. Nevertheless, there is still a solution for arbitrary $\lambda > 0$, so again there is no bifurcation point, $\lambda_c > 0$.

IV. ULTRAVIOLET CUTOFF

A standard way to introduce an ultraviolet cutoff is through the method of Pauli and Villars. The nonlinear equation (3.1) is replaced by

$$\alpha(x) = \frac{3\lambda}{16\pi^2} \int_0^\infty dy \left[\frac{1}{x_{\max} + m^2} - \frac{1}{x_{\max} + \Lambda^2} \right] \frac{y\alpha(y)}{y + \alpha^2(y)} , \tag{4.1}$$

the bifurcation equation by

$$f(x) = \frac{3\lambda}{16\pi^2} \int_0^\infty dy \left[\frac{1}{x_{\max} + m^2} - \frac{1}{x_{\max} + \Lambda^2} \right] f(y) . \tag{4.2}$$

Here $\Lambda \gg m$ is the Pauli-Villars cutoff.

Thanks to the ultraviolet cutoff (*and* the infrared cutoff), the kernel is L^2, and so (4.2) can be treated as a classic Fredholm equation of the homogeneous kind. We know then that if we limit the solutions, f, to L^2, there is only a point set of λ values for which (4.2) has nontrivial solutions, and that there is a smallest value, $\lambda_c > 0$, for which (4.2) has a nontrivial L^2 solution; and this corresponds to the first bifurcation of the trivial solution of (4.1) to a nontrivial solution.

However, we can do much more. There is no reason to limit ourselves to L^2, in the search for solutions of (4.2) and (4.1). Suppose that a solution, f, of (4.2) is merely required to be bounded on $(0, \infty)$. Then we have

$$|f(x)| \le \frac{3\lambda}{16\pi^2} (\Lambda^2 - m^2) \sup_{0<y<\infty} |f(y)| \int_0^\infty \frac{dy}{(x_{\max} + m^2)(x_{\max} + \Lambda^2)} < \frac{C}{(x + m^2)} , \tag{4.3}$$

where the constant C depends on Λ, m, and f.

Consider now the Banach space of bounded, continuous functions $f(x)$ with norm

$$||g|| = \sup_{0<x<\infty} [\, |g(x)| (x + m^2)^\beta] , \tag{4.4}$$

for a fixed $\beta \in (0,1)$. We know from (4.3) that any bounded, continuous solution of (4.2) belongs to this space. Furthermore,

$$\begin{aligned} |f(x)| &\le \frac{3\lambda}{16\pi^2} ||f|| \int_0^\infty \frac{dy}{(y + m^2)^\beta} \left[\frac{1}{x_{\max} + m^2} - \frac{1}{x_{\max} + \Lambda^2} \right] \\ &\le \frac{3\lambda}{16\pi^2} ||f|| \left[\int_0^x \frac{dy}{(x + m^2)(y + m^2)^\beta} + \int_x^\infty \frac{dy}{(y + m^2)^{1+\beta}} \right] \\ &\le \frac{3\lambda}{16\pi^2} ||f|| \frac{(x + m^2)^{-\beta}}{\beta(1-\beta)} . \end{aligned} \tag{4.5}$$

Thus

$$\frac{3\lambda}{16\pi^2}||f|| \geq \beta(1-\beta)||f|| , \tag{4.6}$$

and so, if $f(x)$ is not identically equal to zero, then

$$\frac{3\lambda}{16\pi^2} \geq \beta(1-\beta) \tag{4.7}$$

for any β in (0,1). The maximum of the right-hand side of the inequality (4.7) is attained at $\beta=\frac{1}{2}$, so that

$$\frac{3\lambda}{16\pi^2} \geq \frac{1}{4} . \tag{4.8}$$

The important point is that this lower bound on λ, or equivalently on the bifurcation point λ_c is independent of the Pauli-Villars cutoff Λ. The bounds (4.3), (4.6), and (4.8) also apply to solutions of the nonlinear equation (4.1), as a result of application of the relation

$$\left| \frac{y\alpha(y)}{y+\alpha^2(y)} \right| \leq |\alpha(y)| . \tag{4.9}$$

In other words, Eq. (4.1) has no nontrivial bounded solutions unless inequality (4.8) is satisfied. Furthermore, we show in the Appendix that the nonlinear differential equation (4.1) has no solutions for λ less than the bifurcation point λ_c.

It is interesting to compare this result with that of Sec. III, in which the Pauli-Villars cutoff was absent. In the latter case we had the unique solution (3.6) (aside from a trivial normalization). This solution is never in L^2; but it is in our Banach space, with $\beta=\mathrm{Re}[\gamma_+]$. Clearly, when (4.8) is saturated, $\beta=\frac{1}{2}$ [see (2.8)]. For the cutoff equation (4.2), the solution lies in the Banach space with $\beta \leq 1$. Although the constant, C, of inequality (4.3) explodes as $\Lambda \to \infty$, the inequality (4.7), which is optimized by (4.8), is independent of Λ. There is then an essential difference between the limit $\Lambda \to \infty$ of the solution of the equation with cutoff, and that of the equation without an ultraviolet cutoff.

V. DISCUSSION

In our truncated Dyson-Schwinger equation with the quark-gluon vertex and gluon propagator both free, chiral symmetry is broken spontaneously at some critical coupling $\lambda_c > 0$ if, and only if, both infrared and ultraviolet cutoffs are introduced. We have been able to reproduce and extend the results obtained by Maskawa and Nakajima,[2] using simple arguments. Their analysis is, at best, very involved, relatively unenlightening, and difficult to follow. We have found that Eq. (4.1) and those of a similar type are quite naturally treated as fixed points of mappings in a Banach space of continuous functions that are $O(x^{-1/2})$ at large x. Our approach is applicable to the general system of coupled equations studied in Ref. 2.

APPENDIX

We shall demonstrate that the nonlinear integral equation (4.1) has no solutions for λ less than λ_c, the smallest eigenvalue of the linear integral equation (4.2).

Equation (4.2) is equivalent to the differential equation

$$[p(x)f'(x)]' + \frac{3\lambda}{16\pi^2} f(x) = 0 , \tag{A1}$$

with

$$\frac{1}{p(x)} = \frac{1}{(x+m^2)^2} - \frac{1}{(x+\Lambda^2)^2} . \tag{A2}$$

Along with boundary conditions

$$f'(0)=0 , \tag{A3}$$

$$f(\infty)=0 . \tag{A4}$$

Equations (A1)–(A4) constitute a Sturm-Liouville eigenvalue problem for the function $f(x)$. It follows from standard arguments that, for the smallest eigenvalue λ_c, the eigenfunction $f_c(x)$, defined with $f_c(0) > 0$, is monotonically decreasing and positive on $[0,\infty]$.

Equation (4.1) corresponds to a nonlinear differential equation, which can be cast into the form

$$[p(x)\alpha'(x)]' + \frac{3\lambda}{16\pi^2}\alpha(x) = r(x) \tag{A5}$$

with

$$r(x) = \frac{3\lambda}{16\pi^2} \frac{\alpha^3(x)}{x+\alpha^2(x)} . \tag{A6}$$

The solutions of (4.1) also satisfy the boundary conditions

$$\alpha'(0)=0 , \tag{A7}$$

$$\alpha(\infty)=0 . \tag{A8}$$

For a given $\lambda < \lambda_c$, let us integrate the differential equations (A1) and (A5) from common starting values at $x=0$. That is, we determine $f(\lambda,x)$ and $\alpha(\lambda,x)$ for $x \geq 0$, from initial values

$$f(\lambda,0) = \alpha(\lambda,0) > 0 , \tag{A9}$$

$$\frac{df}{dx}(\lambda,0) = \frac{d\alpha}{dx}(\lambda,0) = 0 . \tag{A10}$$

Define the domain $\mathscr{D}_\lambda$ over which both functions remain positive; viz.,

$$\mathscr{D}_\lambda = \{x \mid x > 0; f(\lambda,y) > 0 \text{ and } \alpha(\lambda,y) > 0 \text{ for } y=[0,x]\} . \tag{A11}$$

The function $r(x)$ is positive on domain $\mathscr{D}_\lambda$. Multiplying Eq. (A1) by $f(x)$ and (A5) by $\alpha(x)$, subtracting, and integrating, we obtain

$$f(\lambda,x)\alpha'(\lambda,x) - \alpha(\lambda,x)f'(\lambda,x) = \frac{1}{p(x)} \int_0^x dy\, r(y) f(\lambda,y) > 0 . \tag{A12}$$

By doing another integration, we obtain

$$\alpha(\lambda,x) > f(\lambda,x) \tag{A13}$$

for $x = \mathscr{D}_\lambda$. Using a similar argument, one can show that

$$f(\lambda,x) > f(\lambda_c,x) \tag{A14}$$

for $x > 0$ and $\lambda < \lambda_c$. Therefore, $\mathscr{D}_\lambda$ consists of all positive x for $\lambda < \lambda_c$, and then $\alpha(\infty) > 0$. The integral equation (4.1) thus has no nontrivial solutions for $\lambda < \lambda_c$. There is a continuous branch of positive solutions $\alpha(\lambda,x)$ of (4.1), where the eigenvalue $\lambda(\lambda \geq \lambda_c)$ increases monotonically with the value of α at $x = 0$.

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