Skyrme-model Lagrangian in quantum mechanics: SU(2) case

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Quantum-mechanical structures of the SU(2) Skyrme model are examined in the framework of the collective-coordinate quantization. This nonlinear model gives us a simple example of quantum mechanics on a curved space. According to the procedure adopted by Sugano, Kimura, and others, we treat the Lagrangian quantum mechanically from the beginning. Some remarks are given on relations between Hamilton and Lagrange formalisms. To the *classical* mass term, a new one is added, which may play a role to stabilize the rotating chiral soliton.

I. INTRODUCTION

Some authors[1–6] have investigated the quantization of nonlinear theories in quantum mechanics. The obtained results can be applied to the Skyrme model[7] so as to treat it quantum mechanically from the beginning. The aim of the present paper is to investigate the quantum structures of the SU(2) Skyrme Lagrangian in the framework of the collective-coordinate formalism proposed by Adkins *et al.*[8]

The usual way of treating the Skyrme model quantum mechanically, which is essentially the same as that adopted by Adkins *et al.*,[8] is as follows.

(i) Start with the Lagrangian[9]

$$L(U_{L\rho};\mathbf{x},t)=\frac{f_\pi{}^2}{4}\mathrm{Tr}(U_{L\rho}U_{L\rho})+\frac{1}{32e_s{}^2}\mathrm{Tr}([U_{L\rho},U_{L\rho}]^2)+\frac{m_\pi{}^2f_\pi{}^2}{4}\mathrm{Tr}(U+U^\dagger-2)\ , \tag{1.1}$$

where $U_{L\rho}=(\partial_\rho U)U^\dagger$ and $(f_\pi)_{\rm expt}\simeq 93$ MeV. It is assumed that the soliton solution $\sigma(\mathbf{x})$ of the hedgehog type

$$\sigma(\mathbf{x})_{\rm hed}=\exp[iF(r)\tau\cdot\hat{\mathbf{x}}],\quad \hat{\mathbf{x}}=\mathbf{x}/r,\quad r=|\mathbf{x}| \tag{1.2a}$$

exists, and the collective coordinate $A(t)$ is introduced[8] by

$$U(\mathbf{x},t)=A(t)\sigma(\mathbf{x})A(t)^\dagger\ . \tag{1.2b}$$

(ii) After some "classical" calculations, one derives

$$L_0{}^c(U_{L\rho})\equiv\int L(U_{L\rho};\mathbf{x},t)d^3x=\frac{a[F]}{2}\sum_{B=1}^{3}w^Bw^B-M_c[F]\ , \tag{1.3a}$$

where

$$A^\dagger\frac{dA}{dt}\equiv\sum_{B=1}^{3}\frac{i}{2}w^B\tau_B\ , \tag{1.3b}$$

$$a[F]\equiv\tfrac{8}{3}\pi f_\pi{}^2\int_0^\infty dr\, r^2s^2\left[1+\frac{1}{e_s{}^2f_\pi{}^2}\left[\frac{s^2}{r^2}+F'^2\right]\right] \tag{1.3c}$$

with $s=\sin F(r)$ and $F'=dF/dr$;

$$M_c[F]\equiv 2\pi f_\pi{}^2\int_0^\infty dr\, r^2\left[\frac{2s^2}{r^2}+F'^2+\frac{1}{e_s{}^2f_\pi{}^2}\frac{s^2}{r^2}\left[\frac{s^2}{r^2}+2F'^2\right]+2m_\pi{}^2(1-c)\right]\ ,\quad c=\cos F\ . \tag{1.3d}$$

The function of $F(r)$ is determined by making use of the minimization condition of $M_c[F]$ with respect to $F(r)$.

(iii) Define the momentum p_b conjugate to q^b [q^b's ($b=1,2,3$) are real parameters which are necessary to define an SU(2) matrix $A(t)$, e.g., three Euler angles]:

$$p_b=\frac{\partial L_0{}^c}{\partial\dot{q}^b}\ . \tag{1.4}$$

(iv) Finally, require the canonical commutation relation between $\{q^b\}$ and $\{p_b\}$, or replace the classical Poissonbrackets relation with the quantum one.[10]

In the procedure described above there are some uncertainties which have some connection with phenomenology such as values of soliton masses and f_π. When we consider some current by introducing an interaction to (1.1), it is not so clear what form of the current is correct quantum mechanically. As far as currents with the first order of $\dot{A}(\equiv dA/dt)$ or w^B are concerned, appropriate symmetrization of "classical" expressions might lead to the correct quantum-mechanical ones. As for the current of $(w^B)^2$ order, there is no definite procedure to get to the quantum-mechanical form from the classical one. In order to remove such uncertainties, we take the viewpoint that the problem should be treated quantum mechanically from the beginning. The quantization problem has an intimate connection with investigations[11] of phenomenological adequacy of the semiclassical approximation.[8] There is a possibility that the first- and second-order $\dot{A}$ corrections to this approximation have effects so large in some kinds of matrix elements that phenomenologically favorable results[8,12] obtained in this approximation are drastically changed.[11,13] So, for the sake of getting more reliable results, it seems also necessary to examine the quantization problem. The present work is confined to considerations of quantum mechanics of the "free" Skyrmion model, and the structures of the electromagnetic and weak currents will be explained in a subsequent paper.

Before going into detail, it is worth noticing the following two points. (a) The hedgehog form of $\sigma(x)$, (1.2a), is consistent with the Euler-Lagrange equation for the semiclassical Skyrme Lagrangian, the solution of which minimizes the classical mass $M_c[F]$, (1.3d) (Ref. 8). It is not appropriate to take the hedgehog form of $\sigma(\mathbf{x})$ before solving equations of motion derived from the quantum-mechanical Lagrangian. Thus in the following consideration we do not use any concrete expression of $\sigma(\mathbf{x})$, but assume only the existence of a soliton solution and utilize its general properties. (b) The second remark is concerned with the form of $A^\dagger\dot{A}$, (1.3b). By using $(a_0,\mathbf{a})$ variables introduced by Adkins *et al.*,[8] where $A=a_0+i\mathbf{a}\cdot\boldsymbol{\tau}$, $A^\dagger\dot{A}$ is expressed as

$$A^\dagger\dot{A}=i(a_0\dot{a}_B-a_B\dot{a}_0)\tau_B+i(\mathbf{a}\times\dot{\mathbf{a}})\cdot\boldsymbol{\tau}+(a_0\dot{a}_0+a_B\dot{a}_B)\ . \tag{1.5}$$

(Here and hereafter we omit $\sum_{B=1}^{3}$ and adopt the summation convention.) The last term $(a_0\dot{a}_0+a_B\dot{a}_B)$ is equal to zero "classically" because of the constraint

$$A^\dagger A=I=a_0a_0+a_Ba_B\ , \tag{1.6}$$

but not equal to zero quantum mechanically. This means that the expansion of $A^\dagger\dot{A}$ in terms of only τ_B such as (1.3b) is not correct quantum mechanically.

The remaining part of the present paper is organized as follows. In Sec. II the fundamental assumptions and useful relations for exploring the quantum-mechanical structure of Skyrme Lagrangian are given. Using the quantum form of $A^\dagger\dot{A}$, we give explicitly the quantum-mechanical Lagrangian in terms of q variables. After defining the momentum operator p_a conjugate to q^a, we require the canonical commutation relations between them. In Sec. III various relations are given which are useful for considering the quantization problem on a curved space. By employing these relations, in Sec. IV a remark on the relation of the present consideration with that done by Kimura[4] is given, and the Hamiltonian form is discussed. In Sec. V the Hamiltonian obtained in Sec. IV is proved anew to lead really to Hamilton equations of motion for variables (q^b,p_b), and their connection with the Lagrange equation is disussed. In Sec. VI, discussions and problems to be explored are given. Especially the physical effect of a new mass term coming from the quantum-mechanical treatment is remarked. In the Appendixes mathematical details are given.

II. FUNDAMENTAL ASSUMPTIONS AND QUANTUM FORM OF SU(2) SKYRME LAGRANGIAN

A. Fundamental assumptions

We start with the Lagrangian $L(U_{Lp};\mathbf{x},t)$, (1.1), of the SU(2) Skyrme model. As already noted, $A(t)[\in\mathrm{SU}(2)]$ is specified by a set of real parameters $\{q^b;b=1,2,3\}$, and since $A^\dagger(\partial A/\partial q^a)$ belongs to the Lie algebra of SU(2), we can write

$$A^\dagger\frac{\partial A}{\partial q^a}=\frac{i}{2}\tau_BC(q)_a{}^B\ . \tag{2.1}$$

Hereafter $\partial/\partial q^a$ is written simply as ∂_a when there is no fear of confusion. We introduce the inverse of $\{C_a{}^B\}$ as

$$C^b{}_EC_b{}^D=\delta_E{}^D,\ \ C^b{}_EC_d{}^E=\delta_d{}^b\ . \tag{2.2}$$

From (2.1), it is easy to derive the useful relation

$$C^b{}_E\partial_bC^a{}_B-C^b{}_B\partial_bC^a{}_E=-\epsilon_{EBF}C^a{}_F \tag{2.3}$$

with the totally antisymmetric tensor ϵ_{EBF}.

The basic assumption is that the commutation relation between q^b and $dq^d/dt\equiv\dot{q}^d$ is given by

$$[\dot{q}^d,q^b]=-if^{db}(q)\ ; \tag{2.4}$$

$f^{db}(q)$ is a function of only q's, which is to be determined after the quantization condition is imposed. f^{ab} is symmetric under the exchange $a\leftrightarrow b$ owing to $[q^a,q^b]=0$.

Here we newly define w^B as

$$w^B\equiv\tfrac{1}{2}\{\dot{q}^a,C_a{}^B\} \tag{2.5}$$

instead of its classical form $w^B=\dot{q}^aC_a{}^B$. Equation (2.5) is the suitable quantum form of w^B, which will be seen from the consideration in the following subsection.

B. Quantum form of $A^\dagger\dot{A}$

First we define the quantum form of $\dot{A}\equiv dA/dt$ as

$$\dot{A}(q)\equiv\tfrac{1}{2}\{\dot{q}^a,\partial_aA(q)\}\ . \tag{2.6}$$

Then we have, from (2.1),

$$\frac{i}{2}\tau_B w^B=\tfrac{1}{2}\{\dot q^a,A^\dagger\partial_a A\}=A^\dagger\dot A+\tfrac{1}{2}[\dot q^a,A^\dagger]\partial_a A\ ;$$

i.e.,

$$A^\dagger\dot A=\frac{i}{2}\tau_B w^B+\frac{i}{8}f^{BB}\ , \tag{2.7a}$$

where

$$f^{BD}(q)\equiv C(q)_a{}^B C(q)_b{}^D f(q)^{ab}\ . \tag{2.7b}$$

Here we have used

$$[\dot q^a,A]=-\mathrm{i}f^{ab}\partial_b A\ . \tag{2.7c}$$

It is easy to prove that $\dot A$ defined by (2.6) has the desired properties:

$$A^\dagger\dot A+\dot A^\dagger A=\dot A A^\dagger+A\dot A^\dagger=0\ . \tag{2.8}$$

As remarked in Sec. I point (b), $A^\dagger\dot A$ is not expressed as (1.3b), but has an additional term of SU(2) singlet. We can use, however, effectively

$$A^\dagger\dot A\overset{\text{eff}}{=}\frac{i}{2}\tau_B w^B\ , \tag{2.9}$$

when the time derivative of $U(\mathbf{x},t)$ is included only in special combinations

$$U_{L\rho}(x)=[\partial_\rho U(x)]U(x)^\dagger$$

and

$$U_{R\rho}(x)=U(x)^\dagger\partial_\rho U(x)\ .$$

The reason is easily seen from the following relations:

$$U_{L4}=\frac{1}{i}A(A^\dagger\dot A-\sigma A^\dagger\dot A\sigma^\dagger)A^\dagger$$
$$=X_{BD}A\tau_D w^B A^\dagger\ , \tag{2.10a}$$

$$U_{R4}=\frac{1}{i}A(\sigma^\dagger A^\dagger\dot A\sigma-A^\dagger\dot A)A^\dagger$$
$$=-X'_{BD}A\tau_D w^B A^\dagger\ , \tag{2.10b}$$

where we have utilized the fact that $\tau_B-\sigma\tau_B\sigma^\dagger$ and $\tau_B-\sigma^\dagger\tau_B\sigma$ can be expanded in terms of τ_B's:

$$\tau_B-\sigma\tau_B\sigma^\dagger=2X_{BD}\tau_D\ , \tag{2.10c}$$

$$\tau_B-\sigma^\dagger\tau_B\sigma=2X'_{BD}\tau_D\ . \tag{2.10d}$$

Properties of X_{BD} and X'_{BD} are summarized in Appendix C.

C. Canonical momentum p_a

For the purpose of defining the canonical momentum p_a conjugate to q^a, we express Lagrangian (1.1) in terms of q^a and $\dot q^a$ variables. First note

$$[w^B,A]=f^{BD}A\frac{\tau_D}{2}\ . \tag{2.11}$$

Employing this relation and

$$[w^B,f^{ED}]=-iC_b{}^B f^{bd}\partial_d f^{ED}\ , \tag{2.12}$$

we can directly confirm the following to hold:

$$L(U_{L\rho};\mathbf{x},t)=\frac{a(\sigma;\mathbf{x})_{BD}}{2}w^B w^D+[(w^B)^0\text{- order term}]\ , \tag{2.13a}$$

where

$$a(\sigma;\mathbf{x})_{BE}\equiv f_\pi{}^2 X(\mathbf{x})_{BD}X(\mathbf{x})_{ED}+\frac{1}{4e_s{}^2}\xi(\mathbf{x})_k{}^F\xi(\mathbf{x})_k{}^H X(\mathbf{x})_{BD}\times X(\mathbf{x})_{EG}\epsilon_{FDJ}\epsilon_{HGJ}\ , \tag{2.13b}$$

$$\frac{\partial\sigma(\mathbf{x})}{\partial x^k}\sigma(\mathbf{x})^\dagger\equiv\frac{i}{2}\tau_B\xi(\mathbf{x})_k{}^B\ . \tag{2.13c}$$

[Because of $\sigma\sigma^\dagger=I$, $(\partial\sigma/\partial x^k)\sigma^\dagger$ is expanded in terms of only τ_B's.] Nonappearance of $(w^B)^1$-order term in the right-hand side (RHS) of (2.13a) is explicitly demonstrated in Appendix A. Similarly, the RHS of (2.13a) can be written

$$L(U_{L\rho};\mathbf{x},t)=\tfrac{1}{2}\dot q^a g(\sigma;\mathbf{x})_{ab}\dot q^b+[(\dot q^a)^0\text{- order term}]\ , \tag{2.14a}$$

where

$$g(\sigma;\mathbf{x})_{ab}\equiv a(\sigma;\mathbf{x})_{BD}C_a{}^B C_b{}^D\ . \tag{2.14b}$$

The proof is explicitly given in Appendix A.

Define

$$a(\sigma)\delta_{BE}\equiv\int d^3x\,a(\sigma;\mathbf{x})_{BE}\ . \tag{2.15}$$

Then we have

$$L_0(U_{L\rho})\equiv\int d^3x\,L(U_{L\rho};\mathbf{x},t)$$
$$=\frac{a(\sigma)}{2}w^B w^B+[(w^B)^0\text{-order term}]\ , \tag{2.16a}$$

and further

$$\frac{a(\sigma)}{2}w^B w^B=\tfrac{1}{2}\dot q^a g_{ab}(q)\dot q^b-v(q)\ . \tag{2.16b}$$

Here, $g_{ab}(q)$ and $v(q)$ are functions of only q's:

$$g_{ab}(q)\equiv\int d^3x\,g(\sigma;\mathbf{x})_{ab}$$
$$=a(\sigma)C_a{}^B C_b{}^B\ , \tag{2.16c}$$

$$v(q)=\frac{a(\sigma)}{4}f^{ad}\partial_a(C_d{}^B f^{be}\partial_e C_b{}^B)-\frac{a(\sigma)}{8}f^{ab}(\partial_a C_b{}^B)f^{ed}\partial_e C_d{}^B\ . \tag{2.16d}$$

As already mentioned in Sec. I we do not utilize any explicit form of $\sigma(\mathbf{x})$ such as the hedgehog one, (1.2a), but only assume existence of the soliton solution.

From (2.16a) and (2.16b), we can define the canonical momentum p_a which is conjugate to q^a as

$$p_a \equiv \frac{\partial L_0(U_{L\rho})}{\partial \dot{q}^a} = \tfrac{1}{2}\{\dot{q}^b, g_{ab}\} \ . \tag{2.17}$$

D. Quantization conditions and a new mass term

We impose the canonical commutation relation

$$[p_a, q^b] = -i\delta_a{}^b \ . \tag{2.18a}$$

Note that for the time being we *do not* require

$$[p_a, p_b] = 0 \ , \tag{2.18b}$$

and we derive, in this subsection, the Lagrangian $L_0(U_{L\rho})$ expressed in terms of q variables; that is, the $(w^B)^0$-order term in (2.16a) is explicitly given.

In order to do this, first we get a condition imposed on f^{ab} appearing in the starting assumption (2.4). From (2.4) and (2.17), one obtains

$$[p_a, q^b] = -i f^{db}(q) g_{da}(q) \ ; \tag{2.19a}$$

therefore,

$$f^{db} g_{da} = \delta_a{}^b \ . \tag{2.19b}$$

Hereafter we write $f^{ab}(q)$ as $g^{ab}(q)$; then (2.4) and (2.19b) are written as

$$[\dot{q}^d, q^b] = -i g^{db} \ , \tag{2.20a}$$

$$g^{db} g_{da} = \delta_a{}^b \ , \tag{2.20b}$$

respectively. Furthermore we have

$$g^{ab} = \frac{1}{a(\sigma)} C^a{}_B C^b{}_B \ , \tag{2.20c}$$

$$\dot{q}^a = \tfrac{1}{2}\{p_b, g^{ba}\} \ . \tag{2.20d}$$

From (2.7a) and (2.11), one easily obtains

$$A^\dagger \dot{A} = \frac{i}{2}\tau_B w^B + \frac{3i}{8a(\sigma)} \ , \tag{2.21}$$

$$[w^B, A] = \frac{1}{a(\sigma)} A \frac{\tau_B}{2} \ . \tag{2.22}$$

By making use of these relations, one can obtain

$$L_0(U_{L\rho}) = \frac{a(\sigma)}{2} w^B w^B - [M_c(\sigma) + \Delta M(\sigma)] \ . \tag{2.23a}$$

Here $M_c(\sigma)$ is the "classical" mass term given by

$$M_c(\sigma) = -\int d^3x \, L(U_{Lk}; \mathbf{x}, t) \ ; \tag{2.23b}$$

$L(U_{Lk}; \mathbf{x}, t)$ is defined by (B1) in Appendix B. $M_c(\sigma)$ reduces to $M[F]$, (1.3d), when the hedgehog form $\sigma(x)_{\text{hed}}$ is substituted for σ. $\Delta M(\sigma)$ as well as $a(\sigma) w^B w^B / 2$ in (2.23a) come from

$$L(U_{L4}; \mathbf{x}, t) \equiv \frac{f_\pi{}^2}{4} \mathrm{Tr}(U_{L4} U_{L4}) + \frac{1}{16 e_s{}^2} \mathrm{Tr}([U_{Lk}, U_{L4}][U_{Lk}, U_{L4}]) \tag{2.24}$$

when (2.22) is employed. With the help of the expression for $\Delta M(\sigma)$ given in Appendix B, we have $\Delta M(\sigma_{\text{hed}}) \equiv \Delta M[F]$ written as

$$\Delta M[F] = \frac{-2\pi}{a[F]^2} \int_0^\infty dr \, r^2 s^2 \times \left[f_\pi{}^2 + \frac{1}{2e_s{}^2} \left[2F'^2 + \frac{s^2}{r^2} \right] \right] \tag{2.25}$$

with $a[F] = a(\sigma_{\text{hed}})$, (1.3c).

It is worthy to remark on the choice of the starting Lagrangian. Although $L(U_{L\rho}; \mathbf{x}, t) = L(U_{R\rho}; \mathbf{x}, t)$ is evident in the classical theory, it is necessary to confirm whether or not this equality holds quantum mechanically. We can easily prove by direct calculations

$$L(U_{L\rho}; \mathbf{x}, t) = L(U_{R\rho}; \mathbf{x}, t) \ . \tag{2.26}$$

Needless to say, the form of $\Delta M(\sigma)$ is unique irrespective of the starting SU(2) Lagrangians which are "classically" equal to (1.1). Details are explained in Appendix D.

III. PROPERTIES OF $\{C_a^E\}$

In this section we summarize properties of $C_a{}^E$'s and $C^a{}_E$'s for later use. In order to do this, it is convenient to consider the relation of $C_a{}^E$'s with the q-coordinate transformation which induces some right SU(2) transformation of SU(2) group element $A(q)$. Because the functional form of A is invariant under such q transformations, we obtain the isometry condition[14] on the "metric" $g_{ab}(q)$, from which useful relations are derived.

A. Relation between $\{C_a^E\}$ and q-coordinate transformation

Let us consider an infinitesimal SU(2) right transformation of $A(q)$:

$$A(q) + \delta_\zeta A(q) = A(q) \left[1 + \frac{i}{2}\tau_B \zeta^B \right] \ , \tag{3.1a}$$

with an infinitesimal parameter ζ^B. This transformation is assumed to be induced by some q-coordinate transformation:

$$q^a \to q'^a = q^a + \zeta^B \rho(q)^a{}_B \ , \tag{3.1b}$$

i.e.,

$$A(q) + \delta_\zeta A(q) = A(q^a + \zeta^B \rho(q)^a{}_B) \ . \tag{3.1c}$$

We have

$$\rho(q)^a{}_B \partial_a A(q) = i A(q) \frac{\tau_B}{2} \ , \tag{3.2a}$$

which leads to

$$\rho(q)^a{}_B C(q)_a{}^D = \delta_B{}^D \ ; \tag{3.2b}$$

therefore,

$$\rho(q)^a{}_B = C(q)^a{}_B \ . \tag{3.2c}$$

The invariance of functional form A under the relevant

q-coordinate transformations means the form invariance of the "metric" (2.16c)

$$g_{ab}(q)=a(\sigma)C(q)_a{}^B C(q)_b{}^B , \tag{2.16c}$$

leading to the isometry[14]

$$g'_{ab}(q')=g_{ab}(q')=\frac{\partial q^d}{\partial q'^a}\frac{\partial q^e}{\partial q'^b}g_{de}(q) . \tag{3.3}$$

The following relations are derived from (3.3):

$$g_{da}\partial_b C^a{}_B+g_{ba}\partial_d C^a{}_B+C^a{}_B\partial_a g_{db}=0 , \tag{3.4a}$$

$$\nabla_b C_a{}^B+\nabla_a C_b{}^B=0 , \tag{3.4b}$$

where

$$\nabla_b C_a{}^B\equiv\partial_b C_a{}^B-\Gamma^e{}_{ba}C_e{}^B , \tag{3.4c}$$

$$\Gamma^e{}_{ba}\equiv\tfrac{1}{2}g^{ed}(\partial_b g_{da}+\partial_a g_{db}-\partial_d g_{ba}) . \tag{3.4d}$$

[Needless to say, (3.4a) and (3.4b) can be directly derived from (2.16c).] From $C_b{}^B=-i\,\mathrm{Tr}(A^\dagger\partial_b A\tau_B)$, one obtains

$$\partial_b C_a{}^B-\partial_a C_b{}^B=\epsilon_{DEB}C_b{}^D C_a{}^E . \tag{3.5}$$

From (3.5) and (3.4b), one obtains

$$\nabla_b C_a{}^B=\tfrac{1}{2}\epsilon_{BDE}C_b{}^D C_a{}^E \tag{3.6a}$$

or

$$\nabla_b C^a{}_B=\partial_b C^a{}_B+\Gamma^a{}_{bd}C^d{}_B =\tfrac{1}{2}\epsilon_{BDE}C_b{}^D C^a{}_E . \tag{3.6b}$$

We define in accordance with Kimura[4] the spin connection $A_{BD,b}$ as

$$\nabla_b C^a{}_B\equiv -A_{BE,b}C^a{}_E ; \tag{3.7a}$$

therefore, in our case we have

$$A_{BE,b}=-\tfrac{1}{2}\epsilon_{BDE}C_b{}^D . \tag{3.7b}$$

B. Local Euclidean frame in q space

The Riemann-Christoffel curvature tensor $R^e{}_{dab}$ is defined by

$$R^e{}_{dab}\equiv\partial_a\Gamma^e{}_{db}-\partial_b\Gamma^e{}_{da}+\Gamma^e{}_{af}\Gamma^f{}_{db}-\Gamma^e{}_{bf}\Gamma^f{}_{da} . \tag{3.8}$$

From the integrability condition[14]

$$R^e{}_{dab}\xi_e+\nabla_a\nabla_b\xi_d-\nabla_b\nabla_a\xi_d=0 \tag{3.9a}$$

with $C^b{}_B g_{be}$ substituted for ξ_e, or directly from

$$\partial_a\partial_b C^e{}_E=\partial_b\partial_a C^e{}_E , \tag{3.9b}$$

one obtains

$$-R^e{}_{dab}=(\partial_a A_{BD,b}-\partial_b A_{BD,a}-A_{BE,a}A_{DE,b}+A_{BE,b}A_{DE,a})C^e{}_D C_d{}^B =\frac{1}{4a(\sigma)}(g_{da}\delta_b{}^e-g_{db}\delta_a{}^e) , \tag{3.10a}$$

which leads to

$$R^{ed}{}_{ab}\equiv g^{df}R^e{}_{fab}=\frac{1}{4a(\sigma)}(\delta_a{}^e\delta_b{}^d-\delta_a{}^d\delta_b{}^e) , \tag{3.10b}$$

$$R\equiv R^{bd}{}_{db}=-\frac{3}{2a(\sigma)} . \tag{3.10c}$$

That $R^e{}_{dab}$ is not identically zero means that SU(2) Skyrme model belongs to the so-called "irreducible case" as first stressed by Kimura.[4] Then we have to attach a local Euclidean frame at each point in the q coordinate space. We introduce the local coordinates $\{Q^B\}$ for the relevant Euclidean frame, and also the vielbeins $\{h^a{}_B\}$ (Ref. 14), which satisfy

$$\begin{aligned} &h(q)^a{}_B h(q)^b{}_B=g^{ab}(q) ,\\ &h(q)_a{}^B h(q)_b{}^B=g_{ab}(q) . \end{aligned} \tag{3.11}$$

Here we introduce P_B

$$P_B\equiv\tfrac{1}{2}\{p_a,h^a{}_B\} . \tag{3.12}$$

Similarly, from

$$dQ^B=\tfrac{1}{2}\{dq^a,h_a{}^B\} , \tag{3.13a}$$

we have

$$\dot{Q}^B=\tfrac{1}{2}\{\dot{q}^b,h_b{}^B\} . \tag{3.13b}$$

All these quantities behave as three-vectors under the rotations

$$h(q)_b{}^B\to h'(q)_b{}^B=\Lambda^B{}_E h(q)_b{}^E \tag{3.14}$$

with q-independent $\Lambda^B{}_E$. Note that, using (2.20d), we can obtain

$$\dot{Q}^B=P^B . \tag{3.15}$$

As a special case, it is allowable to choose

$$\begin{aligned} &h(q)^a{}_B=\frac{1}{\sqrt{a(\sigma)}}C(q)^a{}_B ,\\ &h(q)_a{}^B=\sqrt{a(\sigma)}C(q)_a{}^B . \end{aligned} \tag{3.16}$$

In the next section we explain the difference between this choice and Kimura's, and investigate the quantization condition and also what the "correct" form of the Hamiltonian is.

IV. REMARK ON COMMUTATOR $[p_a,p_b]$

As mentioned at the end of the last section, we will examine, the commutator $[p_a,p_b]$ and the form of the Hamiltonian under the choice (3.16), and make a comment on the difference between our and Kimura's procedures of quantization in the "irreducible" case.[4]

From (3.12), we have

$$[[P_B,h_a{}^D],h_b{}^E]=0 , \tag{4.1a}$$

$$[[P_B,h_a{}^D],[P_E,h_b{}^F]]=0 \tag{4.1b}$$

which are obtained without recourse to $[p_a,p_b]$. By making use only of them, we obtain

$$[p_a,p_b]=\tfrac{1}{2}\{P_B,[P_D,h_b{}^B]h_a{}^D-[P_D,h_a{}^B]h_b{}^D\} +\tfrac{1}{4}(\{[P_B,P_D],h_a{}^B h_b{}^D\}+h_a{}^B[P_B,P_D]h_b{}^D +h_b{}^D[P_B,P_D]h_a{}^B) . \tag{4.2}$$

From (3.6a), for $h_a{}^B$ substituted for $C_a{}^B$, the first part in the RHS of (4.2) becomes

$$\frac{i}{2}\{P_B, A_{BD,a}h_b{}^D - A_{BD,b}h_a{}^D\} . \tag{4.3}$$

Kimura's assertion[4] concerning $[p_a,p_b]$ is based on

$$\begin{aligned} 0 &= [[p_a,p_b],f(q)] \\ &= (A_{BE,a}h_b{}^E - A_{BE,b}h_a{}^E)h^d{}_B\partial_d f(q) \end{aligned} \tag{4.4}$$

for an arbitrary function satisfying $\partial_a\partial_b f = \partial_b\partial_a f$. The first equality is obtained from the Jacobi identity. Kimura[4] derived the second equality by using (4.3) under the tacit assumption

$$[P_B,P_d]=0 \text{ for any } (B,D) . \tag{4.5}$$

On the basis of (4.4), Kimura[4] asserted that, in order to be safe to take $[p_a,p_b]=0$, one has to perform a suitable local rotation of $h(q)_b{}^B$ around a point $P[q]$,

$$h(q)_b{}^B \to h'_b{}^B = \Lambda(q)_{BE}h(q)_b{}^E , \tag{4.6a}$$

and the corresponding transformation

$$\begin{aligned} A(q)'_{BD,b} &= \Lambda(q)_{BE}A(q)_{EF,b}\Lambda(q)_{DF} \\ &\quad - \partial_b\Lambda(q)_{BE}\cdot\Lambda(q)_{DE} , \end{aligned} \tag{4.6b}$$

so that

$$A(q)_{BD,b} \simeq 0 \tag{4.6c}$$

is realized in the very vicinity of the point $P[q]$; only in this sense is one allowed to take $[p_a,p_b]=0$. Note that the derivatives of $A(q)_{BD,b}$ do not necessarily vanish, because $R^e{}_{dab}$ does not always vanish in the "irreducible" case.

It is clear that Kimura's assertion is based on the tacit assumption of (4.5), while (4.5) is no longer correct when we choose (3.16). We can take, however,

$$[p_a,p_b]=0 \tag{4.7}$$

with no contradiction, as will be seen from the following remark.

Here we use for convenience R_B instead of P_B:

$$R_B \equiv -\tfrac{1}{2}\{p_a, C^a{}_B\} \tag{4.8a}$$

or

$$p_a = -\tfrac{1}{2}\{C_a{}^B, R_B\} . \tag{4.8b}$$

We have

$$R_B = -\sqrt{a(\sigma)}P_B \tag{4.9a}$$

and also

$$w^B = \frac{1}{\sqrt{a(\sigma)}}\dot{Q}^B = \frac{-1}{a(\sigma)}R_B . \tag{4.9b}$$

With the help of (2.3), one obtains

$$\begin{aligned} [R_B,R_D] &= -i\epsilon_{BDE}R_E \\ &\quad + \tfrac{1}{4}(\{[p_a,p_b],C^a{}_B C^b{}_D\} \\ &\quad + C^a{}_B[p_a,p_b]C^b{}_D + C^b{}_D[p_a,p_b]C^a{}_B) . \end{aligned} \tag{4.10a}$$

After using (4.7), we have

$$[R_B,R_D] = -i\epsilon_{BDE}R_E , \tag{4.10b}$$

which is clearly different from (4.5). It is simply confirmed that the RHS of (4.2) with $-R_B/\sqrt{a(\sigma)}$ and $\sqrt{a(\sigma)}C_b{}^B$ substituted for P_B and $h_b{}^B$ is equal to zero when (4.10b) is utilized. Thus we can take (4.7) consistently.

V. HAMILTONIAN AND EQUATIONS OF MOTION

A. Form of Hamiltonian

From (4.9b), the Lagrangian (2.23a) is written as

$$L_0 = \tfrac{1}{2}\dot{Q}^B\dot{Q}^B - [M_c(\sigma)+\Delta M(\sigma)] , \tag{5.1}$$

which has the form just like the "standard" one. In the present "irreducible" case, $\dot{Q}^B$ and $\dot{Q}^D$ are not commutable when $B\neq D$. The momentum P_B, which is conjugate to Q^B introduced formally, may be defined by

$$P_B = \frac{\partial L_0}{\partial \dot{Q}^B} , \tag{5.2a}$$

because the formal derivative with respect to $\dot{Q}^B$ leads to

$$P_B = \dot{Q}^B , \tag{5.2b}$$

which is in accordance with (3.15). Thus, the Hamiltonian of the present model is possibly given by

$$\begin{aligned} \mathscr{H} &\equiv P_B\dot{Q}^B - L_0 \\ &= \frac{1}{2a(\sigma)}R_BR_B + [M_c(\sigma)+\Delta M(\sigma)] . \end{aligned} \tag{5.3}$$

Define $H(q,p)$ as

$$H(q,p) \equiv \mathscr{H} . \tag{5.4}$$

Then $H(q,p)$ can be regarded as a Hamiltonian in the sense that we can prove Hamilton equations of motion

$$\frac{\partial H(q,p)}{\partial p_a} = \dot{q}^a, \quad -\frac{\partial H(q,p)}{\partial q^q} = \dot{p}_a , \tag{5.5a}$$

under the condition

$$\dot{R}_B = 0 . \tag{5.5b}$$

Needless to say, this condition is consistent with $H(q,p)$ to be a Hamiltonian, because we have

$$[R_B, H(q,p)] = 0 \tag{5.6}$$

from the commutation relation (4.10b). Explicit derivations of (5.5a) will be given in the next subsection.

In the rest of this subsection, we mention the factor Z introduced by Lin, Lin, and Sugano.[2] For the sake of avoiding confusion arising from various forms of Z, we introduce $V(q)$ defined by

$$V(q) \equiv \tfrac{1}{2}\{p_a,\dot{q}^a\} - \frac{1}{a(\sigma)}R_BR_B ; \tag{5.7a}$$

thus

$$H(q,p) = K - V(q) , \tag{5.7b}$$

where

$$K \equiv \tfrac{1}{2}\{p_a,\dot{q}^a\} - L_0 \; . \tag{5.7c}$$

One can easily obtain

$$V(q) = \frac{-1}{4a(\sigma)}(\partial_a C^b{}_B)(\partial_b C^a{}_B) \; . \tag{5.8}$$

Utilizing (3.6b), we have

$$V(q) = Z_0(q) - \tfrac{1}{4}R \tag{5.9a}$$

with

$$Z_0(q) \equiv \frac{-1}{4}\Gamma^b{}_{ad}\Gamma^a{}_{be}g^{de} \; , \tag{5.9b}$$

while the first form of Z introduced by Lin, Lin, and Sugano[2] is given by

$$Z(q) = -\tfrac{1}{16}g^{ab}g^{cd}g^{ef}(\partial_a g_{cd})(\partial_b g_{ef}) - \tfrac{1}{4}\partial_a(g^{ab}g^{cd}\partial_b g_{cd}) - \tfrac{1}{4}\partial_a\partial_b g^{ab} \; . \tag{5.10}$$

Employing

$$\partial_d g^{ab} = -\Gamma^a{}_{de}g^{eb} - \Gamma^b{}_{de}g^{ea} \; , \tag{5.11a}$$

one can show $Z(q) = Z_0(q) - R/4$, i.e.,

$$V(q) = Z(q) \; . \tag{5.11b}$$

Therefore, we see

$$H(q,p) = \tfrac{1}{2}\{p_a,\dot{q}^a\} - L_0 - Z(q) \; . \tag{5.12}$$

It may be worthwhile to point out the following two points about Z.

(1) With the help of (5.11a), $Z_0(q)$ is easily rewritten as

$$Z_0(q) = -\tfrac{1}{16}g^{ab}(\partial_a g^{de})(\partial_b g_{de}) + \tfrac{1}{8}g^{ab}(\partial_a g^{de})(\partial_d g_{be}) \; , \tag{5.13}$$

which is equal to the second expression of Z given by Sugano.[3]

(2) By direct calculations we see the following identity to hold:

$$\tfrac{1}{4}g^{-1/4}\{g_{ab},\dot{q}^b\}g^{1/2}g^{ad}\{g_{de},\dot{q}^e\}g^{-1/4} = \tfrac{1}{2}\{p_a,\dot{q}^a\} - Z = \frac{1}{a(\sigma)}R_B R_B \; , \tag{5.14}$$

where $g \equiv \det[g_{ab}]$. Thus our Hamiltonian is written as

$$H(q,p) = \tfrac{1}{2}g^{-1/4}p_a g^{1/2}g^{ab}p_b g^{-1/4} + M_c(\sigma) + \Delta M(\sigma) \; . \tag{5.15}$$

B. Hamilton and Lagrange equations

In the "irreducible" case, $\mathscr{H}$ (5.3) is the Hamiltonian and we can derive (5.5a) from the Hamilton equations for (Q^B,P_D), as was done by Lin, Lin, and Sugano.[2] As mentioned in Sec. V A we derive in the following the Hamilton equations of motion (5.5a) under the condition (5.5b). Instead of R_B, w^B, $C_a{}^B$, and $C^a{}_B$, we use for convenience P_B, $\dot{Q}^B$, $h^a{}_B$, and $h_a{}^B$ given by (3.12), (3.13b), and (3.16). In the following derivation we are allowed to use the commutation relations

$$[p_a,q^b] = -i\delta_a{}^b, \quad \text{others} = 0 \; , \tag{5.16}$$

and other equalities derived from (5.16). We will explain step by step as follows.

(i) We adopt the notation

$$\left.\frac{\partial H}{\partial p_a}\right|_q \; , \tag{5.17}$$

which means the differentiation of H with respect to p_a by keeping $\{q^b\}$ to be constant. Then, one obtains

$$\left.\frac{\partial H(q,p)}{\partial p_a}\right|_q = \frac{1}{2}\left.\left\{P_B,\frac{\partial P_B}{\partial p_a}\right\}\right|_q = \tfrac{1}{4}\{\{p_b,h^b{}_B\},h^a{}_B\} = \tfrac{1}{2}\{p_b,g^{ba}\} = \dot{q}^a \; , \tag{5.18}$$

which is the first equation of (5.5a).

(ii) It is convenient to utilize the following five formulas derived from (5.16):

$$[P_B,F(q)] = -ih^a{}_B\partial_a F(q) \; , \tag{5.19a}$$

$$\{\dot{q}^b,\partial_b h_a{}^B\} = \{\dot{Q}^D,h^b{}_D\partial_b h_a{}^B\} \; , \tag{5.19b}$$

because

$$\text{LHS} = \tfrac{1}{2}\{\{\dot{Q}^D,h^b{}_D\},\partial_b h_a{}^B\} = \{\dot{Q}^D,h^b{}_D\partial_b h_a{}^B\} + \tfrac{1}{2}[[h^b{}_D,\dot{Q}^D],\partial_b h_a{}^B] \; ,$$

where the second term vanishes due to $\dot{Q}^d = P_D$ and (5.19a);

$$[P_B,P_D] = \frac{i}{\sqrt{a(\sigma)}}\epsilon_{BDE}P_E \; , \tag{5.19c}$$

$$h^d{}_E\partial_d h^b{}_B - h^d{}_B\partial_d h^b{}_E = \frac{-1}{\sqrt{a(\sigma)}}\epsilon_{EBF}h^b{}_F \; , \tag{5.19d}$$

$$\dot{p}_a = \tfrac{1}{4}\{P_B,\{P_D,h^b{}_D\partial_b h_a{}^B\}\} \; , \tag{5.19e}$$

because

$$\dot{p}_a = \frac{1}{2}\frac{d}{dt}\{P_B,h_a{}^B\} = \tfrac{1}{2}\{P_B,\dot{h}_a{}^B\} \quad [\text{due to (5.5b)}] = \tfrac{1}{4}\{P_B,\{\dot{q}^b,\partial_b h_a{}^B\}\} = \tfrac{1}{4}\{P_B,\{P_D,h^b{}_D\partial_b h_a{}^B\}\} \quad [\text{due to (5.19b)}] \; .$$

(iii) Next we examine $\partial H/\partial q^a|_p$:

$$\left.\frac{\partial H(q,p)}{\partial q^a}\right|_p = \frac{i}{2}\{P_B,[p_a,P_B]\} = \frac{i}{4}\{P_B,\{[P_D,P_B],h_a{}^D\}\} + \frac{i}{4}\{P_B,\{P_D,[h_a{}^D,P_B]\}\} \; . \tag{5.20a}$$

Because of (5.19c),

$$\text{first term of (5.20a)} = \frac{-1}{4\sqrt{a(\sigma)}}\epsilon_{DBE}\{P_B,\{P_E,h_a{}^D\}\} = \frac{-1}{8\sqrt{a}}\epsilon_{DBE}[[P_B,P_E],h_a{}^D]$$
$$= \frac{-i}{4a}[P_D,h_a{}^D]$$
$$= \frac{-1}{4a}h^b{}_D\partial_b h_a{}^D \text{ [due to (5.19a)] ;} \tag{5.20b}$$

$$\text{second term of (5.20a)} = -\tfrac{1}{4}\{P_B,\{P_D,h^b{}_B\partial_b h_a{}^D\}\} \text{ [due to (5.19a)]}$$
$$= -\tfrac{1}{4}\{P_B,\{P_D,h^b{}_D\partial_b h_a{}^B\}\} - \tfrac{1}{4}[[P_B,P_D],h^b{}_B\partial_b h_a{}^D]$$
$$= -\dot{p}_a - \frac{i}{4\sqrt{a}}\epsilon_{BDE}[P_E,h^b{}_B\partial_b h_a{}^D] \text{ [due to (5.19e) and (5.19c)]}$$
$$= -\dot{p}_a - \frac{1}{4\sqrt{a}}\epsilon_{BDE}h^d{}_E\partial_d(h^b{}_B\partial_b h_a{}^D) \text{ [due to (5.19a)]}$$
$$= -\dot{p}_a - \frac{1}{4\sqrt{a}}\epsilon_{BDE}\partial_b h_a{}^D\cdot\tfrac{1}{2}(h^d{}_E\partial_d h^b{}_B - h^d{}_B\partial_d h^b{}_E)$$
$$= -\dot{p}_a + \frac{1}{4a}h^b{}_D\partial_b h_a{}^D \text{ [due to (5.19d)] .} \tag{5.20c}$$

Thus we obtain the second equation of (5.5a).

In the end of this section, we add a remark on the Lagrange equation, according to the procedure given by Lin *et al.*[2] The Lagrangian (5.1) is written as

$$L_0 = \tfrac{1}{2}\dot{q}^a g_{ab}\dot{q}^b - v(q) - [M_c(\sigma) + \Delta M(\sigma)] \tag{5.21}$$

with $v(q)$ given by (2.16d). From

$$-\dot{p}_a = \left.\frac{\partial H(q,p)}{\partial q^a}\right|_p$$
$$= \left.\frac{\partial}{\partial q^a}[\tfrac{1}{2}\{p_b,\dot{q}^b\} - (L_0+Z)]\right|_p , \tag{5.22a}$$

we have

$$-\dot{p}_a = \frac{1}{2}\left\{p_b,\left.\frac{\partial\dot{q}^b}{\partial q^a}\right|_p\right\} - \left.\frac{1}{2}\frac{\partial}{\partial q^a}(\dot{q}^b g_{bd}\dot{q}^d)\right|_p + \frac{\partial}{\partial q^a}(v - Z)$$
$$= -[\tfrac{1}{2}\dot{q}^b(\partial_a g_{bd})\dot{q}^d + \partial_a(-v+Z)] - \tfrac{1}{4}(\partial_a g^{fb})\partial_f(g^{de}\partial_d g_{be})$$
$$= -\left.\frac{\partial(L_0+Z)}{\partial q^a}\right|_{\dot{q}} - \tfrac{1}{4}(\partial_a g^{fb})\partial_f(g^{de}\partial_d g_{be}) , \tag{5.22b}$$

which leads to

$$\frac{d}{dt}\frac{\partial(L_0+Z)}{\partial\dot{q}^a} - \frac{\partial(L_0+Z)}{\partial q^a} = \tfrac{1}{4}(\partial_a g^{fb})\partial_f(g^{de}\partial_d g_{be}) , \tag{5.22c}$$

or

$$\frac{d}{dt}\frac{\partial L_0}{\partial\dot{q}^a} - \frac{\partial L_0}{\partial q^a} = \frac{\partial Z}{\partial q^a} + \tfrac{1}{4}(\partial_a g^{fb})\partial_f(g^{de}\partial_d g_{be}) . \tag{5.22d}$$

VI. DISCUSSIONS AND FURTHER OUTLOOK

We have examined, on the basis of the collective-coordinate formalism,[8] the SU(2) Skyrme Lagrangian, which gives one of the simple examples of quantum-mechanical systems on a curves space. We have treated quantum mechanically the collective coordinate from the beginning. The SU(2) Skyrme model amounts to one of the "irreducible" cases as remarked by Kimura.[4] It was pointed out that in our case the standard form of the Lagrangian $L_0 = \dot{Q}^B\dot{Q}^B/2 - W(Q)$ and $P_B = \dot{Q}^B$ are obtained already at early stages, and that we can impose consistently the commutation relation $[p_a,p_b]=0$. We have shown that the last point is different from Kimura's assertion,[4] since his tacit assumption $[P_B,P_D]=0$ for any (B,D) is not valid in our case.

As noted at the end of Sec. II, the "classical" mass term $M_c(\sigma)$ is modified due to the appearance of $\Delta M(\sigma)$ as a quantum effect. For the hedgehog configuration

$\sigma(\mathbf{x})=\sigma(\mathbf{x})_{\rm hed}$, $\Delta M(\sigma_{\rm hed})\equiv\Delta M[F]$ given by (2.25) is rewritten as

$$\Delta M[F]=\frac{-3}{4a[F]}+\frac{1}{a[F]^2}\frac{\pi}{e_s^2}\int_0^\infty dr\, s^4 . \quad (6.1)$$

The eigenvalue of Hamiltonian (5.4) for $I^2=J^2=l(l+1)$ state is

$$H_l[F]=M_c[F]+\Delta M[F]+\frac{l(l+1)}{2a[F]} . \quad (6.2)$$

We require the variational condition so as to minimize H_l with respect to F, and obtain

$$\frac{\delta M_c}{\delta F}+\frac{1}{a[F]^2}\left[\frac{3}{4}-\frac{1}{a[F]}\frac{2\pi}{e_s^2}\int_0^\infty dr\, s^4 -\frac{l(l+1)}{2}\right]\frac{\delta a[F]}{\delta F}+\frac{1}{a[F]^2}\frac{4\pi}{e_s^2}s^3c=0 , \quad (6.3a)$$

where

$$\frac{\delta M_c}{\delta F}\equiv\frac{\partial M_c}{\partial F(r)}-\frac{d}{dr}\frac{\partial M_c}{\partial F(r)'} . \quad (6.3b)$$

As in Ref. 15, we can linearize (6.3a) due to the boundary condition $F(r\to\infty)\to 0$, obtaining the following differential equation which determines the asymptotic behavior of F with l:

$$r^2F_l''+2rF_l'-2F_l-\mu_l^2r^2F_l=0 , \quad (6.4a)$$

where

$$\mu_l^2=m_\pi^2+\frac{1}{a[F_l]^2}\left[1-\frac{8\pi}{3e_s^2}\frac{1}{a[F_l]}\int_0^\infty dr\, s^4 -\frac{2l(l+1)}{3}\right] . \quad (6.4b)$$

Since

$$\mu_{l=0}^2(m_\pi^2\to 0)=\frac{1}{a[F_0]^3}\frac{8\pi}{3}\int_0^\infty dr\, r^2s^2 \times\left[f_\pi^2+\frac{1}{e_s^2}F_0'^2\right] , \quad (6.5a)$$

we see that, even for $m_\pi=0$, the solution exists which behaves asymptotically like $F_0(r)\sim\exp(-\mu_0 r)/r$. Further we have

$$\mu_{l=1/2}^2(m_\pi\to 0)=\frac{1}{a[F_{1/2}]^3}\frac{4\pi}{3f_\pi e_s^3}J[F_{1/2}] \quad (6.5b)$$

with

$$J[F]\equiv\int_0^\infty dz\, z^2\bar{s}^2(1+\bar{F}'^2-\bar{s}^2/z^2) ,$$
$$z=f_\pi e_s r,\ \ \bar{F}(z)=F(r),\ \ \bar{s}=\sin\bar{F} . \quad (6.5c)$$

Therefore, as already pointed out in another paper,[16] the new mass term may play an important role in stabilizing the rotating chiral soliton. For a reasonable trial form of $F_{1/2}(r)$, one can confirm that $J[F_{1/2}]$ is positive.

We have shown that $H(q,p)$, (5.4), can be regarded as Hamiltonian in the sense that it leads to Hamilton equations. While, even when the starting Lagrangian is the same classically, different Hamiltonians including a term proportional to the curvature are proposed.[4,17] This problem will be discussed elsewhere.

In order to examine the static properties of the nucleon, we have to introduce the electromagnetic and the weak current as well as the baryon-number current, and to express them in terms of R_B's and $D^{ad}(A)_{BD}$'s, where $D^{ad}(A)_{BD}$ is defined by

$$D^{ad}(A)_{BD}={\rm Tr}(A^\dagger\tau_BA\tau_D)/2 . \quad (6.6)$$

There exist some problems: e.g., ambiguity of current forms, the conservation law of the electromagnetic current, and the existence of the hedgehog configuration $\sigma(\mathbf{x})_{\rm hed}$.

The extension of the present formalism to other cases, especially to the SU(3) case, is worthy of examination. Some trouble may arise due to the Wess-Zumino term.[18] In separate papers, we will investigate these problems.

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APPENDIX A: NONAPPEARANCE OF $(w^B)^1$- AND $(\dot{q}^a)^1$-ORDER TERMS IN $L(U_{L\rho};\mathbf{x},t)$

Employing X_{BE} and $\xi_k{}^B$ defined by (2.10c) and (2.13c), we rewrite $L(U_{L4};\mathbf{x},t)$, (2.24), as

$$L(U_{L4};\mathbf{x},t)=(Y^{(1)}_{BE,KJ}+Y^{(2)}_{BE,KJ}){\rm Tr}(Aw^Bw^E\tau_K\tau_JA^\dagger) , \quad (A1)$$

where

$$Y^{(1)}_{BE,KJ}\equiv\frac{f_\pi^2}{4}X_{BK}X_{EJ} , \quad (A2)$$

$$Y^{(2)}_{BE,KJ}\equiv\frac{1}{16e_s^2}\xi_k{}^F\xi_k{}^HX_{BD}X_{EG}\epsilon_{FDK}\epsilon_{HGJ} . \quad (A3)$$

Substituting $w^BA-f^{BD}A\tau_D/2$ and $A^\dagger w^B-f^{BD}\tau_DA^\dagger/2$ for Aw^B and $w^BA^\dagger$ in (A2), we obtain

$$\begin{aligned}{\rm Tr}(Aw^B\tau_K\tau_Jw^EA^\dagger) &=2w^Bw^E\delta_{KJ}-i(f^{BM}w^E+w^Bf^{EM})\epsilon_{MKJ}\\ &\quad+\tfrac{1}{2}f^{BM}f^{EN}(\delta_{MK}\delta_{JN}-\delta_{MJ}\delta_{KN}+\delta_{MN}\delta_{KJ}) ,\end{aligned} \quad (A4)$$

from which we easily see that the $(w^B)^2$-order term in $L(U_{L4};\mathbf{x},t)$ is given by (2.13a) with (2.13b).

The $(w^B)^1$-order term in $L(U_{L4};\mathbf{x},t)$ is

$$(Y^{(1)}_{BE,KJ}+Y^{(2)}_{BE,KJ})(-i)(f^{BM}w^E+w^Bf^{EM})\epsilon_{MKJ}=\frac{1}{2i}\left[\sum_{i=1}^{2}Y^{(i)}_{BE,KJ}\right]\{[w^B,f^{EM}]-[w^E,f^{BM}]\}\epsilon_{MKJ}$$
$$=-\frac{1}{2}\left[\sum_{i=1}^{2}Y^{(i)}_{BE,KJ}\right]f^{bd}(C_b{}^B\partial_d f^{EM}-C_b{}^E\partial_d f^{BM})\epsilon_{MKJ}\ . \tag{A5}$$

Here we have utilized the symmetry property

$$Y^{(i)}_{BE,KJ}(\mathbf{x})=Y^{(i)}_{EB,JK}(\mathbf{x}),\quad i=1,2 \tag{A6}$$

and (2.12). Thus, we see that no $(w^B)^1$-order terms appear in $L(U_{L\rho};\mathbf{x},t)$. As will be seen in Appendix B, (A5) vanishes after that quantization condition is imposed.

Similarly, we can prove (2.16b) by utilizing the symmetry property

$$a(\sigma;\mathbf{x})_{BE}=a(\sigma;\mathbf{x})_{EB}\ . \tag{A7}$$

We have

$$\tfrac{1}{2}a(\sigma;\mathbf{x})_{BE}w^Bw^E=\tfrac{1}{2}a(\sigma;\mathbf{x})_{BE}\left[\dot q^{\,b}C_b{}^BC_d{}^E\dot q^{\,d}+\tfrac{1}{4}f^{be}f^{dg}(\partial_eC_b{}^B)(\partial_gC_d{}^E)+\frac{i}{2}[-\dot q^{\,b}C_b{}^Bf^{de}\partial_eC_d{}^E+f^{be}(\partial_eC_b{}^B)C_d{}^E\dot q^{\,d}]\right];$$

$(\dot q^{\,b})^1$-order terms

$$a(\sigma;\mathbf{x})_{BE}[\dot q^{\,b},C_b{}^Bf^{de}\partial_eC_d{}^E]\ , \tag{A8}$$

reducing to a $(\dot q^{\,b})^0$-order term.

APPENDIX B: FORMS OF MASS TERMS IN $L_0(U_{L\rho})$

First we give the "classical" mass $M_c(\sigma)$ coming from

$$L(U_{Lk};\mathbf{x},t)\equiv\frac{f_\pi{}^2}{4}\mathrm{Tr}(U_{Lk}U_{Lk})+\frac{1}{32e_s{}^2}\mathrm{Tr}([U_{Lj},U_{Lk}]^2)+\tfrac{1}{4}m_\pi{}^2f_\pi{}^2\mathrm{Tr}(U+U^\dagger-2)\ . \tag{B1}$$

It is easy to derive

$$L(U_{Lk};\mathbf{x},t)=-\frac{f_\pi{}^2}{8}\xi_k{}^B\xi_k{}^B-\frac{1}{64e_s{}^2}[(\xi_j{}^B\xi_j{}^B)^2-\xi_j{}^B\xi_k{}^B\xi_j{}^D\xi_k{}^D]\ , \tag{B2}$$

$$M_c(\sigma)\equiv-\int d^3x\,L(U_{Lk};\mathbf{x},t)\ . \tag{B3}$$

If we define $\xi^{(h)}(x)_k{}^B$ as $\xi_k{}^B$ for the hedgehog configuration $\sigma(\mathbf{x})_{\text{hed}}$, we have

$$\xi^{(h)}(x)_k{}^B=\frac{\sin(2F)}{r}\delta_{Bk}+\left[-\frac{\sin(2F)}{r}+2F'\right]\hat{\mathbf{x}}_B\hat{\mathbf{x}}_k+\frac{2s^2}{r}\epsilon_{kEB}\hat{\mathbf{x}}_E\ ; \tag{B4}$$

$$\xi^{(h)}(x)_j{}^B\xi^{(h)}(x)_k{}^B=4\left[\frac{s^2}{r^2}\delta_{jk}+\left[\frac{-s^2}{r^2}+F'^2\right]\hat{\mathbf{x}}_j\hat{\mathbf{x}}_k\right]\ . \tag{B5}$$

It is easy to confirm

$$M_c(\sigma_{\text{hed}})=M_c[F]\quad[\text{Eq. (1.3d)}] \tag{B6}$$

to hold.

Next we consider the new contribution $\Delta M(\sigma)$ from $L(U_{L4};\mathbf{x},t)$, (2.24), to the mass term. As shown in Sec. II, we can take

$$f^{ab}=g^{ab}=\frac{1}{a(\sigma)}C^a{}_BC^b{}_B\quad[\text{Eq.(2.20c)}] \tag{B7}$$

after imposing the quantization condition (2.18a). Then, from (2.7b) one gets

$$f^{BE}=\frac{1}{a(\sigma)}\delta_{BE}\ . \tag{B8}$$

We see, therefore, that (A5) vanishes and that the last term on the RHS of (A4) is equal to

$$\frac{1}{2a(\sigma)^2}(\delta_{BK}\delta_{EJ}-\delta_{BJ}\delta_{EK}+\delta_{BE}\delta_{KJ})\ , \tag{B9}$$

which leads to

$$\Delta M(\sigma)=\frac{-1}{2a(\sigma)^2}\int d^3x[(Y^{(1)}_{BE,BE}-Y^{(1)}_{BE,EB}+Y^{(1)}_{BB,EE})+(Y^{(1)}_{\ldots,\ldots}\to Y^{(2)}_{\ldots,\ldots})]\ . \tag{B10}$$

As already given in Ref. 16, $\Delta M[F]$ for the hedgehog configuration becomes equal to (2.25).

APPENDIX C: PROPERTIES OF X_{BD}, X'_{BD}, AND $\xi_k{}^B$

(i) From the definitions (2.10c) and (2.10d), we obtain

$$X_{BD}=\tfrac{1}{4}\mathrm{Tr}[(\tau_B-\sigma\tau_B\sigma^\dagger)\tau_D]=X'_{DB} \ . \quad \text{(C1)}$$

We can prove

$$X_{BD}X_{ED}=X'_{BD}X'_{ED}=X_{DB}X_{DE} \ , \quad \text{(C2)}$$

by making use of (C1) and

$$X'_{BD}=-\tfrac{1}{2}X_{BE}\mathrm{Tr}(\sigma^\dagger\tau_E\sigma\tau_D) \ . \quad \text{(C3)}$$

We can also derive

$$X_{BD}X_{ED}=\tfrac{1}{2}(X_{BE}+X'_{BE})=X^{(s)}_{BE} \ , \quad \text{(C4)}$$

where $X^{(s)}_{BD}$ is the symmetric part of X_{BD}; $X^{(s)}_{BD}=(X_{BD}+X_{DB})/2$. From (C3) we have

$$\begin{aligned}X'_{BD}&=-\tfrac{1}{2}X_{BE}\mathrm{Tr}[\tau_E(\tau_D-2X_{DF}\tau_F)]\\&=-X_{BD}+2X_{BE}X_{DE} \ , \end{aligned}\quad \text{(C5)}$$

leading to (C4). Of course, the above equality is satisfied by $X^{(h)}_{BD}$ ($=X_{BD}$ for the hedgehog σ_{hed});

$$X^{(h)}_{BD}=s^2(\sigma_{BD}-\hat{\mathbf{x}}_B\hat{\mathbf{x}}_D)+sc\hat{\mathbf{x}}_E\epsilon_{EBD} \ . \quad \text{(C6)}$$

(ii) Next we consider properties of $\xi_k{}^B$ defined by (2.13c) and ξ'^B_k, defined by

$$\sigma^\dagger(\mathbf{x})\frac{\partial\sigma(\mathbf{x})}{\partial x^k}\equiv\frac{i}{2}\tau_B\xi'(\mathbf{x})_k{}^B \ . \quad \text{(C7)}$$

We may employ, if necessary,

$$\sigma=\pi_0+i\pi_B\tau_B$$

with

$$(\pi_0)^2+\pi_B\pi_B=1 \ . \quad \text{(C8)}$$

It is easy to confirm

$$\xi_k{}^B\xi_j{}^B=\xi'^B_k\xi'^B_j \ . \quad \text{(C9)}$$

Furthermore, we have

$$\begin{aligned}\xi_k{}^EX_{BE}&=\tfrac{1}{2}(\xi_k{}^B-\xi'^B_k)\\&=-\xi'^E_kX_{EB} \ . \end{aligned}\quad \text{(C10)}$$

This is because

$$\begin{aligned}\xi_k{}^B&=\frac{1}{i}\mathrm{Tr}(\partial_k\sigma\cdot\sigma^\dagger\tau_B)\\&=\tfrac{1}{2}\mathrm{Tr}(\sigma\tau_E\sigma^\dagger\tau_B)\xi'^E_k=-2X_{EB}\xi'^E_k+\xi'^B_k \ . \end{aligned}\quad \text{(C11)}$$

It is needless to mention that the hedgehog $\xi_k{}^{(h)B}$, (B4), satisfies the above equations.

We add some formulas for a later convenience:

$$\xi'^B_k\xi'^D_kX'_{BD}=-\xi'^B_k\xi_k{}^DX_{BD}=\xi_k{}^B\xi_k{}^DX_{DB} \ , \quad \text{(C12)}$$

$$\begin{aligned}\xi'^B_k\xi'^E_kX'_{BD}X'_{ED}&=\xi'^B_k\xi'^E_kX'_{DB}X'_{DE}\\&=\xi_k{}^B\xi_k{}^EX_{BD}X_{ED} \ , \end{aligned}\quad \text{(C13)}$$

$$\xi'^F_kX'_{BD}\epsilon_{FBD}=-\xi_k{}^FX_{BD}\epsilon_{FBD} \ . \quad \text{(C14)}$$

The last formula is easily proved when (C8) is utilized.

APPENDIX D: PROOF OF $L(U_{L4};\mathbf{x},t)=L(U_{R4};\mathbf{x},t)$

We define

$$L(U_{R4};\mathbf{x},t)\equiv\frac{f_\pi{}^2}{4}\mathrm{Tr}[U_{R4},U_{R4}]+\frac{1}{16e_s{}^2}\mathrm{Tr}[[U_{Rk},U_{R4}]^2] \ . \quad \text{(D1)}$$

Then $a(\sigma,\mathbf{x})'_{BE}$, which is the coefficient of $w^Bw^E/2$ in $L(U_{R4};\mathbf{x},t)$, is written, corresponding to (2.13b), as

$$a(\sigma,\mathbf{x})'_{BE}=f_\pi{}^2X'_{BD}X'_{ED}+\frac{1}{4e_s{}^2}\xi'^F_k\xi_k{}'^HX'_{BD}X'_{EG}\epsilon_{FDK}\epsilon_{HGK} \ . \quad \text{(D2)}$$

Because the last term is rewritten as

$$\frac{1}{4e_s{}^2}(\xi'^F_k\xi'^F_kX'_{BD}X'_{ED}-\xi'^D_kX'_{BD}\xi'^F_kX'_{EF})=\frac{1}{4e_s{}^2}(\xi_k{}^F\xi_k{}^FX_{BD}X_{ED}-\xi_k{}^DX_{BD}\xi_k{}^FX_{EF}) \ ,$$

we easily derive

$$a(\sigma;\mathbf{x})'_{BE}=a(\sigma;\mathbf{x})_{BE} \ . \quad \text{(D3)}$$

Similarly, the contribution to the mass term from $L(U_{R4};\mathbf{x},t)$, $\Delta M(\sigma)'$, is proved to be equal to $\Delta M(\sigma)$. $\Delta M(\sigma)'$ is written as

$$\Delta M(\sigma)'=\frac{-1}{2a(\sigma)^2}\int d^3x[(Y^{(1)'}_{BE,BE}-Y^{(1)'}_{BE,EB}+Y^{(1)'}_{BB,EE})+(Y^{(1)'}_{\dots,\dots}\to Y^{(2)'}_{\dots,\dots})] \ , \quad \text{(D4)}$$

where

$$Y^{(1)'}_{BE,KL}\equiv\frac{f_\pi{}^2}{4}X'_{BK}X'_{EL} \ , \quad \text{(D5)}$$

$$Y^{(2)'}_{BE,KL}\equiv\frac{1}{16e_s{}^2}\xi'^F_k\xi'^H_kX'_{BD}X'_{EG}\epsilon_{FDK}\epsilon_{HGL} \ . \quad \text{(D6)}$$

Because of (C1) and (C2), the $Y^{(1)\prime}$ contribution is equal to $Y^{(1)}$ contribution to $\Delta M(\sigma)$. The same situation holds also for the $Y^{(2)\prime}$ contribution, because

$$\begin{aligned}&(Y^{(2)'}_{BE,BE}-Y^{(2)'}_{BE,EB}+Y^{(2)'}_{BB,EE})16e_s{}^2\\&=\xi'^F_k\xi'^H_kX'_{BD}X'_{EG}(\epsilon_{FDB}\epsilon_{HGE}-\epsilon_{FDE}\epsilon_{HGB})\\&\quad+(\xi'^F_k\xi'^F_kX'_{BD}X'_{BD}-\xi'^D_kX'_{BD}\xi'^F_kX'_{BF}) \ ; \end{aligned}\quad \text{(D7)}$$

all terms are equal to the corresponding ones without primes.

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$$U_{L\rho}U_{L\rho}=\sum_{k=1}^{3}U_{Lk}U_{Lk}+U_{L4}U_{L4}=U_{Lk}U_{Lk}-U_{L0}U_{L0}\ .$$

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