Parity-violating one-loop six-point function in type-I superstring theory

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We evaluate in closed form the parity-violating one-loop six-point function with external gauge bosons in type-I superstring theory. The amplitude is proven finite for arbitrary internal symmetry. The axial-vector anomaly is obtained for general internal symmetry and is shown to vanish in the case of SO(32).

I. INTRODUCTION

The current revival of interest in string theories as candidates for the fundamental description of the nature of matter and its interactions is based on the many desirable characteristics which string theories apparently possess. The revival was initiated by the discovery of Green and Schwarz[1] that string theories with certain internal symmetries have four-point functions which are finite at the one-loop level. Another desirable feature of strings is that they can be made to be supersymmetric. A third virtue apparently associated with string theories is that they contain modes which can be identified as gravitons; i.e., these theories potentially contain a description of the quantum theory of gravity.

Because these theories offer the promise of a comprehensive description of all of the known interactions[2] while at the same time avoiding the defects that are inherent in pointlike gauge field theories, they must be tested in every possible way. String theories have not yet been developed to the point where their predictions can be easily tested by experiment. Therefore, tests of these theories must consist of searches for anomalies, inconsistencies, infinities, and other undesirable features of candidates for a fundamental theory of interactions. Some work of this type has already been carried out. For example, the parity-conserving parts of certain five- and higher N-point functions have been shown to be finite at the one-loop level.[3,4]

Apart from the vanishing of the axial-vector anomaly proven by Green and Schwarz,[5] little, if anything, has been published concerning the behavior of parity-violating loop graphs in superstring theories. Indeed, the light-cone gauge, in which most of the recent string calculations have been made, seems at present to be inherently ambiguous with respect to parity-violating effects. For this reason, the calculation of Ref. 5 was made using the older Lorentz-covariant formulation. In Ref. 5 although the six-point function itself was not calculated, its gauge projection was shown to vanish in the case of an SO(32) internal symmetry. The proof was given using two different regulator schemes which, although leading to a consistent null result for SO(32), gave differing results for the value of the chiral anomaly in the case of a general internal symmetry. We obtain unambiguous results for arbitrary internal symmetry.

In this paper we use a Lorentz-covariant formulation to investigate the parity-violating part of the one-loop six-point function with external bosons and internal fermions described by a type-I string theory. We use the technique developed in Ref. 6 for the general dual model. In Sec. II we discuss the general case of parity-violating loops with N external particles. In Sec. III we specialize to the six-point function and obtain an expression for this function in terms of Jacobi θ functions. Finiteness is shown for arbitrary internal symmetry, but SO(32) is required for anomaly cancellation.

In the final section we discuss how our work affects the status of type-I strings with an SO(32) internal symmetry in particular and string theories in general as fundamental descriptions of the nature of matter.

II. TRACE CALCULATIONS IN PARITY-VIOLATING LOOPS

The calculation of the parity-violating loop amplitudes in superstring theory differs in some respects from that of the parity-conserving amplitudes. The planar amplitude with N external gauge bosons with momenta k_i and polarizations ζ_i is defined in the F_2 formulation by

$$A_N^{11}=-d^Dp\,\tfrac{1}{2}\left\langle p,0\left|\left[\mathrm{Tr}\Gamma^{11}\prod_{i=1}^{N}\frac{1}{F_0}V_{F_2}(k_i,1)\right]\right|p,0\right\rangle \times\delta^D\left[\sum_i k_i\right] . \tag{2.1}$$

This is multiplied by the Chan-Paton internal-symmetry factor for the planar loop

$$F_P = \left[\mathrm{Tr} \prod_{i=1}^{N} \lambda_{a_i} \right] (\mathrm{Tr}\lambda^0) \ . \tag{2.2}$$

The second factor in F_P, the trace of the unit matrix λ^0, comes from the string end point to which no external gauge bosons are attached. The amplitude contains a D-dimensional loop momentum integration ($D=10$ in superstring theory) and a trace over the higher modes of string excitation.

The vertex V_{F_2} and the inverse propagator F_0 are defined in terms of the string coordinate

$$Q_\mu(\rho,a,a^\dagger) = q_{0\mu} - ip_{0\mu}\ln\rho + \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} (a_\mu^{n\dagger}\rho^n + a_\mu^n \rho^{-n}) \ , \tag{2.3}$$

the conjugate momentum

$$P_\mu(\rho,a,a^\dagger) = i\rho \frac{d}{d\rho} Q_\mu(\rho,a,a^\dagger) \ , \tag{2.4}$$

and the Ramond Γ_μ

$$\Gamma_\mu(\rho,b,b^\dagger) = \gamma_\mu + \sqrt{2} i \gamma^{11} \sum_{n=1}^{\infty} (b_\mu^{n\dagger}\rho^n + b_\mu^n\rho^{-n}) \ . \tag{2.5}$$

Namely,

$$V_{F_2}(k,\rho) = e^{ik\cdot Q(\rho)} \frac{\zeta\cdot\Gamma(\rho)}{\sqrt{2}i} \tag{2.6}$$

and

$$F_0 = \frac{1}{\sqrt{2}i} \oint \frac{dz}{2\pi i z} \Gamma_\mu(z) P_\mu(z) \ . \tag{2.7}$$

The ten-dimensional Dirac matrices satisfy

$$\{\gamma_\mu,\gamma_\nu\} = -2g_{\mu\nu} \ , \tag{2.8}$$

$$\{\gamma_\mu,\gamma^{11}\} = 0 \ . \tag{2.9}$$

The Γ^{11} is constructed in analogy with Dirac's γ_5 to anticommute with all Γ_μ:

$$\{\Gamma^{11},\Gamma_\mu(\rho)\} = 0 \ . \tag{2.10}$$

The explicit form of Γ^{11} is

$$\Gamma^{11} = \gamma^{11} \exp\left[i\pi \sum_{n=1}^{\infty} b^{n\dagger} b^n \right] \ . \tag{2.11}$$

The operator F_0 satisfies

$$F_0{}^2 = L_0 = P_0{}^2/2 + \sum_{n=1}^{\infty} n(a^{n\dagger}\cdot a^n + b^{n\dagger}\cdot b^n) \tag{2.12}$$

and

$$\{F_0, V_{F_2}(k,\rho)\} = V_{F_1}(k,\rho) \ , \tag{2.13}$$

where the vertex for gauge-boson emission from a fermion line in the F_1 formulation is

$$V_{F_1}(k,\rho) = e^{ik\cdot Q(\rho)}[\zeta\cdot P(\rho) - k\cdot\Gamma(\rho)\zeta\cdot\Gamma(\rho)/2] \ . \tag{2.14}$$

Using (2.12) and (2.13), the F_0's in Eq. (2.1) can be pulled to the right leaving one in the hth position and discarding terms which are negligible according to the canceled propagator argument. Possible residual effects due to these terms are discussed briefly in the conclusions. To preserve manifest cyclic symmetry, we write the amplitude as an average over the N possible final positions of the F_0:

$$A_N^{11} = -d^D p \frac{\delta^D\left[\sum k\right]}{2} \frac{1}{N} \times \sum_{h=1}^{N} \left\langle p,0 \left| \mathrm{Tr}\Gamma^{11} \left[\frac{F_0}{L_0} V_{F_2}(k_h,1) \right] \times \prod_{\substack{i=1 \\ i\neq h}}^{N} \left[\frac{1}{L_0} V_{F_1}(k_i,1) \right] \right| p,0 \right\rangle \ . \tag{2.15}$$

In (2.15) the first term in large parentheses is understood to be inserted into the vacant hth position in the following product. We write therefore

$$A_N^{11} = \frac{1}{N} \sum_{h=1}^{6} A_N^{11}(h) \ . \tag{2.16}$$

In the following, we deal explicitly with the $h=1$ term, resymmetrizing at the end. We make the usual substitution

$$\frac{1}{L_0} = \int_0^1 dx\, x^{L_0-1} \ . \tag{2.17}$$

Then, pulling the $x_i{}^{L_0}$ ($i\geq 2$) to the right, one obtains

$$A_N^{11}(1) = -\frac{d\Omega}{2} d^D p\, \delta^D\left[\sum k\right] \times \left\langle p,0 \left| \mathrm{Tr}\Gamma^{11} w^{L_0} F_0 V_{F_2}(k_1,1) \times \prod_{i=2}^{N} V_{F_1}(k_i,\rho_i) \right| p,0 \right\rangle \ , \tag{2.18}$$

where

$$d\Omega = \frac{dw}{w} \prod_{i=2}^{N} \frac{d\rho_i}{\rho_i} \ , \tag{2.19}$$

$$\rho_i = x_2 \cdots x_i, \quad i\geq 2 \ , \tag{2.20}$$

$$\rho_1 = 1 \ , \tag{2.21}$$

$$w = x_1 x_2 \cdots x_N \ . \tag{2.22}$$

As in the case of parity-conserving amplitudes, the loop momentum integration in Eq. (2.18) is equivalent to a trace over a zeroth mode with vanishing frequency. That is, we write (2.3) as

$$Q_\mu(\rho,a,a^\dagger) = \sum_{n=0}^{\infty} \gamma_n (a_\mu^{n\dagger}\rho^{n+\epsilon} + a_\mu^n \rho^{-n-\epsilon}) \tag{2.23}$$

with

$$\gamma_n = \left[\frac{\Gamma(n+2\epsilon)}{n!} \right]^{1/2} ; \tag{2.24}$$

then

$$A_N^{11}(1) = -\frac{d\Omega}{2}\,\mathrm{Tr}\Gamma^{11} w^{L_0} F_0 V_{F_2}(k_1,1)\prod_{i=2}^{N} V_{F_1}(k_i,\rho_i)\ . \tag{2.25}$$

After taking the trace in Eq. (2.25) including that over excitations in the zeroth mode, one takes the limit $\epsilon \to 0$, recovering in the process the D-dimensional momentum-conserving δ function.

For the F_0 in Eq. (2.25), we will substitute the form (2.7), with P_μ given by Eqs. (2.4) and (2.23). We write L_0 in the form

$$L_0 = \sum_{m=0}^{\infty}(m+\epsilon)a_\mu^{m\dagger}a_\mu^m + \sum_{m=1}^{\infty} m b_\mu^{m\dagger} b_\mu^m\ . \tag{2.26}$$

The trace in Eq. (2.25) can be written as a vacuum expectation value (VEV)

$$\mathrm{Tr}\Gamma^{11} w^{L_0} O(a^n, a^{n\dagger}, b^n, b^{n\dagger}) = 2^{D/2}\langle 0 \,|\, e^{a'^n a^n} e^{b'^n b^n}\Gamma^{11} w^{L_0} O(a^n,a^{n\dagger},b^n,b^{n\dagger}) e^{a^{n\dagger}a'^{n\dagger}} e^{b^{n\dagger}b'^{n\dagger}} \,|\, 0\rangle\ . \tag{2.27}$$

The primed oscillators, which do not appear in the operator whose trace is desired, have the function of providing the correct sum over normalized Fock-space states in the unprimed oscillators. The VEV in Eq. (2.27) is defined to include a normalized trace over the Dirac matrices, i.e.,

$$\langle 0 \,|\, f(\gamma_\mu) \,|\, 0\rangle \equiv 2^{-D/2}\,\mathrm{Tr} f(\gamma_\mu)\ . \tag{2.28}$$

Pulling some of the operators in Eq. (2.27) to the left, we have, for any operator O constructed from the elementary oscillators,

$$\begin{aligned}\mathrm{Tr}\Gamma^{11} w^{L_0} O(a^n,a^{n\dagger},b^n,b^{n\dagger}) &= 2^{D/2}\langle 0 \,|\, \gamma^{11} e^{a'^n a^n w^{n+\epsilon}} e^{-b'^n b^n w^n} e^{a^{n\dagger}a'^{n\dagger}} e^{b^{n\dagger}b'^{n\dagger}} O(a^n + a'^{n\dagger}, a^{n\dagger}, b^n + b'^{n\dagger}, b^{n\dagger}) \,|\, 0\rangle \\ &= 2^{D/2}\prod_{n=0}^{\infty}(1-w^{n+\epsilon})^{-D}\prod_{n=1}^{\infty}(1-w^n)^D \langle 0 \,|\, \gamma^{11} O(A^n, \overline{A^n}, B^n, \overline{B^n}) \,|\, 0\rangle\ ,\end{aligned} \tag{2.29}$$

where

$$A^n = \frac{a^n}{1-w^{n+\epsilon}} + a'^{n\dagger}\ , \tag{2.30}$$

$$\overline{A^n} = a^{n\dagger} + \frac{a'^n w^{n+\epsilon}}{1-w^{n+\epsilon}}\ , \tag{2.31}$$

$$B^n = \frac{b^n}{1-w^n} + b'^{n\dagger}\ , \tag{2.32}$$

$$\overline{B^n} = b^{n\dagger} - \frac{b'^n w^n}{1-w^n}\ . \tag{2.33}$$

We see that, unlike the case of the parity-conserving loop, the fermionic oscillators as well as the bosonic oscillators are singular at $w=1$. Note, however, that the canonical commutation and anticommutation relations are preserved. Without affecting the loop amplitude, one can also write the more symmetric expressions

$$A^n = \frac{1}{(1-w^{n+\epsilon})^{1/2}}(a^n + a'^{n\dagger} w^{(n+\epsilon)/2})\ , \tag{2.34}$$

$$\overline{A^n} = \frac{1}{(1-w^{n+\epsilon})^{1/2}}(a^{n\dagger} + a'^n w^{(n+\epsilon)/2})\ , \tag{2.35}$$

$$B^n = \frac{1}{(1-w^n)^{1/2}}(b^n + b'^{n\dagger} w^{n/2})\ , \tag{2.36}$$

$$\overline{B^n} = \frac{1}{(1-w^n)^{1/2}}(b^{n\dagger} - b'^n w^{n/2})\ . \tag{2.37}$$

In this representation, the A and $\overline{A}$ are Hermitian conjugates.

In Eq. (2.29) bosonic and fermionic contributions to the partition function are seen to cancel leaving only the zeroth-mode (loop integration) factor. The effects of the projection operator onto states of positive norm therefore also cancel. Thus

$$A_N^{11}(1) = -d\Omega\, 2^{D/2-1}(-\epsilon \ln w)^{-D}\oint \frac{dz}{2\pi i z}\frac{1}{\sqrt{2}i}\left\langle 0 \left| \gamma^{11} P_\mu(z,A,\overline{A})\Gamma_\mu(z,B,\overline{B}) V_{F_2}(k_1,1,A,\overline{A})\prod_{i=2}^{N} V_{F_1}(k_i,\rho_i,A,\overline{A}) \right| 0\right\rangle\ . \tag{2.38}$$

In the vertices of Eq. (2.38), we have written $A, \overline{A}$ to represent collectively fermionic or bosonic oscillators. That is, the a_μ^n and $a_\mu^{n\dagger}$ of the original vertices have been replaced by the A_μ^n and $\overline{A_\mu^n}$ of Eqs. (2.30) and (2.31) or Eqs. (2.34) and (2.35) while the b_μ^n and $b_\mu^{n\dagger}$ have been replaced by B_μ^n and $\overline{B_\mu^n}$ of Eqs. (2.32) and (2.33) or (2.36) and (2.37).

The VEV of Eq. (2.38) is evaluated by pulling the $a_\mu^n, a_\mu'^n, b_\mu^n, b_\mu'^n$ to the right where they annihilate the vacuum. In the first step of this process, the generalized plane-wave factors are separated out yielding

$$A_N^{11}(1)=d\Omega\, 2^{D/2-2}A_N^0 \oint \frac{dz}{2\pi i z}\left\langle 0\left|\gamma^{11}[P_\mu(z)+B_\mu(z)]\Gamma_\mu(z)\zeta_1\cdot\Gamma(\rho_1)\prod_{i=2}^{N}J(\rho_i)\right|0\right\rangle . \tag{2.39}$$

Here

$$A_N^0=(-\epsilon\ln w)^{-D}\left\langle 0\left|\prod_{i=1}^{N}e^{ik_i\cdot Q(\rho_i,A,\overline{A})}\right|0\right\rangle , \tag{2.40}$$

$$B_\mu(z)=\sum_{j=1}^{N} ik_{j_\nu}\langle 0\,|\,P_\mu(z,A,\overline{A})Q_\nu(\rho_j,A,\overline{A})\,|\,0\rangle , \tag{2.41}$$

and

$$J(\rho_i)=\zeta_i\cdot P(\rho_i,A,\overline{A})+\zeta_i\cdot B(\rho_i)+\frac{k_i\cdot\Gamma(\rho_i,B,\overline{B})\zeta_i\cdot\Gamma(\rho_i,B,\overline{B})}{(-2)} . \tag{2.42}$$

The VEV's of Eqs. (2.40) and (2.41) are expressible in terms of Jacobi θ functions:

$$\begin{aligned}A_N^0&=(-\epsilon\ln w)^{-D}\prod_{i<j}\exp[-\langle 0\,|\,k_i\cdot Q(\rho_i)k_j\cdot Q(\rho_j)\,|\,0\rangle]\\&=(-\epsilon\ln w)^{-D}\prod_{i<j}\exp\left[-k_i\cdot k_j\left[\frac{1}{\epsilon^2\ln w}-\frac{\ln w}{12}-\ln\psi(\rho_j/\rho_i,w)\right]\right]\\&=\left[\frac{\tau}{i}\right]^{-D/2}\delta^D\left[\sum k_i\right]\prod_{i<j}\psi(\rho_j/\rho_i,w)^{k_i\cdot k_j} ,\end{aligned} \tag{2.43}$$

where

$$\psi(x,w)=-2\pi i\, e^{i\pi\nu^2/\tau}\frac{\theta_1(\nu\,|\,\tau)}{\theta_1'(0\,|\,\tau)} \tag{2.44}$$

and

$$\nu=\frac{\ln x}{2\pi i} , \tag{2.45}$$

$$\tau=\frac{\ln w}{2\pi i} . \tag{2.46}$$

The correlation of a P with a Q is

$$\langle 0\,|\,P_\mu(\rho_i,A,\overline{A})Q_\nu(\rho_j,A,\overline{A})\,|\,0\rangle\equiv g_{\mu\nu}G(\rho_i/\rho_j,w) , \tag{2.47}$$

where

$$G(\rho_i/\rho_j,w)=-i\rho_i\frac{\partial}{\partial\rho_i}\ln\psi(\rho_j/\rho_i,w) . \tag{2.48}$$

Partial derivatives with respect to ρ_i are to be taken at constant w and vice versa. The properties of the Jacobi θ function are such that

$$G(x^{-1},w)=-G(x,w) . \tag{2.49}$$

For $|x|<1$, we have the series expansion

$$G(x,w)=i\left[\frac{1}{2}-\frac{\ln x}{\ln w}+\sum_{n=1}^{\infty}\frac{x^n-(w/x)^n}{1-w^n}\right] . \tag{2.50}$$

Using momentum conservation, we may write

$$B_\mu(z)=\sum_{j=2}^{N}k_{j_\mu}C_j(z) \tag{2.51}$$

with

$$C_j(z)=iG(z/\rho_j)-iG(z/\rho_1) . \tag{2.52}$$

The evaluation of the amplitudes of Eq. (2.39) depends also on the correlations of two P_μ and of two Γ_μ:

$$\langle 0\,|\,P_\mu(\rho_i,A,\overline{A})P_\nu(\rho_j,A,\overline{A})\,|\,0\rangle=g_{\mu\nu}\chi^P(\rho_j/\rho_i,w) , \tag{2.53}$$

$$\langle 0\,|\,\Gamma_\mu(\rho_i,B,\overline{B})\Gamma_\nu(\rho_j,B,\overline{B})\,|\,0\rangle=-2g_{\mu\nu}\chi_0^{11}(\rho_j/\rho_i,w) . \tag{2.54}$$

The relation (2.4) implies

$$\chi^P(\rho_j/\rho_i,w)=i\rho_j\frac{\partial}{\partial\rho_j}G(\rho_i/\rho_j,w) . \tag{2.55}$$

For $|x|<1$,

$$\chi^P(x,w)=-\frac{1}{\ln w}+\sum_{n=1}^{\infty}n\frac{x^n+(w/x)^n}{1-w^n} . \tag{2.56}$$

The fermionic correlation in the parity-violating loop satisfies

$$\chi_0^{11}(x,w)=-iG(x,w)+\frac{\ln x}{\ln w} . \tag{2.57}$$

In $D=10$ dimensions, each nonvanishing term in Eq. (2.39) is proportional to a vacuum expectation value of at least $10\Gamma_\mu$. These are in turn each proportional to the totally antisymmetric, Lorentz-covariant, ten-dimensional ϵ tensor

$$\left\langle 0 \left| \gamma^{11} \prod_{j=1}^{2M} \Gamma_{\mu_j}(z_j) \right| 0 \right\rangle = \sum_{i_1,i_2,\ldots,i_{10}=1,\ i_1<i_2<\cdots<i_{10}}^{2M} (-1)^{P_1} \epsilon_{\mu_{i_1}\mu_{i_2}\cdots\mu_{i_{10}}} \left\langle 0 \left| \prod_{j=1\neq i_m}^{2M} \Gamma_{\mu_j}(z_j) \right| 0 \right\rangle . \tag{2.58}$$

The parity P_1 is given by

$$P_1 = \sum_{j=1}^{10} (i_j - j) = -55 + \sum_{j=1}^{10} i_j . \tag{2.59}$$

The vacuum expectation value without the γ^{11} is

$$\left\langle 0 \left| \prod_{j=1\neq i_m}^{2M} \Gamma_{\mu_j}(z_j) \right| 0 \right\rangle = \sum_{\text{perms}} (-1)^P \prod_{l=1,\ j_n\neq i_m}^{M} [g_{\mu_{j_{2l-1}}\mu_{j_{2l}}} \chi_0^{11}(z_{j_{2l}}/z_{j_{2l-1}}, w)] . \tag{2.60}$$

The summation is over all permutations of the $2M-10$ indices $\mu_j \to \mu_{j_n}$ such that $j_{2l-1} < j_{2l}$. P is the parity of the permutation.

It is convenient to introduce a shorthand notation for the ten-dimensional ϵ tensor partially or fully contracted with Lorentz vectors:

$$\begin{aligned} \epsilon \left[\prod_{l=1}^{10} P_l \right] &\equiv \epsilon_{\mu_1\mu_2\cdots\mu_{10}} P_{1\mu_1} P_{2\mu_2} P_{3\mu_3} \cdots P_{10\mu_{10}} , \\ \epsilon \left[\mu_1\mu_2 \prod_{l=3}^{10} P_l \right] &\equiv \epsilon_{\mu_1\mu_2\mu_3\cdots\mu_{10}} P_{3\mu_3} \cdots P_{10\mu_{10}} . \end{aligned} \tag{2.61}$$

An important property (we thank Professor S. Jones for providing a simple proof of this identity) of the ϵ tensor for any vectors k and p_l is

$$k_\mu \epsilon \left[\prod_{l=1}^{10} p_l \right] = \sum_{i=1}^{10} (-1)^{i-1} p_{i\mu} \epsilon \left[k \prod_{l=1\neq i}^{10} p_l \right] . \tag{2.62}$$

In the evaluation of Eq. (2.39), one encounters some easily performed contour integrals around the origin in z.

$$\oint \frac{dz}{2\pi i z} C_j(z) = -\frac{\ln\rho_j}{\ln w} , \tag{2.63}$$

$$\begin{aligned} \oint \frac{dz}{2\pi i z} C_j(z) \chi_0^{11}(\rho_i/z) = {} & \frac{(\ln\rho_j)(\ln\rho_j - 2\ln\rho_i - \ln w)}{2\ln^2 w} \\ & + w\frac{\partial}{\partial w} \ln \frac{\psi(\rho_j/\rho_i, w)}{\psi(\rho_1/\rho_i, w)} . \end{aligned} \tag{2.64}$$

III. THE ONE-LOOP PARITY-VIOLATING SIX-POINT FUNCTION

In the remainder of this paper, we will confine our attention to the parity-violating six-point function. Further investigation of the general N-point function will be left for a future publication. With $N=6$, Eq. (2.39) becomes

$$\begin{aligned} A_6^{11}(1) = \tfrac{1}{2} d\Omega\, A_6^0 \oint \frac{dz}{2\pi i z} \Bigg[& \sum_{i=2}^{6} [\zeta_{i\mu} \chi^P(\rho_i/z) + B_\mu(z) \zeta_i \cdot B(\rho_i)] \epsilon \left[\mu \zeta_1 \prod_{l=2\neq i}^{6} k_l \zeta_l \right] \\ & - \tfrac{1}{2} B_\mu(z) \left\langle 0 \left| \gamma^{11} \Gamma_\mu(z) \zeta_1 \cdot \Gamma(\rho_1) \prod_{l=2}^{6} [k_l \cdot \Gamma(\rho_l) \zeta_l \cdot \Gamma(\rho_l)] \right| 0 \right\rangle \Bigg] . \end{aligned} \tag{3.1}$$

For $B_\mu(z)$, we may substitute Eq. (2.51). In each term of the remaining VEV, a momentum vector k_j appears twice. Since the ϵ tensor is totally antisymmetric, the nonzero terms from Eqs. (2.58) and (2.60) are sharply reduced in number. Upon evaluation, one finds

$$\begin{aligned} A_6^{11}(1) = \tfrac{1}{2} d\Omega\, A_6^0 \oint \frac{dz}{2\pi i z} \Bigg[& \sum_{i=2}^{6} [\zeta_{i\mu} \chi^P(\rho_i/z) + k_{i\mu} C_i(z) \zeta_i \cdot B(\rho_i)] \epsilon \left[\mu \zeta_1 \prod_{l=2\neq i}^{6} k_l \zeta_l \right] \\ & - \sum_{j=2}^{6} C_j(z) \sum_{i=2\neq j}^{6} (k_j \cdot \zeta_i k_{i\mu} - k_j \cdot k_i \zeta_{i\mu}) [\chi_0^{11}(\rho_i/z) - \chi_0^{11}(\rho_i/\rho_j)] \epsilon \left[\mu \zeta_1 \prod_{l=2\neq i}^{6} k_l \zeta_l \right] \\ & + \sum_{j=2}^{6} k_j \cdot \zeta_1 C_j(z) [\chi_0^{11}(\rho_1/z) - \chi_0^{11}(\rho_1/\rho_j)] \epsilon \left[\prod_{l=2}^{6} k_l \zeta_l \right] \Bigg] . \end{aligned} \tag{3.2}$$

The identity (2.62) implies that

$$k_j \cdot \zeta_1 \epsilon \left[\prod_{l=2}^{6} k_l \zeta_l \right] = \sum_{i=2\neq j}^{6} (k_j \cdot k_i \zeta_{i\mu} - k_j \cdot \zeta_i k_{i\mu}) \epsilon \left[\zeta_1 \mu \prod_{l=2\neq i}^{6} k_l \zeta_l \right] , \tag{3.3}$$

reducing the last two terms of (3.2) to a parallel double summation. The term in the first square brackets in Eq. (3.2) may be similarly reduced using (2.63) and the fact [deducible from (2.40), (2.41), (2.43), (2.47), and (2.48)] that

$$A_N^0 k_i\cdot B(\rho_i)=\rho_i\frac{\partial}{\partial\rho_i}A_N^0\,, \tag{3.4}$$

$$\begin{aligned}A_6^{11}(1)=&\frac{-d\Omega}{2\ln w}\sum_{i=2}^{N}\zeta_{i\mu}\rho_i\frac{\partial}{\partial\rho_i}(A_6^0\ln\rho_i)\epsilon\left[\mu\zeta_1\prod_{l=2\neq i}^{6}k_l\zeta_l\right]\\&-\oint\frac{dz}{2\pi iz}\frac{d\Omega}{2}A_6^0\sum_{i=2}^{6}[\zeta_{i\mu}k_i\cdot B(\rho_i)-k_{i\mu}\zeta_i\cdot B(\rho_i)]C_i(z)\epsilon\left[\mu\zeta_1\prod_{l=2\neq i}^{6}k_l\zeta_l\right]\\&-\oint\frac{dz}{2\pi iz}\frac{d\Omega}{2}A_6^0\sum_{j=2}^{6}\sum_{i=2\neq j}^{6}C_j(z)(k_j\cdot\zeta_ik_{i\mu}-k_j\cdot k_i\zeta_{i\mu})\epsilon\left[\mu\zeta_1\prod_{l=2\neq i}^{6}k_l\zeta_l\right]\\&\times[\chi_0^{11}(\rho_i/z)-\chi_0^{11}(\rho_i/\rho_j)-\chi_0^{11}(\rho_1/z)+\chi_0^{11}(\rho_1/\rho_j)]\,.\end{aligned} \tag{3.5}$$

As a perfect derivative, the first term in (3.5) gives a negligible contribution since the amplitudes A_N^0 are defined as analytic continuations from the region of spacelike Mandelstam invariants. This prescription is the basis of the "canceled propagator" argument. Then, substituting into (3.5) the definition of the $B(\rho_i)$, we have

$$A_6^{11}=\frac{d\Omega}{2}A_6^0\sum_{i,j=2,\ i\neq j}^{6}T_{ij}(1)X_{ij}^{(1)}(\rho,w)\,, \tag{3.6}$$

where

$$T_{ij}(h)=k_j\cdot k_i\epsilon_{ih}+k_j\cdot\zeta_i\epsilon_i\,, \tag{3.6a}$$

$$\epsilon_{ih}\equiv\epsilon\left[\zeta_i\zeta_h\prod_{l=1\neq i,h}^{6}k_l\zeta_l\right],\quad \epsilon_{ii}\equiv 0\,, \tag{3.6b}$$

$$\epsilon_i\equiv\epsilon\left[\prod_{l=1\neq i}^{6}k_l\zeta_l\right], \tag{3.6c}$$

and

$$\begin{aligned}X_{ij}^{(1)}=\oint\frac{dz}{2\pi iz}\{&-C_i(z)[iG(\rho_i/\rho_j)-iG(\rho_i/\rho_1)]\\&+C_j(z)[\chi_0^{11}(\rho_i/z)-\chi_0^{11}(\rho_i/\rho_j)\\&-\chi_0^{11}(\rho_1/z)+\chi_0^{11}(\rho_1/\rho_j)]\}\,.\end{aligned} \tag{3.7}$$

The required contour integrals are given by Eqs. (2.63) and (2.64). Thus, also making use of Eqs. (2.48) and (2.57), one finds

$$X_{ij}^{(1)}=\left[w\frac{\partial}{\partial w}+\sum_{l=2}^{6}\frac{\ln\rho_l}{\ln w}\rho_l\frac{\partial}{\partial\rho_l}\right]\ln\frac{\Psi(\rho_j/\rho_i)}{\Psi(\rho_j/\rho_1)\Psi(\rho_i/\rho_1)}\,. \tag{3.8}$$

The tensor $T_{ij}(1)$ satisfies

$$\sum_{i,j=2,\ i\neq j}^{6}T_{ij}=\sum_{i=1}^{6}T_{ij}=\sum_{j=1}^{6}T_{ij}=0\,. \tag{3.9}$$

We may, therefore, write

$$A_6^{11}=\frac{d\Omega}{2}A_6^0\sum_{i,j=1,\ i\neq j}^{6}T_{ij}(1)\left[w\frac{\partial}{\partial w}+\sum_{l=2}^{6}\frac{\ln\rho_l}{\ln w}\rho_l\frac{\partial}{\partial\rho_l}\right]\ln\Psi(\rho_j/\rho_i)\,. \tag{3.10}$$

The amplitude of Eq. (3.10) is nonsingular at $w=0$ behaving as

$$A_6^{11}\sim\frac{dw}{w}(\ln w)^{-5}\,. \tag{3.11}$$

To investigate its behavior at $w=1$, we make use of the Jacobi transformation

$$\rho_i'=e^{(\ln\rho_i)2\pi i/\ln w}\,, \tag{3.12}$$

$$w'=e^{4\pi^2/\ln w}\,. \tag{3.13}$$

As $w\to 1$, w' goes rapidly to zero. The volume elements are related by

$$d\Omega=d\Omega'\tau^7\,, \tag{3.14}$$

where $d\Omega'$ has the same form in primed variables as $d\Omega$ has in the unprimed variables [Eq. (2.19)].

The ψ function of Eq. (2.44) satisfies

$$\psi(\rho_j/\rho_i,w)=\tau\hat{\psi}(\rho_j'/\rho_i',w')\,, \tag{3.15}$$

where $\tau=(\ln w)/2\pi i$ and

$$\hat{\psi}(x,w)=-2\pi i\frac{\theta_1\left[\frac{\ln x}{2\pi i}\middle|\frac{\ln w}{2\pi i}\right]}{\theta_1'\left[0\middle|\frac{\ln w}{2\pi i}\right]}\,. \tag{3.16}$$

$\hat{\psi}$ has a power-series expression about the origin in w.

Thus using momentum conservation, we have, for the A_N^0 of Eq. (2.43),

$$A_N^0=\left[\frac{\tau}{i}\right]^{-D/2}\delta^D\left[\sum k_i\right]\prod_{i<j}\hat{\psi}(\rho_j'/\rho_i',w')^{k_i\cdot k_j} . \tag{3.17}$$

Equations (3.13) and (2.48) imply

$$G(\rho_i/\rho_j,w)=\frac{1}{\tau}\hat{G}(\rho_i'/\rho_j',w') , \tag{3.18}$$

where

$$\hat{G}(x,w)=-ix\frac{\partial}{\partial x}\ln\hat{\psi}(x,w) . \tag{3.19}$$

The behavior of X_{ij} under the Jacobi transformation may then be determined using

$$w\frac{d}{dw}=\frac{1}{\tau^2}\left[w'\frac{\partial}{\partial w'}+\frac{\ln\rho'}{\ln w'}\rho'\frac{\partial}{\partial\rho'}\right] \tag{3.20}$$

and

$$\frac{\ln\rho_i}{\ln w}=-\frac{1}{\tau}\frac{\ln\rho_i'}{\ln w'} . \tag{3.21}$$

Note that partial derivatives with respect to w are at constant ρ and partials with respect to w' are at constant ρ'. Therefore,

$$X_{ij}(\rho_i,\rho_j,w)=\frac{1}{\tau^2}\hat{X}_{ij}(\rho_i',\rho_j',w') , \tag{3.22}$$

where

$$\hat{X}_{ij}(\rho_i',\rho_i',w')=w'\frac{\partial}{\partial w'}\ln\frac{\hat{\psi}(\rho_j'/\rho_i',w')}{\hat{\psi}(\rho_1'/\rho_i',w')\hat{\psi}(\rho_j'/\rho_1',w')} . \tag{3.23}$$

$\hat{X}$ is obtained from the X of Eq. (3.8) by replacing ψ by $\hat{\psi}$ and by discarding the G terms.

In deriving Eq. (3.23), we have also discarded terms independent of both ρ_i and ρ_j since

$$\sum_{i,j=2,\ i\neq j}^{6}\epsilon\left[\mu\zeta_1\prod_{l=2\neq i}^{6}k_l\right](k_j\cdot\zeta_i k_{i\mu}-k_j\cdot k_i\zeta_{i\mu})=k_1\cdot\zeta_1\epsilon\left[\prod_{l=2}^{6}k_l\zeta_l\right]=0 . \tag{3.24}$$

The vanishing of Eq. (3.24) holds for (unphysical) longitudinal polarizations ζ_l, as well as for (physical) transverse polarizations. This equation follows from Eq. (2.62).

Collecting factors from Eqs. (3.14), (3.17), and (3.22) and restoring the symmetrization of Eq. (2.16), the one-loop, planar, parity-violating, six-point function is

$$A_6^{11}=i\frac{d\Omega'}{2}\delta^D\left[\sum k_i\right]\prod_{i<j}\hat{\psi}(\rho_j'/\rho_i',w')^{k_ik_j}\frac{1}{6}\sum_{h=1}^{6}\sum_{i,j=1,\ i\neq j}^{6}T_{ij}(h)w'\frac{\partial}{\partial w'}\ln\hat{\Psi}(\rho_j'/\rho_i',w') . \tag{3.25}$$

The integrand is seen to be finite and analytic at $w'=0$, since $\hat{X}$ vanishes at that end point. The planar parity-violating amplitude is, therefore, finite by itself as is the Möbius strip loop discussed below.

We turn now to the evaluation of the one-loop parity-violating six-point function where the world sheet of the circulating string forms a Möbius strip instead of an annulus. With a single twist between emission of the sixth and first gauge boson, the amplitude is obtained from (3.6) by replacing w by $-w$ in A_6^0 and X_{ij} except in logarithms, all of which arise from zeroth-mode terms. In addition, the Chan-Paton factor of Eq. (2.2) is modified to read

$$F_M=-\left[\mathrm{Tr}\prod_{i=1}^{N}\lambda_{\alpha_i}\right] .$$

In the Möbius loop, there is only one string boundary. The extra minus sign here is characteristic of an orthogonal group internal symmetry.[2]

The modular group transformation relating the behavior of the Möbius loop near $w=1$ to that near $w=0$ is

$$\rho_i''=\exp\left[i\pi\frac{\ln\rho_i}{\ln w}\right] , \tag{3.26a}$$

$$\nu''=\ln\rho''/2\pi i , \tag{3.26b}$$

$$w''=e^{\pi^2/\ln w} , \tag{3.26c}$$

$$\tau''=\ln w''/2\pi i . \tag{3.26d}$$

In terms of the double primed variables, we have

$$d\Omega=d\Omega''(2\tau)^7 , \tag{3.27}$$

$$\psi(\rho_j/\rho_i,-w)=(2\tau)\hat{\psi}(\rho_j''/\rho_i'',-w'') , \tag{3.28}$$

$$G(\rho_j/\rho_i,-w)=(2\tau)^{-1}\hat{G}(\rho_j''/\rho_i'',-w'') , \tag{3.29}$$

$$A_N^0(\rho_i,-w)=\left[\frac{\tau}{i}\right]^{-D/2}\delta^D\left[\sum k_i\right]\times\prod_{i<j}\hat{\psi}(\rho_j''/\rho_i'',-w'')^{k_i\cdot k_j} , \tag{3.30}$$

$$w\frac{\partial}{\partial w}=(2\tau)^{-2}\left[w''\frac{\partial}{\partial w''}+\frac{\nu''}{\tau''}\rho''\frac{\partial}{\partial\rho''}\right] , \tag{3.31}$$

$$\frac{\ln\rho_i}{\ln w}=-(2\tau)^{-1}\frac{\ln\rho_i''}{\ln w''}\,, \tag{3.32}$$

$$X_{ij}(\rho_i,\rho_j,-w)=\frac{1}{(2\tau)^2}\hat{X}_{ij}(\rho_i'',\rho_j'',-w')\,. \tag{3.33}$$

In comparing Eqs. (3.27)–(3.33) with their analogs from the planar loop, we see an extra factor of 2 accompanying each τ multiplier except in Eq. (3.30). Thus when factors are collected to form the Möbius strip analog of Eq. (3.25), the τ's cancel as before, leaving, however, an extra factor of $2^{D/2}$. That is

$$A_{6,M}^{11}=2^{D/2-1}i\,d\Omega''\delta^D\left[\sum k_i\right]\prod_{i<j}\hat{\psi}(\rho_j''/\rho_i'',-w'')^{k_ik_j}\sum_{i,j=1,\,i\neq j}^{6}\tfrac{1}{6}\sum_{h=1}^{6}T_{ij}(h)w''\frac{\partial}{\partial w''}\ln\hat{\Psi}(\rho_j''/\rho_i'',-w'')\,. \tag{3.34}$$

If we add all the one-loop graphs with an odd number of twists, the amplitude retains the form of Eq. (3.34) except that the ρ_i'' variables undergo an ordered integration around the full unit circle. The integration ranges of the primed and double primed variables are then identical so that, in adding the planar loop of Eq. (3.25) to the Möbius loop of (3.34), we assume we may suppress the distinction. (See, however, Ref. 7.)

The full six-point function in one-loop order with zero or an odd number of twists is, including Chan-Paton factors,

$$\begin{aligned}A_6^{11}=\frac{i\,d\Omega'}{2}\left[\mathrm{Tr}\prod_{i=1}^{6}\lambda_{a_i}\right]\delta^D\left[\sum k_i\right]\tfrac{1}{6}\sum_{h=1}^{6}\sum_{i,j=1,\,i\neq j}^{6}T_{ij}(h)\Bigg[&(\mathrm{Tr}\lambda^0)\prod_{l<m}\hat{\Psi}(\rho_m'/\rho_l',w')^{k_l\cdot k_m}w'\frac{\partial}{\partial w'}\ln\hat{\Psi}(\rho_j'/\rho_i',w')\\&-2^{D/2}\prod_{l<m}\hat{\Psi}(\rho_m'/\rho_l',-w')^{k_l\cdot k_m}w'\frac{\partial}{\partial w'}\ln\hat{\Psi}(\rho_j'/\rho_i',-w')\Bigg]\,.\end{aligned} \tag{3.35}$$

Using the expression in Eq. (3.35), we can now derive an expression for the anomaly for arbitrary internal-symmetry groups. The anomalies associated with the parity-violating amplitudes cancel for an SO(32) internal symmetry. To see this replace the nth polarization vector by its corresponding momentum

$$\lim_{\zeta_n\to k_n}T_{ij}(h)=\epsilon_n k_j\cdot k_i\delta_{hn}\,.$$

Then

$$\begin{aligned}A_6^{11}\underset{\zeta_n\to k_n}{\longrightarrow}\epsilon_n\frac{i\,d\Omega'}{12}&\left[\mathrm{Tr}\prod_{i=1}^{6}\lambda_{a_i}\right]\delta^D\left[\sum k_i\right]\\&\times\Bigg[\prod_{l<m}\hat{\Psi}(\rho_m'/\rho_l',w')^{k_l\cdot k_m}w'\frac{\partial}{\partial w'}\sum_{i,j=1,\,i\neq j}^{6}k_i\cdot k_j\ln\hat{\Psi}(\rho_j'/\rho_i',w')(\mathrm{Tr}\lambda^0)\\&\quad-\prod_{l<m}\hat{\Psi}(\rho_m'/\rho_l',-w')^{k_l\cdot k_m}w'\frac{\partial}{\partial w'}\sum_{i,j=1,\,i\neq j}^{6}k_j\cdot k_i\ln\hat{\Psi}(\rho_j'/\rho_i',-w')2^{D/2}\Bigg]\,.\end{aligned} \tag{3.36}$$

And thus under the gauge projection $\zeta_n\to k_n$

$$A_6^{11}\to\epsilon_n\frac{i\,d\Omega'}{6}\left[\mathrm{Tr}\prod_{i=1}^{6}\lambda_{a_i}\right]\delta^D\left[\sum k_i\right]w'\frac{\partial}{\partial w'}\left[(\mathrm{Tr}\lambda^0)\prod_{l<m}\hat{\Psi}(\rho_m'/\rho_l',w')^{k_l\cdot k_m}-2^{D/2}\prod_{l<m}\hat{\Psi}(\rho_m'/\rho_l',-w')^{k_l\cdot k_m}\right] \tag{3.37}$$

$$=\frac{\epsilon_n}{6}i\prod_{j=2}^{6}\frac{d\rho_j'}{\rho_j'}\left[\mathrm{Tr}\prod_{i=1}^{6}\lambda_{a_i}\right]\delta^D\left[\sum k_i\right]\prod_{l<m}\hat{\Psi}(\rho_m'/\rho_l',0)^{k_l\cdot k_m}(\mathrm{Tr}\lambda^0-2^{D/2})\,. \tag{3.38}$$

The contribution from the $w'=1$ end point vanishes. Equation (3.38) shows that the anomaly cancels for an SO(32) internal symmetry in the critical dimension $D=10$. The form of Eq. (3.38) is in agreement with that of Green and Schwarz.

IV. CONCLUSION

In Eq. (3.35) we have a closed-form expression for the one-loop parity-violating six-point function which is manifestly finite for the arbitrary internal-symmetry group due to the form of Eq. (3.23). This is the first proof of finiteness for the parity-violating amplitudes.

In Eq. (3.35) there is no logarithmic dependence on w', as is probably required for factorization of the parity-violating contributions to graviton exchange. In a future publication, we will explore the questions of finiteness and anomaly cancellation for the parity-violating N-point function.

We have also shown, without need of diverse regularization schemes[5,8] (since the amplitude is finite), that the anomalies associated with the annulus and Möbius strip diagrams cancel for an SO(32) internal symmetry. This is reassuring since for a general internal-symmetry group different regularization schemes appeared previously to result in different values for the anomaly.

Our results do not include possible effects from surface terms such as canceled propagator terms which have been discarded in accordance with usual practice in calculating string amplitudes. Such effects, if nonzero, would not modify our demonstration of finiteness for the parity-odd six-point function but could affect anomaly cancellation. We expect that unitarity requires the absence of such contributions from the symmetrized amplitude.

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