Spinor loop anomalies with very general local fermion Lagrangians

Jooha Minn, Jewan Kim, and Choonkyu Lee
Department of Physics, Seoul National University, Seoul, 151, Korea
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Structures of gauge and gravitational anomalies are analyzed assuming arbitrary local, possibly derivative-type, couplings between spin-$\frac{1}{2}$ fields and general Bose fields in the theory. In $2n$-dimensional spacetime we show that inclusion of such general local couplings does not lead to any new spinor loop anomaly, besides the ones already identified keeping only minimal gauge and gravitational couplings to spin-$\frac{1}{2}$ fields. In addition to the proof based on regularization, a topological understanding of the results is provided as well. Also discussed are physical implications of our findings.

I. INTRODUCTION

During the last two decades gauge field theories have emerged as the natural framework for the description of elementary-particle dynamics. Indeed, from the standard $SU(3)\times SU(2)\times U(1)$ model of strong and electroweak interactions to a majority of attempts which are directed to go beyond it, the gauge-theory framework has been used. Besides, gravitational interactions seem to also be based on the gauge principles of general coordinate invariance and local Lorentz invariance. In the development of gauge theories, chiral anomalies[1] associated with quantized spinor matter fields have played some important roles ever since their discovery. This is because global or gauge symmetries of the classical action may not be maintained in the full quantum theory when anomalies are present, and hence their presence can lead to significant physical consequences.[2–4] The chiral-anomaly problem is not restricted to four spacetime dimensions. With a revived interest in the Kaluza-Klein idea recently, people have analyzed the structure of gauge and gravitational anomalies also in general $2n$-dimensional spacetime.[5] As another important development, we should here mention recent works by Zumino and others;[6–10] they have made it possible to understand the expression for spinor anomalies using topological arguments and the index theorems[11,12] well known in mathematics.

In the flat four-dimensional spacetime the full general expression for Abelian or non-Abelian anomalies was first obtained by Bardeen,[13] through a direct Feynman-diagram analysis of the one-loop fermion effective action in the presence of arbitrary external gauge, scalar, and pseudoscalar fields. He found that all apparent anomalies involving the scalar and pseudoscalar fields could be removed by appropriate local counterterms and therefore should be regarded just as artifacts of the particular computational (or regularization) method to evaluate the diagrams. Recently Bardeen and Deo[14] showed, again by direct Feynman-diagram analysis, that there exists no genuine anomaly involving tensor fields either. Then without resorting to any detailed perturbative calculation, two of us[15] have been able to generalize this result further: of all possible *nonderivative* couplings of the spinor field with arbitrary external Bose fields considered in $2n$-dimensional spacetime, only the pure gauge field anomalies amount to the true obstruction to gauge invariance and no genuine anomaly involving non-gauge-type external fields exists.

In the present paper we shall expand and generalize the work of Ref. 15 in several directions. First, we do away with the nonderivative coupling restriction—we consider all possible spinor loop anomalies, assuming arbitrary local couplings between the quantized spinor fields and general external Bose fields. Couplings may be of a nonrenormalizable type and interaction terms involving (covariant) derivatives, which act on the spinor fields, are also allowed. We establish that none of these additional interactions bring in new anomalies; viz., the anomaly determined keeping only *minimal* gauge couplings (with the given spinor fields) is all. Also discussed in the present paper are possible spinor loop anomalies when such general spinor field theories are considered in curved spacetime (with or without torsion). We again establish analogous statements concerning the effects of additional local couplings on the gauge and gravitational anomaly[16] structure of the theory. To reach these conclusions, a certain kind of regularization can be used judiciously as in Ref. 15. After presenting this method, an alternative proof along the line of Ref. 16 (together with some topological considerations) will be given in addition.

This paper is organized as follows. In Sec. II we analyze spinor loop anomalies in the context of a general local fermion Lagrangian in flat spacetime, and prove our assertion by employing a suitable regularization. In Sec. III we consider possible spinor loop anomalies in curved spacetime in the same vein. In Sec. IV we present our alternative proof (for spinor field theories set in flat Euclidean spacetime). Our conclusion is also examined from the viewpoint that sees the gauge anomaly as the manifestation of a topological obstruction to defining a gauge-invariant effective action. Finally, we discuss our findings and elaborate on their physical implications in Sec. V.

II. ON SPINOR ANOMALIES IN FLAT SPACETIME

In flat spacetime, let us consider a theory consisting of a quantized spin-$\frac{1}{2}$ field (with arbitrary internal degrees of freedom) which has general local couplings to various external Bose fields and self-couplings. The Lagrangian density may be assumed to have the form (in $D=2n$ dimensions)

$$\mathscr{L}(x)=\bar{\psi}(x)[i\gamma^{\mu}D_{\mu}+f(D_{\mu},\Gamma(x))]\psi(x)+\mathscr{L}'(x)\ , \qquad (2.1)$$

where $(\psi,\bar{\psi})$ are dynamical freedoms of our theory and we shall take them to be $2^{D/2}$-component Dirac spinor fields. Here, $D_{\mu}\equiv\partial_{\mu}-iB_{\mu}(x)$ is the gauge-covariant derivative, $\Gamma(x)=\{\Gamma^{i}(x)\}$ denotes collectively various possible external Bose fields (no restriction on spins) other than B_{μ}, and $f(D_{\mu},\Gamma(x))$ is a differential operator of the general form

$$\begin{aligned}f(D_{\mu},\Gamma(x))=\ &a_0(\Gamma(x))+a_1(\Gamma(x))^{\mu}D_{\mu}\\ &+a_2(\Gamma(x))^{\mu\nu}D_{\mu}D_{\nu}+\cdots\\ &+a_r(\Gamma(x))^{\mu_1\cdots\mu_r}D_{\mu_1}D_{\mu_2}\cdots D_{\mu_r}\ . \end{aligned}\qquad (2.2)$$

In Eq. (2.2) coefficients $a_k(\Gamma(x))$'s can have polynomial dependences on $\Gamma(x)$ and will be matrix valued in the internal and Dirac spin spaces in general. Any local interaction term which cannot be represented by the first term in the right-hand side of Eq. (2.1) (e.g., four-Fermi interactions) will be included in the piece $\mathscr{L}'(x)$. Note that $\mathscr{L}'(x)$ can have dependences on external fields $B_{\mu}(x)$ and $\Gamma(x)$ and may involve covariant derivatives too. The gauge field $B_{\mu}(x)$ will have the structure

$$\begin{aligned}B_{\mu}(x)&=V_{\mu}(x)\times 1+A_{\mu}(x)\gamma_{D+1}\\ &\equiv B_{\mu}^{a}(x)T^{a}\quad\left[\gamma_{D+1}\equiv -i^{n+1}\prod_{\mu=0}^{D-1}\gamma^{\mu}\right]\ ,\end{aligned}\qquad (2.3)$$

where $V_{\mu}(x),A_{\mu}(x)$ are some Hermitian matrices purely in the internal symmetry space (and hence not involving Dirac-spin indices) and γ_{D+1} is the D-dimensional counterpart of the usual γ_5 in four spacetime dimensions. Gauge group generators T^{a}, defined through Eq. (2.3), may involve γ_{D+1} (while B_{μ}^{a} do not).

We want to study the quantum failure of gauge invariance in the theory given above. Our Lagrangian (2.1), as a c-number Lagrangian, will here be assumed to possess "gauge invariance" in the sense that it remains unchanged if we make infinitesimal field transformations

$$\begin{aligned}&\delta_{\Lambda}\psi(x)=i\Lambda(x)\psi(x)\ ,\\ &\delta_{\Lambda}\bar{\psi}(x)=i\bar{\psi}(x)\gamma^{0}\Lambda(x)\gamma^{0}\end{aligned}\qquad (2.4)$$

with gauge function $\Lambda(x)=\Lambda^{a}(x)T^{a}$, and simultaneously consider external field variations $[\delta_{\Lambda}B_{\mu}^{a}(x),\delta_{\Lambda}\Gamma^{i}(x)]$ in accordance with

$$\begin{aligned}&\delta_{\Lambda}B_{\mu}(x)=\partial_{\mu}\Lambda(x)+i[\Lambda(x),B_{\mu}(x)]\equiv D_{\mu}\Lambda(x)\ ,\\ &\delta_{\Lambda}[\gamma^{0}a_k(\Gamma(x))^{\mu_1\cdots\mu_k}]\\ &\qquad =i[\Lambda(x),\gamma^{0}a_k(\Gamma(x))^{\mu_1\cdots\mu_k}]\ \ (k=0,1,2,\ldots,r)\ .\end{aligned}\qquad (2.5)$$

Fields belonging to $\Gamma(x)$ will transform gauge covariantly (or gauge invariantly, in special cases) since $\Gamma(x)$ consists of "non-gauge-type" external Bose fields only. A (gauge-covariant) field strength constructed out of B_{μ} may well be included in $\Gamma(x)$; it can be regarded just as another external tensor field in the theory. Also note that we have written our Lagrangian using Dirac spinors (rather than Weyl spinors). This is no loss of generality since any Weyl spinor may be paired without altering the physics content of the theory with a noninteracting (i.e., free) opposite-chirality Weyl spinor to form a Dirac spinor. [Note that Majorana-mass-term-like couplings may be incorporated as a part of $\mathscr{L}'(x)$ in the Lagrangian (2.1).]

With the general form (2.1) as the spinor-field-dependent part of the full Lagrangian, the theory will be nonrenormalizable except for very special cases. For renormalizable theories involving Dirac spinors in $D=4$, for instance, we may set $\mathscr{L}'(x)=0$ and assume the structure

$$\begin{aligned}&a_0(\Gamma(x))=m_0+S(x)+i\gamma_5P(x)\ ,\\ &a_k(\Gamma(x))=0\ \ \text{for } k\geq 1\ ,\end{aligned}\qquad (2.6)$$

with matrices $S(x)$ and $P(x)$ describing scalar and pseudoscalar couplings, respectively. But, as emphasized by various authors[5] who studied chiral anomalies in higher spacetime dimensions, renormalizability of a theory is really a separate issue from the *gauge consistency* of a given theory. Even in $D=4$, it is a perfectly legitimate question to talk of possible chiral anomalies and gauge consistency with *effective* field theories which need not be renormalizable. In $D=4$ the simplest possible nonrenormalizable interaction will have the form of antisymmetric tensor couplings,[14] i.e.,

$$\bar{\psi}(x)\sigma_{\mu\nu}f^{\mu\nu}(x)\psi(x)$$

with tensor field matrix $f^{\mu\nu}(x)=-f^{\nu\mu}(x)$. (Note that in $D=4$ we have the identity $\gamma_5\sigma^{\mu\nu}=\frac{1}{2}i\epsilon^{\mu\nu\lambda\delta}\sigma_{\lambda\delta}$.) Beyond that, one can imagine derivative couplings [i.e., $a_k(\Gamma(x))\neq 0$ for some $k\geq 1$ in our notation], four-Fermi-type interactions, etc. We want to know whether these nonrenormalizable interactions can lead to new chiral anomalies. We believe that anomalies and the gauge consistency of a theory are more fundamental issues than smooth ultraviolet behaviors of the theory.[17] It is with this philosophy that we have kept totally general local couplings between external Bose fields (of arbitrary integer spin) and spin-$\frac{1}{2}$ fields in our Lagrangian (2.1), putting aside the renormalizability issue. We have taken all Bose fields of the theory to be external here since quantum nature of those variables plays only a secondary role in studying spinor anomalies.[18] (See also the second paragraph below.)

In the presence of external fields $B_{\mu}(x)$ and $\Gamma(x)$, we may define the effective action functional $W[B_{\mu},\Gamma]$ formally by the path integral[19]

$$e^{iW[B_{\mu},\Gamma]}=\int[\mathscr{D}\psi][\mathscr{D}\bar{\psi}]\exp\left[i\int d^{D}x\ \mathscr{L}(x)\right]\ . \qquad (2.7)$$

When expanded as a power series in B_{μ} and Γ, $W(B_{\mu},\Gamma)$ will play the role of a generating functional for (unrenormalized) connected spinor loop Feynman diagrams with any given number of external B_{μ} and/or Γ legs. When the Lagrangian includes only terms bilinear in spinor fields, $W[B_\mu,\Gamma]$ may be identified with the spinor one-loop effective action (see Fig. 1). But, if we allow interaction vertices involving more than two spinor fields through the piece $\mathscr{L}'(x)$, *multiloop* contributions to $W[B_\mu,\Gamma]$ should be also considered and some typical Feynman diagrams in that case are shown in Fig. 2. Naively, gauge invariance of our Lagrangian $\mathscr{L}(x)$ would imply

$$\delta_\Lambda W[B_\mu,\Gamma]=0 \tag{2.8}$$

for infinitesimal external field variations $(\delta_\Lambda B_\mu^a(x),\delta_\Lambda \Gamma^i(x))$ represented by Eq. (2.5). But, for our general Lagrangian (2.1), the functional $W[B_\mu,\Gamma]$ is ill defined without being properly regularized and the relation (2.8), inferred at purely formal level, may not bear the truth. Representing the properly regularized effective action functional by $W_R[B_\mu,\Gamma]$, we may here define the anomaly $\mathscr{A}_R[B_\mu,\Gamma;\Lambda]$ simply as

$$\delta_\Lambda W_R[B_\mu,\Gamma]\equiv\int d^Dx\left[(D_\mu\Lambda)^a(x)\frac{\delta W_R[B_\mu,\Gamma]}{\delta B_\mu^a(x)}+\delta_\Lambda\Gamma^i_{(x)}\frac{\delta W_R[B_\mu,\Gamma]}{\delta\Gamma^i(x)}\right]=\mathscr{A}_R[B_\mu,\Gamma;\Lambda]\ , \tag{2.9}$$

viz., the anomaly represents the obstruction to gauge invariance of the regularized effective action.

In the full quantum theory where B_μ and Γ are also dynamical degrees of freedom, we will have to integrate the expression (2.7) further over field configurations of $B_\mu(x)$ and $\Gamma(x)$. (Note that, as far as integration over Fermi field variables is concerned, fields B_μ and Γ may be regarded as "external" even in this case.) Here, without gauge invariance for the spinor effective action $W_R[B_\mu,\Gamma]$, we cannot secure that only transverse parts of gauge fields B_μ (together with Γ) make physical degrees of freedom. If $W_R[B_\mu,\Gamma]$ has an anomaly under gauge transformations, the unitarity of the theory will be thus in jeopardy; i.e., the theory becomes inconsistent. We may also use fields B_μ and Γ for a different purpose, i.e., to study possible anomalies in conservation laws[2] and in current algebra.[4] Let us describe this application briefly. Among the fields represented by B_μ and Γ, we may regard a part of them as nondynamical external fields (or spectator fields) in the full quantum theory context and use them just as external sources for various kind of "currents" (involving spinor fields). Here, especially important will be currents directly related to exact (or softly broken) global symmetries of the dynamical part of the Lagrangian; for them we would have conservation laws. Matrix elements involving those currents can be related to functional derivatives of the effective action with respect to appropriate spectator fields (e.g., $\{\delta W_R[B_\mu,\Gamma]/\delta B_\mu^a(x)\}$) and, as results of global symmetries in the Lagrangian, they are subject to so-called Ward-Takahashi identities. Here again, naive conservation laws and naively inferred Ward-Takahashi relations (for correlation functions involving a multiple number of currents, especially) may break down due to chiral anomalies. The correct relations incorporating spinor anomaly effects can be easily found with the help of the anomaly equation (2.9) (or its integrated form in the sense of Ref. 4), taking $\Lambda(x)$ to be a local generalization of phase parameters connected to global symmetries in question. Note that conservation law anomalies and current-algebra anomalies can be extracted from the same equation determining the gauge consistency of a theory, Eq. (2.9); the anomaly equation has dual applications. In this paper B_μ will be referred to as "gauge fields" regardless of its eventual role in full theory.

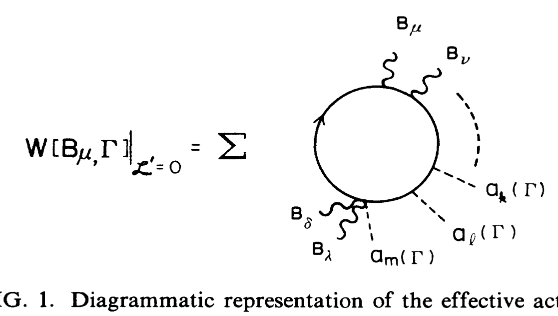


FIG. 1. Diagrammatic representation of the effective action with $\mathscr{L}'(x)=0$. Solid lines represent usual free-fermion propagators.

FIG. 2. Examples of multiloop contributions to $W[B_\mu,\Gamma]$ with $\mathscr{L}'(x)\neq 0$.

Our goal is to identify the full expression for $\mathscr{A}_R[B_\mu,\Gamma;\Lambda]$, assuming the general spinor Lagrangian (2.1). Only the expression in the limit of sending away regularization is of interest, and any regularization preserving the locality of the theory and Poincaré symmetry may be used for calculation. According to our definition of the anomaly, it will in general depend on the regularization procedure chosen. However, once $\mathscr{A}_R[B_\mu,\Gamma;\Lambda]$ has been identified with the help of any particular (well-defined) regularization, different choices of regularization can be accounted for by allowing arbitrary *local counterterms* to the regularized effective action $W_R[B_\mu,\Gamma]$. Indeed in Ref. 13 Bardeen first used a particular symmetric point-splitting method to compute anomalies in the presence of general renormalizable couplings in four dimensions [i.e., for the case (2.6), with $\Gamma(x)=(S(x),P(x))$]. The resulting anomaly expression involved not only gauge fields B_μ but also scalar fields $S(x)$ and pseudoscalar fields $P(x)$. But Bardeen was able to eliminate all $S(x)$ and $P(x)$ dependences from his anomaly expression by exploiting fully this freedom of adding local counterterms to the effective action; *the* *genuine anomaly involves only gauge fields B_μ*, and $S(x)$- or $P(x)$-dependent terms in the anomaly were merely artifacts of the regularization chosen. [By the same strategy it has recently been established[14] that including additional tensor couplings of the general form $\bar{\psi}(x)\sigma_{\mu\nu}f^{\mu\nu}(x)\psi(x)$ to the analysis of Ref. 13 leads to no new (genuine) anomaly either.] Of course it is only the final reduced expression—here, involving gauge fields B_μ only—for the anomaly which is relevant physically and for topological considerations in gauge theories with fermions.

For our general spinor Lagrangian it will be impractical to find anomalies following Bardeen's procedure. It is not so much because we are working in general $D=2n$ dimensions, but more due to the fact that (by allowing arbitrary derivative couplings and also the $\mathscr{L}'$ term) we here have to deal with an *infinite* number of ill-defined Feynman diagrams which can be potential sources for anomalies. In case there exist an infinite number of "anomalous" Feynman diagrams in a particular regularization scheme chosen, $\mathscr{A}_R[B_\mu,\Gamma;\Lambda]$ will be given by the spacetime integral of a certain infinite-order local polynomial in $B_\mu(x)$ and $\Gamma(x)$. Even with such an expression for $\mathscr{A}_R[B_\mu,\Gamma;\Lambda]$ at hand, there waits another formidable task of sorting out genuine anomalies from it by manipulating with general local counterterms. We thus reverse the game—we ask whether a special (intermediate) regularization procedure exists which makes the anomaly calculation simplest and keeps superfluous regularization-dependent anomalies to a minimum. By this strategy it is possible to arrive at the complete solution of the problem without any detailed Feynman-diagram analysis.

Employing a simple regularization scheme and perturbation theory, we shall here prove that it is possible to set

$$\mathscr{A}_R[B_\mu,\Gamma;\Lambda]=\mathscr{A}_{\min}[B_\mu;\Lambda]\ , \tag{2.10}$$

where $\mathscr{A}_{\min}[B_\mu;\Lambda]$ represents the spinor anomaly expression in the theory with minimal gauge couplings only, i.e., based on the Lagrangian

$$\mathscr{L}_{\min}(x)=\bar{\psi}(x)i\gamma^\mu D_\mu\psi(x)\quad [D_\mu=\partial_\mu-iB_\mu(x)]\ . \tag{2.11}$$

An immediate corollary of Eq (2.10) is that non-gauge-type (or gauge-covariantly transforming) fields $\Gamma(x)$ simply play no role as for *genuine* anomalies, irrespectively of the way they interact with spinor fields (as long as couplings are local). If anomalies involving, say, external scalar or tensor fields are found using a particular computational or regularization scheme, they must be all superfluous; i.e., they can be eliminated by adding suitable local counterterms to the effective action. With the expression $\mathscr{A}_{\min}[B_\mu;\Lambda]$ above, there are still ambiguities related to the addition of local counterterms to the (regularized) effective action for the "minimal" theory (2.11). To establish Eq. (2.10), it suffices to consider the following partially regularized theory:

$$\mathscr{L}_{\bar{R}}(x)=\bar{\psi}(x)\left\{i\gamma^\mu D_\mu\left[1+\left[-\frac{1}{M^2}D^2\right]^q\right]+f(D_\mu,\Gamma(x))\right\}\psi(x)+\mathscr{L}'(x)\ , \tag{2.12}$$

where q is some positive integer, $D^2=D_\mu D^\mu$, and M is a large parameter with the dimension of mass (which we may eventually let approach infinity). [This kind of regularization has been used in our earlier work[15] and also in Sec. II of Ref. 16, to deal with theories which may be recognized as special cases of our general Lagrangian (2.1).] The corresponding effective action functional $W_{\bar{R}}[B_\mu,\Gamma]$, specified by Eq. (2.7) using the Lagrangian $\mathscr{L}_{\bar{R}}(x)$, will be less ill defined than $W[B_\mu,\Gamma]$ with a suitable choice of q (but, as we shall see below, not yet completely well defined).

We may study the effective action $W_{\bar{R}}[B_\mu,\Gamma]$ using perturbation theory. In choosing the unperturbed Lagrangian, an important point to consider is whether or not resulting Feynman diagrams would have a well-defined meaning, i.e., whether they would correspond to convergent integrals. With that in mind, it should be desirable to include the highest dimensional term[20] of the Lagrangian (2.12) in the unperturbed Lagrangian. We shall here take the value of q sufficiently large so that the piece

$$\bar{\psi}(x)i\gamma^\mu\partial_\mu\left[-\frac{1}{M^2}\partial^2\right]^q\psi(x)$$

may be that highest dimensional term, and consider the expression

$$\bar{\psi}(x)\left\{i\gamma^\mu\partial_\mu\left[1+\left[-\frac{1}{M^2}\partial^2\right]^q\right]\right\}\psi(x) \tag{2.13}$$

as defining the unperturbed Lagrangian. All other terms in the Lagrangian (2.12) define interaction vertices.[21] Dividing the Lagrangian as such, we can obtain the Feynman diagram representation of the effective action in a straightforward manner; it will be a valid representation of $W_{\bar{R}}[B_\mu,\Gamma]$ when fields (B_μ,Γ), coefficients $a_k(\Gamma)$'s [see Eq. (2.2)], and couplings appearing in the $\mathscr{L}'$ term can be viewed relatively "small." In momentum space the free-fermion Feynman propagator based on the unperturbed Lagrangian (2.13) will read

$$iS_F(p)_R=\frac{M^{2q}}{(p^2)^q+M^{2q}}\frac{i\gamma^\mu p_\mu}{p^2+i\epsilon}\ . \tag{2.14}$$

Note that

$$S_F(p)_R=O\left[\left[\frac{1}{p^2}\right]^{q+1}\right]$$

for very large spacelike momentum p; but, for small p (compared to M), we recover the usual free-fermion propagator.

We may now divide the set of all Feynman diagrams contributing to the effective action $W_{\bar{R}}[B_\mu,\Gamma]$ into the following three distinct groups.

Type I. All one-loop spinor diagrams which are derivable entirely from the truncated theory based on the Lagrangian

$$\mathscr{L}_{\text{trunc}}(x)=\bar{\psi}(x)i\gamma^\mu D_\mu\left[1+\left[-\frac{1}{M^2}D^2\right]^q\right]\psi(x)\ . \tag{2.15}$$

Type II. One-loop spinor diagrams which do not be-long to type I, in the theory (2.12) with $\mathcal{L}'(x)$ set to zero; for any type-II Feynman diagram, there will be at least one vertex which is attached to $a_k(\Gamma)$'s.

Type III. All other (in general multiloop) Feynman diagrams possible in the theory (2.12); any type-III Feynman diagram will involve at least one $\mathcal{L}'$ vertex.

Based on this grouping, the effective action $W_{\bar{R}}[B_\mu,\Gamma]$ can be written as the sum

$$W_{\bar{R}}[B_\mu,\Gamma]=W_{\bar{R}}[B_\mu]_{\rm I}+W_{\bar{R}}[B_\mu,\Gamma]_{\rm II}+W_{\bar{R}}[B_\mu,\Gamma]_{\rm III}\ , \tag{2.16}$$

where $W_{\bar{R}}[B_\mu]_{\rm I}$ denotes the contribution due to all type-I Feynman diagrams, etc. (See Fig. 3 for diagrammatic illustrations.)

It should be noted that if we choose q sufficiently large, all type-II and type-III Feynman diagrams correspond to convergent integrals and hence $W_{\bar{R}}[B_\mu,\Gamma]_{\rm II}$ and $W_{\bar{R}}[B_\mu,\Gamma]_{\rm III}$ become well defined. The same cannot be said for type-I diagrams (i.e., for one-loop spinor diagrams with external B_μ legs only)—$W_{\bar{R}}[B_\mu]_{\rm I}$ remains ill defined, regardless of the value of q. This comes about as follows. Despite the fact that the fermion propagator in the theory (2.12) takes the form (2.14), no real gain in convergence results for type-I Feynman diagrams since we now have extra momentum factors at vertices which originate from $[-(1/M^2)D^2]^q$. In fact one sees that type-I Feynman diagrams with the total number of attached B_μ legs not larger than $D=2n$ are superficially divergent regardless of the value of q. In contrast, vertices attached to the $a_k(\Gamma)$ legs and also $\mathcal{L}'$ vertices are still just those of the unregularized theory (2.1); for these vertices, our regularization does not generate extra momentum factors which might upset the gain in convergence by having the regularized propagator (2.14). As a result any spinor loop

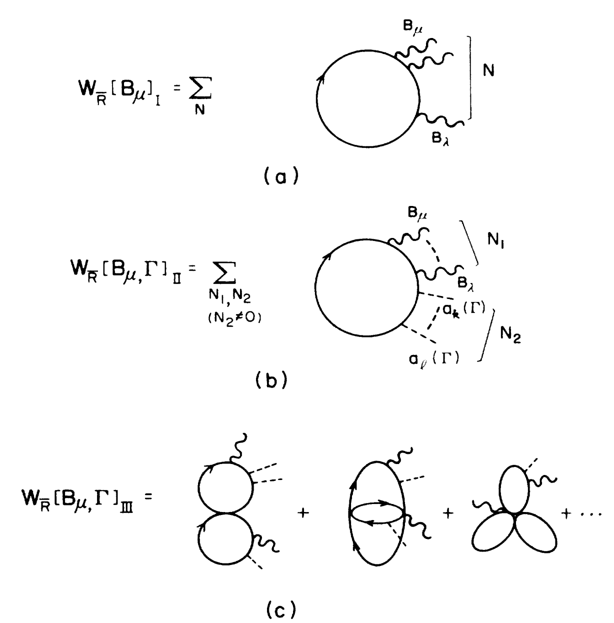


FIG. 3. Feynman-diagram representations for $W_{\bar{R}}[B_\mu]_{\rm I}$, $W_{\bar{R}}[B_\mu,\Gamma]_{\rm II}$, and $W_{\bar{R}}[B_\mu,\Gamma]_{\rm III}$.

diagram involving at least one $a_k(\Gamma)$ leg or $\mathcal{L}'$ vertex (but an arbitrary number of B_μ legs) can be made well defined in the theory (2.12) by choosing a sufficiently large q value (for a given spinor Lagrangian and for given D). Thus if we restrict our attention to that part of the effective action described by type-II and type-III diagrams (i.e., $W_{\bar{R}}[B_\mu,\Gamma]_{\rm II}+W_{\bar{R}}[B_\mu,\Gamma]_{\rm III}$), regularization we have introduced through the Lagrangian (2.12) is sufficient to make it well defined.

The Lagrangian of the form (2.12) can be considered for a general, γ_{D+1}-dependent, gauge field matrix $B_\mu(x)$. Furthermore, $\mathcal{L}_{\bar{R}}(x)$ possesses manifestly the gauge invariance of the original theory, and the separation we have made in Eq. (2.16) for the effective action is clearly a gauge-invariant one.[22] We may thus immediately conclude that under arbitrary, chiral or nonchiral, gauge transformations (2.5), we have

$$\delta_\Lambda(W_{\bar{R}}[B_\mu,\Gamma]_{\rm II}+W_{\bar{R}}[B_\mu,\Gamma]_{\rm III})=0\ ; \tag{2.17}$$

i.e., the functional $W_{\bar{R}}[B_\mu,\Gamma]_{\rm II}+W_{\bar{R}}[B_\mu,\Gamma]_{\rm III}$ is fully gauge invariant. Of course, no meaningful statement of this sort is possible for the ill-defined piece $W_{\bar{R}}[B_\mu]_{\rm I}$. As $M\rightarrow\infty$, all M-dependent pieces in the effective action $W_{\bar{R}}[B_\mu,\Gamma]$ are nothing more than local counterterms involving fields B_μ or Γ (or a pure constant term); viz., the locality of the theory is preserved in the limit $M\rightarrow\infty$. But our gauge-invariant Lagrangian $\mathcal{L}_{\bar{R}}(x)$ provides only a partial regularization of the theory; some additional regularization, which *may not be gauge invariant* but at least does no harm to the locality of the theory and Poincaré symmetry, should be brought in to make type-I Feynman diagrams well defined. We may here restrict the use of such additional regularization solely to the piece $W_{\bar{R}}[B_\mu]_{\rm I}$, and not touch $W_{\bar{R}}[B_\mu,\Gamma]_{\rm II}$ or $W_{\bar{R}}[B_\mu,\Gamma]_{\rm III}$. (This implies that this additional regularization is not introduced at the Lagrangian level.) Note that up to local counterterms, the truncated Lagrangian (2.15) (for $M\rightarrow\infty$) defines the same theory as the minimal theory Lagrangian (2.11).

The precise way how $W_{\bar{R}}[B_\mu]_{\rm I}$ is regularized is not important for us. All we are concerned with is that for the general spinor Lagrangian (2.1), we do have a well-defined regularized effective action of the form

$$W_R[B_\mu,\Gamma]=W_R[B_\mu]_{\rm I}+W_{\bar{R}}[B_\mu,\Gamma]_{\rm II}+W_{\bar{R}}[B_\mu,\Gamma]_{\rm III}\ , \tag{2.18}$$

where $W_R[B_\mu]_{\rm I}$ represents a suitably regularized effective action for the minimal theory (2.11). Note that depending on specific regularizations chosen, the functional $W_R[R_\mu]_{\rm I}$ may change by certain local counterterms involving gauge fields B_μ. Here, even taking fully such freedom into account, no gauge-invariant (and well-defined) functional for $W_R[B_\mu]_{\rm I}$ may actually exist, i.e., $\delta_\Lambda W_R[B_\mu]_{\rm I}\neq 0$. If we now set

$$\delta_\Lambda W_R[B_\mu]_{\rm I}=\mathcal{A}_{\rm min}[B_\mu;\Lambda] \tag{2.19}$$

and use Eqs. (2.18) and (2.17), we obtain

$$\delta_\Lambda W_R[B_\mu,\Gamma]=\delta_\Lambda W_R[B_\mu]_{\rm I}=\mathcal{A}_{\rm min}[B_\mu;\Lambda]\ . \tag{2.20}$$

This proves the content of Eq. (2.10). The full spinor anomaly in our general fermion field theory (2.1) is thus provided by the expression $\mathscr{A}_{\min}[B_\mu;\Lambda]$—the spinor anomaly for the corresponding minimal theory (2.11); various additional interactions present in the Lagrangian (2.1) do not introduce any new obstruction to defining the gauge-invariant spinor effective action.

The structure of $\mathscr{A}_{\min}[B_\mu;\Lambda]$ is well known. In $D=4$ a direct Feynman-diagram analysis gives the result[13,2]

$$\mathscr{A}_{\min}[B_\mu,\Lambda]=-\int d^4x\,\Lambda^a(x)\left[D_\mu^{ab}\frac{\delta W_R[B_\mu]_{\rm I}}{\delta B_\mu^b(x)}\right]$$
$$=\frac{1}{32\pi^2}\int d^4x\,\epsilon^{\mu\nu\lambda\delta}\,{\rm tr}\left[\gamma_5\Lambda(x)\partial^\mu\left[-\frac{2}{3}B^\nu\partial^\lambda B^\delta+\frac{i}{3}B^\nu B^\lambda B^\delta\right]\right], \quad (2.21)$$

up to terms which can be expressed as a gauge variation of a certain local functional of B_μ. The ambiguities mentioned here amount to adding certain local counterterms to the effective action $W_R[B_\mu]_{\rm I}$, and indeed that freedom should be exploited to obtain the usual *gauge-invariant* form[1,3] for the abelian [or $U_A(1)$-type] current anomaly on the basis of Eq. (2.21). [The situation is slightly different in the case of non-Abelian current anomalies.[9,10] See Eq. (2.24) and ensuing discussions.] To find the formula for $\mathscr{A}_{\min}[B_\mu,\Lambda]$ in general $D=2n$ dimensions, it is advantageous to take the differential geometric approach developed in Ref. 6 (and in the last of Ref. 5). Briefly, the construction goes as follows. First note that the minimal theory (2.11), if one wishes, may be always rewritten using only left-handed Weyl spinors (by considering conjugate spinors instead for right-handed ones), and hence we may well restrict our consideration to the case

$$B_\mu(x)=B_\mu^a(x)\lambda^a\frac{1-\gamma_{D+1}}{2}$$

here. Then in a space of $2n+2$ dimensions (i.e., two dimensions higher than the spacetime dimension), we introduce the matrix one-form (gauge connection) $B=-iB_\mu^a\lambda^a dx^\mu$, and the field strength two-form $F=dB+B\wedge B$. The $(n+1)$th Chern character will be proportional to ${\rm tr}F^{n+1}$ ($F^{n+1}\equiv F\wedge\cdots\wedge F$, $n+1$ times); it defines a closed $(2n+2)$-form and is gauge invariant. Since ${\rm tr}F^{n+1}$ is closed, it should be possible to write

$${\rm tr}F^{n+1}=d\omega_{2n+1}^0(B,F) \quad (2.22)$$

with a $(2n+1)$-form $\omega_{2n+1}^0(B,F)$ (known as the Chern-Simons secondary characteristic class). The gauge invariance of ${\rm tr}F^{n+1}$ in turn allows us to write

$$\delta_\Lambda\omega_{2n+1}^0(B,F)=d\omega_{2n}^1(\Lambda,B,F) \quad (2.23)$$

with some $2n$-form $\omega_{2n}^1(\Lambda,B,F)$. It has been shown that up to a normalization factor, this $2n$-form $\omega_{2n}^1(\Lambda,B,F)$ determines the anomaly $\mathscr{A}_{\min}[B_\mu;\Lambda]$ in $2n$ dimensions, and there also exists a systematic procedure to find $\omega_{2n}^1(\Lambda,B,F)$ explicitly.[6] The normalization factor may be fixed by resorting to the Atiyah-Singer index theorem[11] in a $(2n+2)$-dimensional space.

The anomaly we have just considered is the so-called consistent anomaly,[10] and it is basically the quantity $-D_\nu^{ab}J_b^\nu(x;B_\mu)$ with the definition [see Eq. (2.21)]

$$J_a^\nu(x;B_\mu)=\frac{\delta W_R[B_\mu]_{\rm I}}{\delta B_\nu^a(x)}\ . \quad (2.24)$$

The consistent anomaly assigns a non-gauge-covariant expression to $-D_\nu^{ab}J_b^\nu$, and hence the current defined by (2.24) will not be gauge covariant in general. In the literature[23] a *gauge-covariant form* for the anomaly has also been given. It involves a gauge-covariant current $\tilde{J}_a^\nu$, related to the above current by

$$\tilde{J}_a^\nu(x;B_\mu)=J_a^\nu(x;B_\mu)+X_a^\nu(x;B_\mu) \quad (2.25)$$

with a suitable covariantizing local counterterm X_a^ν. The "covariant" anomaly is defined through $-D_\nu^{ab}\tilde{J}_b^\nu$, and it should be noted that the counterterm X_a^ν here is in general not attributable to local counterterm ambiguities in the effective action. In Refs. 9 and 10 a detailed account is given for the nature of this covariant anomaly; the authors proved especially the existence of an appropriate local expression $X_a^\nu(x;B_\mu)$. Here we shall make just one comment related to the last statement—there is a trivial way to see the existence of a gauge-covariant current $\tilde{J}_a^\nu$, consistent with the relation (2.25). All that is needed is to modify the Lagrangian (2.11) to the form

$$\mathscr{L}_{\min}(x)=\bar\psi(x)\left\{i\gamma^\mu D_\mu\left[1+\left[-\frac{1}{M^2}D^2\right]^q\right]-i\gamma^\mu\tilde{B}_\mu\right\}\psi(x)$$
$$[D_\mu=\partial_\mu-iB_\mu,\ \ \tilde{B}_\mu(x)=\tilde{B}_\mu^a(x)T^a] \quad (2.26)$$

with a large regularization parameter M and to set

$$\tilde{J}_a^\nu(x;B_\mu)=\left.\frac{\delta\tilde{W}_{\bar R}[B_\mu,\tilde{B}_\mu]}{\delta\tilde{B}_\nu^a(x)}\right|_{\tilde{B}_\mu=0}, \quad (2.27)$$

where $\tilde{W}_{\bar R}[B_\mu,\tilde{B}_\mu]$ is the effective action corresponding to the theory (2.26). (Note that $\tilde{B}_\nu^a$ here serves the role of an external source for the current $\bar\psi\gamma^\mu T^a\psi$.) The right-hand side of Eq. (2.27) will be well defined for sufficiently large positive integer q, while defining a gauge-covariant quantity by construction. Also for $M\to\infty$ one should have no difficulty in proving that this current $\tilde{J}_a^\nu$ differs from J_a^ν [defined by Eq. (2.24)] by some local counterterm at most. We have thus shown that one can use the current defined by Eq. (2.27) to discuss the covariant anomaly.

III. ON SPINOR ANOMALIES IN CURVED SPACETIME

As a straightforward generalization of the fermion field theory (2.1) to curved space, we can consider the (c-number) action

$$S=\int d^Dx\,(\det e)\{\bar{\psi}(x)[iE_m{}^\mu\gamma^m D_\mu+f(e^m{}_\mu,D_\mu,\Gamma)]\times P_\pm\psi(x)+\mathscr{L}'(x)\} \tag{3.1}$$

with the covariant derivative (acting on a Lorentz spinor)

$$D_\mu=\partial_\mu-iB_\mu-\frac{i}{4}\omega_{lm,\mu}\sigma^{lm}\quad\left[\sigma^{lm}\equiv\frac{i}{2}[\gamma^l,\gamma^m]\right], \tag{3.2}$$

where $e^m{}_\mu$ is the vielbein field ($l,m,\ldots$ represent local Lorentz indices) with its inverse matrix denoted by $E_m{}^\mu$, γ^m are purely numerical $2^{D/2}$-dimensional Dirac γ matrices obeying $\{\gamma^l,\gamma^m\}=-2\eta^{lm}I$ (η^{lm}: the flat spacetime metric), and $P_\pm$ denotes the chirality projection operator appropriate to each given spinor field component [i.e. $(1\pm\gamma_{D+1}/2)$, or 1 with both chiralities]. Spacetime dimension D, as long as it is even, will be left open. The metric tensor is given in terms of the vielbein field by $g_{\mu\nu}=e^m{}_\mu e^n{}_\nu\eta_{mn}$, and we shall follow the usual rule in raising or lowering coordinate or local Lorentz indices attached to tensors. The quantity $\omega_{lm,\mu}$, appearing in our definition of D_μ above, will be identified with

$$\omega_{lm,\mu}=\tfrac{1}{2}e^n{}_\mu(\xi_{lmn}+\xi_{mnl}-\xi_{nlm}),\quad [\xi^l_{mn}\equiv E_m{}^\mu E_n{}^\nu(\partial_\mu e^l{}_\nu-\partial_\nu e^l{}_\mu)], \tag{3.3}$$

i.e., in agreement with the expression for the spin connection when there is no torsion. As in the flat-space case [see Eq. (2.1)], terms involving $f(e^m{}_\mu,D_\mu,\Gamma)$ and $\mathscr{L}'(x)$ in the action (3.1) stand for various "nonminimal" local couplings which are possible for spin-$\frac{1}{2}$ fields in the presence of general external Bose fields (including gravitational ones). Also note that having introduced the chirality projection $P_\pm$ explicitly, we may assume without any loss of generality that the gauge field matrix B_μ in Eq. (3.2) includes no part multiplied by γ_{D+1} [cf. Eq. (2.3)].

Our Eq. (3.3) needs some explanations. Taking $\omega_{lm,\mu}$ as such should not be interpreted in the same sense that our discussions will be restricted to the torsion-free space. For curved space with torsion, one can express the full spin connection $\tilde{\omega}_{lm,\mu}$ in the form

$$\tilde{\omega}_{lm,\mu}=\omega_{lm,\mu}+e_{l\nu}e_{m\lambda}K^{\nu\lambda}{}_\mu, \tag{3.4}$$

where $K^{\nu\lambda}{}_\mu$ is the contortion tensor, $K^{\nu\lambda}{}_\mu=\frac{1}{2}(C^{\nu\lambda}{}_\mu+C^\lambda{}_\mu{}^\nu+C_\mu{}^{\lambda\nu})$, constructed from the torsion tensor $C^\lambda{}_{\mu\nu}(=-C^\lambda{}_{\nu\mu})$. (Notations here are those of Ref. 24. These authors investigated the role of torsion field to the spinor anomaly structure in $D=4$.) Naturally, with the spin connection given by the form (3.4), we will have to include local couplings of spinor fields with the torsion term in our theory. For the anomaly consideration, we here find it convenient to regard all such torsion-field-dependent couplings as *parts of nonminimal couplings* in our general action (3.1), while keeping only the Riemannian spin connection part (3.3) in the definition of D_μ in Eq. (3.2). [This makes sense because $\tilde{\omega}_{lm,\mu}-\omega_{lm,\mu}$ defines a *tensor* under local Lorentz transformations (while being a coordinate vector).]

For the sake of clarity, we may add some explanations on the structure of nonminimal coupling terms in the action (3.1). By Γ we are representing general external bosonic *tensor* fields to which spinors can couple; Γ includes not only various non-gauge-type Bose fields considered in the flat-space case but also tensors of purely gravitational origin, such as Riemann curvature tensor and torsion tensor. Here a tensor refers to a quantity transforming covariantly or invariantly under all three of general coordinate transformations, local Lorentz transformations, and ordinary gauge transformations. Our action S will be, of course, required to be invariant under the three kinds of transformations just mentioned. Subject to that condition, $f(e^m{}_\mu,D_\mu,\Gamma)$ can be a general differential operator of the form

$$f(e^m{}_\mu,D_\mu,\Gamma)=a_0(e^m{}_\mu(x),\Gamma(x))+a_1(e^m{}_\mu(x),\Gamma(x))^lD_l+\cdots+a_r(e^m{}_\mu(x),\Gamma(x))^{l_1l_2\cdots l_r}D_{l_1}D_{l_2}\cdots D_{l_r}, \tag{3.5}$$

where $D_l\equiv E_l{}^\mu D_\mu$ (with D_μ appropriately generalized so that it may act on a general Lorentz tensor spinor), and coefficients a_k's will be in general matrix valued in the internal and Lorentz spin spaces. Note that the first term $a_0(e^m{}_\mu(x),\Gamma(x))$ alone will be sufficient to describe all possible nonderivative, local, bilinear spinor couplings in curved space. (This includes for instance general Yukawa couplings [set $a_0P_\pm=m_0+S(x)+i\gamma_{D+1}P(x)$], Pauli-type antisymmetric tensor couplings [set $a_0P_\pm=\sigma^{lm}E_l{}^\mu E_m{}^\nu f_{\mu\nu}(x)$, with tensor field matrix $f_{\mu\nu}(x)=-f_{\nu\mu}(x)$], and the spinor-torsion coupling term in Riemann-Cartan space.[24]) Local interaction terms involving more than two spinor fields are represented (as in the flat-space case) by the $\mathscr{L}'$ piece in our action (3.1). We shall not concern ourselves with the nonrenormalizability of the theory by the reasoning already given in Sec. II.

In the action (3.1), only the spinor fields $(\psi,\bar{\psi})$ are dynamical degrees of freedom. Let us denote the corresponding effective action functional defined as in Eq. (2.7) by $W[e^m{}_\mu,B_\mu,\Gamma]$. It needs to be suitably regularized, and we will write the regularized expression by $W_R[e^m{}_\mu,B_\mu,\Gamma]$. The question is whether or not the three kinds of invariance properties assumed for our c-number action can be maintained by the regularized effective action; if not, we have anomalies. In principle three separate anomaly equations can be considered here, namely,

$$\delta_\xi^{(E)}W_R[e^m{}_\mu,B_\mu,\Gamma]=\mathscr{G}_R^{(E)}[e^m{}_\mu,B_\mu,\Gamma;\xi], \tag{3.6a}$$

$$\delta_\theta^{(L)}W_R[e^m{}_\mu,B_\mu,\Gamma]=\mathscr{G}_R^{(L)}[e^m{}_\mu,B_\mu,\Gamma;\theta], \tag{3.6b}$$

$$\delta_\Lambda W_R[e^m{}_\mu,B_\mu,\Gamma]=\mathscr{A}_R[e^m{}_\mu,B_\mu,\Gamma;\Lambda], \tag{3.6c}$$

where infinitesimal variation $\delta_\xi^{(E)}$ is associated with (active) general coordinate transformation[9] or Einstein transformation (Ref. 10), $\delta_\theta^{(L)}$ with local Lorentz transformation, and δ_Λ with ordinary gauge transformation. Infinitesimal Einstein transformations operate on tensors as

Lie derivatives; for instance, we have [in connection with the infinitesimal general coordinate transformation $x^\mu \rightarrow x'^\mu = x^\mu - \xi^\mu(x)$]

$$\delta_\xi^{(E)}\phi(x)=\xi^\mu(x)\partial_\mu\phi(x) \quad [\phi(x)\text{: scalar field}] ,$$
$$\delta_\xi^{(E)}e^m{}_\mu(x)=\xi^\nu(x)\partial_\nu e^m{}_\mu(x)+e^m{}_\nu(x)\partial_\mu\xi^\nu(x) . \tag{3.7}$$

On the other hand, the vielbein field transforms under local frame rotations specified by infinitesimal parameter $\theta_{mn}(x) = -\theta_{nm}(x)$ according to

$$\delta_\theta^{(L)}e^m{}_\mu(x)=\theta^m{}_l(x)e^l{}_\mu(x) \quad (\theta^m{}_l=\eta^{mn}\theta_{nl}) . \tag{3.8}$$

Below $\mathscr{G}_R^{(E)}[e^m{}_\mu,B_\mu,\Gamma;\xi]$ will be called the Einstein anomaly (following Ref. 10), $\mathscr{G}_R^{(L)}[e^m{}_\mu,B_\mu,\Gamma;\theta]$ the Lorentz anomaly, and $\mathscr{A}_R[e^m{}_\mu,B_\mu,\Gamma;\Lambda]$ the gauge anomaly. The first two make gravitational anomalies.

We shall now prove that it is possible to set (up to arbitrary local counterterms which may be added to the effective action)

$$\mathscr{G}_R^{(E)}[e^m{}_\mu,B_\mu,\Gamma;\xi]=\mathscr{G}_{\min}^{(E)}[e^m{}_\mu,B_\mu;\xi] , \tag{3.9a}$$

$$\mathscr{G}_R^{(L)}[e^m{}_\mu,B_\mu,\Gamma;\theta]=\mathscr{G}_{\min}^{(L)}[e^m{}_\mu,B_\mu;\theta] , \tag{3.9b}$$

$$\mathscr{A}_R[e^m{}_\mu,B_\mu,\Gamma;\Lambda]=\mathscr{A}_{\min}[e^m{}_\mu,B_\mu;\Lambda] , \tag{3.9c}$$

where the right-hand sides, indicated by the subscript min, represent the anomaly expressions obtained solely on the basis of the *minimal theory* action

$$S_{\min}=\int d^Dx\,(\det e)\bar\psi(x)iE_m{}^\mu\gamma^m D_\mu P_\pm\psi(x)$$
$$\left[D_\mu=\partial_\mu-iB_\mu-\frac{i}{4}\omega_{lm,\mu}\sigma^{lm}\right] . \tag{3.10}$$

There is a simple intermediate regularization scheme in which the statement embodied in Eq. (3.9) is realized manifestly. In a direct analogy to the flat spacetime case dealt with in Sec. II, we may here consider a partial regularization of the theory (3.1) by the action

$$S_{\bar R}=\int d^Dx\,(\det e)\left[\bar\psi(x)\left\{iE_m{}^\mu\gamma^m D_\mu\left[1+\left[-\frac{1}{M^2}D^2\right]^q\right]+f(e^m{}_\mu,D_\mu,\Gamma)\right\}P_\pm\psi(x)+\mathscr{L}'(x)\right] , \tag{3.11}$$

where q is some positive integer, M a large regularization mass parameter, and

$$D^2\equiv\eta^{lm}D_lD_m \quad \text{(acting on a Lorentz spinor)}$$
$$=\frac{1}{(\det e)}D_\mu[(\det e)g^{\mu\nu}D_\nu] \quad \left[D_\mu=\partial\mu-iB_\mu-\frac{i}{4}\omega_{lm,\mu}\sigma^{lm}\right] . \tag{3.12}$$

We also define the truncated theory by the action [cf. Eq. (2.15)]

$$S_{\text{trunc}}=\int d^Dx\,(\det e)\left\{\bar\psi(x)iE_m{}^\mu\gamma^m D_\mu\left[1+\left[-\frac{1}{M^2}D^2\right]^q\right]P_\pm\psi(x)\right\} . \tag{3.13}$$

Now let us denote the effective action functional corresponding to the theory (3.11) as $W_{\bar R}[e^m{}_\mu,B_\mu,\Gamma]$. It may be studied using standard perturbation theory. We shall here choose q to be sufficiently large. Then, as will be explained below, we can secure a well-defined integral for any Feynman diagram contributing to $W_{\bar R}[e^m{}_\mu,B_\mu,\Gamma]$ which cannot be reckoned with entirely within the truncated theory (3.13).

For a perturbation-theoretic analysis of the theory (3.11), we had better write the vielbein field as $e^m{}_\mu(x)=\delta^m{}_\mu+h^m{}_\mu(x)$; small $h^m{}_\mu$ will correspond to a weak gravitational field. Then one can easily obtain the Feynman-diagrammatic representation of the effective action $W_R[e^m{}_\mu,B_\mu,\Gamma]$, assuming that (i) external fields $(h^m{}_\mu(x),B_\mu(x),\Gamma(x))$ are small, i.e., may be considered perturbatively and also (ii) all nonminimal couplings—represented by $(\det e)\bar\psi f(e^m{}_\mu,B_\mu,\Gamma)P_\pm\psi$ and $(\det e)\mathscr{L}'$—enter the theory with small coupling constants. Some typical Feynman diagrams are shown in Fig. 4. Note that aside from appropriate chirality projection factors, the undressed spinor propagator will be precisely that given in Eq. (2.14). Despite the complexity (in connection with gravitational couplings especially), it is not difficult to sort out all ill-defined Feynman diagrams in the theory (3.11) by means of power counting. First consider (one-loop) Feynman diagrams which do not involve any nonminimal vertices, i.e., those associated with the truncated theory (3.13) entirely. For those diagrams the degree of superficial divergence is given by the formula (independently of the value of q)

$$r=D-N_B-N_\omega , \tag{3.14}$$

where N_B denotes the total number of external B_μ legs

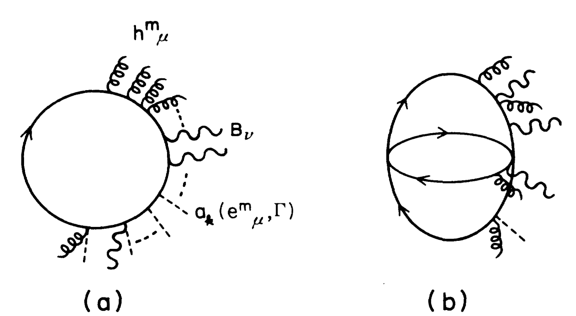
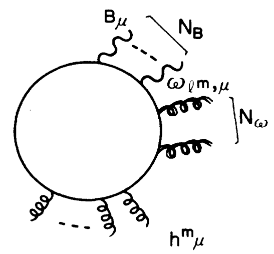


FIG. 4. Some typical Feynman diagrams contributing to $W_{\bar R}[e^m{}_\mu,B_\mu,\Gamma]$. Multiloop contributions exist only when $\mathscr{L}'(x)\neq 0$.

and N_ω the number of spin connection legs attached to the diagram. [With the action (3.13), $h^m{}_\mu$'s enter the interaction Lagrangian specifically through the spin connection $\omega_{lm,\mu}$ or through other vielbein field dependences. For power counting we find it convenient to distinguish the two cases; and spin-connection legs should be understood in that sense.] If $r \geq 0$, the Feynman diagram is superficially divergent and hence ill defined yet. In view of the fact that the number of external $h^m{}_\mu$ legs—excluding those counted as the spin-connection legs—do not enter the formula (3.14) at all, we have clearly an infinite number of superficially divergent Feynman diagrams in the theory (3.13) (see Fig. 5). The situation does not change as we increase the value of q.

How about Feynman diagrams which involve *at least one nonminimal vertices?* Any one-loop or higher-loop Feynman diagram belonging to this latter class will actually become well defined in the theory (3.11) (i.e., $r<0$) if we choose the integer q sufficiently large. (As in the flat-space discussion, we are here assuming that there is an upper bound in dimensions of all nonminimal coupling terms present.[20]) The same reasoning which we have used in Sec. II applies here. All vertex factors associated with nonminimal couplings remain unchanged by modifying the original theory to the form (3.11). At every vertex part associated with nonminimal couplings, we thus have the situation that vertex factors are unaffected but attached spinor propagators are regularized ones. This means that, given any one-particle-irreducible Feynman diagram including nonminimal vertices, we can always reduce the degree of superficial divergence (for all subgraphs and for the whole graph) to some negative value by choosing q to be sufficiently large.

The relations (3.9a)–(3.9c) now follow immediately. Evidently, our partially regularized action (3.11) does possess manifestly the general covariance, local Lorentz invariance, and ordinary gauge invariance of the original action. Together with what we have shown above, this means that the part of the effective action which is represented by all Feynman diagrams involving at least one nonminimal vertex can be properly regularized (i.e., made well defined) without jeopardizing any of the three invariance properties mentioned. Therefore, there cannot be any *genuine* gravitational or gauge anomaly involving, partly, or wholly, nonminimal local couplings in the theory (such as general Yukawa-type couplings, antisymmetric tensor couplings, couplings involving the torsion field, various higher derivative couplings, and even four-Fermi-type couplings). This is true for general $D=2n$ dimensions. From $W_{\bar{R}}[e^m{}_\mu, B_\mu, \Gamma]$, only the part which can



$(0 \leq N_B + N_\omega \leq D)$

FIG. 5. Superficially divergent Feynman diagrams in the truncated theory (3.13).

be entirely related to the truncated theory (3.13) may generate anomalies with regards to some of the three invariances; for that part of the effective action, ill-defined contributions still remain and there may not exist any acceptable regularization procedure for them which does no harm to the three invariances. Up to local counterterms our truncated theory (3.13) (with $M\to\infty$), of course, defines the same theory as the minimal theory (3.10). The full content of Eqs. (3.9a)–(3.9c) has been now established. It asserts that if one finds gravitational or gauge anomalies depending on any kind of nonminimal local couplings by a calculational procedure using some particular regularization scheme, they are merely artifacts of the regularization chosen and can be eliminated by suitable local counterterms to the effective action.

The structure of gravitational and gauge anomalies in the minimal theory (3.10) has been analyzed in the pioneering work by Alvarez-Gaumé and Witten.[16] Now, with some additional inputs by others (especially Refs. 9 and 10), we have a more or less complete understanding on the subject. In what follows brief discussions on some of this development shall be given. First of all, with both gravitational and (real or spectator) gauge fields present together, there exist much arbitrariness—connected with adding some local counterterms to the effective action—in the forms of the three anomaly functionals. Here the choice we find particularly convenient is the one with the gravitational anomalies, i.e., $\mathcal{G}^{(E)}_{\min}[e^m{}_\mu, B_\mu;\xi]$ and $\mathcal{G}^{(L)}_{\min}[e^m{}_\mu, B_\mu;\theta]$, *made completely independent of* B_μ (at the expense of making the expression for $\mathcal{A}_{\min}[e^m{}_\mu, B_\mu;\Lambda]$ complicated). How does one know that such choice is possible? There is a simple way to realize that.

Consider (partially) regularizing the minimal theory (3.10) by

$$S_{\min(\bar{R})} = \int d^Dx\,(\det e)\bar{\psi}(x)\left\{iE_m{}^\mu\gamma^m\nabla_\mu\left[1+\left[-\frac{1}{M^2}\nabla^2\right]^q\right]-iB_\mu\right\}P_\pm\psi(x)$$

$$\left[\nabla_\mu \equiv \partial_\mu - \frac{i}{4}\omega_{lm,\mu}\sigma^{lm},\quad \nabla^2 \equiv \frac{1}{(\det e)}\nabla_\mu[(\det e)g^{\mu\nu}\nabla_\nu]\right] \qquad (3.15)$$

with some positive integer q. This regularization explicitly harms ordinary gauge invariance, but still preserves general covariance and local Lorentz symmetry manifestly. With the regularized action (3.15), any one-loop spinor diagram which involves at least one external B_μ leg becomes well defined if one chooses q to be sufficiently large. This can be shown by power-counting analysis [similar to what we have done with the theory (3.11)].[25] Besides the regularization introduced through the action (3.15), a separate (in general not symmetry-preserving) regularization may be employed exclusively for those still ill-defined spinor loop diagrams involving purely gravitational legs. If we denote the effective action thus constructed as $W_{\min(R)}[e^m{}_\mu,B_\mu]$, we will then have the gravitational anomaly equations of the form

$$\begin{aligned}\delta_\xi^{(E)}W_{\min(R)}[e^m{}_\mu,B_\mu]&=\delta_\xi^{(E)}W_{\min(R)}[e^m{}_\mu,B_\mu=0]\\&\equiv\mathcal{G}^{(E)}[e^m{}_\mu;\xi]\,,\\\delta_\theta^{(L)}W_{\min(R)}[e^m{}_\mu,B_\mu]&=\delta_\theta^{(L)}W_{\min(R)}[e^m{}_\mu,B_\mu=0]\\&\equiv\mathcal{G}^{(L)}[e^m{}_\mu;\theta]\,,\end{aligned}\qquad(3.16)$$

since the piece

$$W_{\min(R)}[e^m{}_\mu,B_\mu]-W_{\min(R)}[e^m{}_\mu,B_\mu=0]$$

is invariant (by construction) under general coordinate transformations and local Lorentz transformations. While the gravitational anomaly expressions get simplified, the corresponding gauge anomaly

$$\mathcal{A}_{\min(R)}[e^m{}_\mu,B_\mu;\Lambda]=\delta_\Lambda W_{\min(R)}[e^m{}_\mu,B_\mu]=-\int d^Dx\,(\det e)\Lambda^a(x)D_\nu^{ab}J_b^\nu[x;e^m{}_\mu,B_\mu]$$
$$\left[J_a^\nu[x;e^m{}_\mu,B_\mu]\equiv\frac{1}{(\det e)}\frac{\delta W_{\min(R)}[e^m{}_\mu,B_\mu]}{\delta B_\nu^a(x)}\right]\,,\qquad(3.17)$$

will acquire a complicated functional dependence on gauge and gravitational fields. [Here the covariant derivative D_ν^{ab} (in $D_\nu^{ab}J_b^\nu$) includes the usual Christoffel connection term together with the gauge connection.] In the language of Ref. 16, we have constructed the effective action $W_{\min(R)}[e^m{}_\mu,B_\mu]$ such that all *mixed* anomalies—i.e., anomalies from spinor loop diagrams with external legs consisting partly of gauge fields and partly of gravitational fields—may contribute to ordinary gauge transformation anomalies,[26] but not to gravitational anomalies.

There are also ambiguities concerning the expressions $\mathcal{G}^{(E)}[e^m{}_\mu;\xi]$ and $\mathcal{G}^{(L)}[e^m{}_\mu;\theta]$, which represent pure gravitational anomalies. We may best characterize them by defining the energy-momentum "tensor" (really its expectation value)

$$T^\nu{}_n[x;e^m{}_\mu]=\frac{1}{(\det e)}\frac{\delta W_{\min(R)}[e^m{}_\mu,B_\mu=0]}{\delta e^n{}_\nu(x)}\,.\qquad(3.18)$$

(It may be the case that $T^\nu{}_n[x;e^m{}_\mu]$ is not a true tensor. See the last paragraph of this section.) Then, using the transformation laws (3.7) and (3.8), one can show that

$$\begin{aligned}\mathcal{G}^{(E)}[e^m{}_\mu;\xi]=\int d^Dx\,(\det e)\xi^\nu(x)&\\\times\{\nabla_\mu T_\nu{}^\mu[x;e^m{}_\mu]&\\+\omega_{ln,\nu}(x)T^{ln}[x;e^m{}_\mu]\}\,,&\qquad(3.19a)\end{aligned}$$

$$\mathcal{G}^{(L)}[e^m{}_\mu;\theta]=\int d^Dx\,(\det e)\theta_{nl}(x)T^{ln}[x;e^m{}_\mu]\,.\qquad(3.19b)$$

Since $\theta_{nl}=-\theta_{ln}$, nonvanishing Lorentz anomaly clearly implies that $T^{ln}-T^{nl}\neq0$. On the other hand, from Eq. (3.19a), we see that nonvanishing Einstein anomaly (with a manifestly *symmetric* energy-momentum tensor) can be related to the nonconservation of the energy-momentum tensor $\nabla_\mu T_\nu{}^\mu\neq0$. In Ref. 16 it has been shown (by a simple group theory argument) that we may have nonvanishing gravitational anomalies only in $D=4k+2$ ($k=0,1,2,\ldots$) dimensions. With Weyl spinors in $D=4k+2$ dimensions, we do have gravitational anomalies, either in the form of the Lorentz anomaly (i.e., $T^{ln}-T^{nl}\neq0$) or in the form of the Einstein anomaly (i.e., $\nabla_\mu T_\nu{}^\mu\neq0$). Authors of Refs. 9 and 10 have clarified the situation by showing that with the help of suitable local counterterms to the effective action, one can shift entirely the Einstein anomaly to the Lorentz anomaly and vice versa.

For the explicit form of the gravitational anomaly, it is again convenient to use the language of differential geometry. Actually, up to overall normalization, they can be expressed in terms of the $2n$-form ω_{2n}^1 (here $D=2n=4k+2$), defined through relations analogous to Eqs. (2.22) and (2.23) with the arguments appropriately changed. For instance, the Lorentz anomaly is determined by $\omega_{2n}^1(\theta,\omega,R)$, with the spin connection one-form $\omega^l{}_m=\omega^l{}_{m,\mu}dx^\mu$ and the curvature two-form $R=d\omega+\omega\wedge\omega$. (For the Einstein anomaly, the Christoffel connection $\Gamma^\mu{}_\nu=\Gamma^\mu{}_{\lambda\nu}dx^\lambda$ may be used instead.) Here $\omega_{2n}^1(\theta,\omega,R)$ satisfies the relations

$$P_{2n+2}(R)=d\omega_{2n+1}^0(\omega,R)$$

and

$$\delta_\theta^{(L)}\omega_{2n+1}^0(\omega,R)=d\omega_{2n}^1(\theta,\omega,R)\,,$$

with $P_{2n+2}(R)$ denoting the $(2n+2)$-form piece of the so-called Dirac genus. For further details, readers are referred to Ref. 9. Also the full structure of mixed anomalies (all contained in $\mathcal{A}_{\min(R)}[e^m{}_\mu,B_\mu;\Lambda]$ above) being rather complicated, shall not be discussed here and the interested reader is referred to the literature.[16,9] For a consistent gauge theory with spinors in curved spacetime, one must make sure that all these anomalies—gravitational and gauge—be absent.

The things discussed so far are *consistent* anomalies; when anomalies are present, $J_a^\nu[x;e^m{}_\mu,B_\mu]$ and $T^\nu{}_n[x;e^m{}_\mu]$ defined above do not really define true tensors (with respect to the three kinds of transformations being considered). It has been demonstrated[9,10] that by adding suitable covariantizing local counterterms to $(J_a^\nu, T^\nu{}_n)$, it is possible to construct related true tensors $(\widetilde{J}_a^\nu, \widetilde{T}^\nu{}_n)$. (Note that these covariant quantities can no longer be written as appropriate functional derivatives of the effective action.) If the anomalies are defined using the currents $(\widetilde{J}_a^\nu, \widetilde{T}^\nu{}_n)$, the results will be the *covariant* (guage or gravitational) anomalies. Actually, the existence of the covariant currents $(\widetilde{J}_a^\nu, \widetilde{T}_n^\nu)$ can be seen most directly by the method described at the end of Sec. II. We shall briefly explain how the covariant energy-momentum tensor $\widetilde{T}^\nu{}_n$ can be secured using that method. First we define the quantity $\mathcal{T}^\nu{}_n(x;e^m{}_\mu,\psi,\bar\psi)$ by

$$\mathcal{T}^\nu{}_n(x;e^m{}_\mu,\psi,\bar\psi)=\frac{1}{\det e}\frac{\delta}{\delta e^n{}_\nu(x)} \times\left[\int d^Dx\,(\det e)\bar\psi(iE_m{}^\mu\gamma^m\nabla_\mu)P_\pm\psi\right]. \tag{3.20}$$

We then modify our minimal theory action (with B_μ set to zero) to the form

$$\widetilde{S}_{\min}=\int d^Dx\,(\det e)\left\{\bar\psi(x)iE_m{}^\mu\gamma^m\nabla_\mu\left[1+\left[-\frac{1}{M^2}\nabla^2\right]^q\right]P_\pm\psi(x)+\widetilde{h}^n{}_\nu(x)\mathcal{T}^\nu{}_n(x;e^m{}_\mu,\psi,\bar\psi)\right\}, \tag{3.21}$$

where $\widetilde{h}^n{}_\nu(x)$ serves the role of an external source for the quantity $\mathcal{T}^\nu{}_n$. If $\widetilde{W}_{\bar R}[e^m{}_\mu,\widetilde{h}^m{}_\mu]$ denotes the corresponding effective action, we may now set

$$\widetilde{T}^\nu{}_n[x;e^m{}_\mu]=\frac{1}{(\det e)}\frac{\delta\widetilde{W}_{\bar R}[e^m{}_\mu,\widetilde{h}^m{}_\mu]}{\delta\widetilde{h}^n{}_\nu(x)}\Bigg|_{\widetilde{h}^m{}_\mu=0}. \tag{3.22}$$

For sufficiently large integer q, this quantity $\widetilde{T}^\nu{}_n[x;e^m{}_\mu]$ will be well defined and defines a true tensor by construction. Moreover, in the limit $M\to\infty$, $\widetilde{T}^\nu{}_n[x;e^m{}_\mu]$ will not differ from the quantity $T^\nu{}_n[x;e^m{}_\mu]$ [defined by Eq. (3.18)] more than some local counterterm. Hence, in $\widetilde{T}^\nu{}_n[x;e^m{}_\mu]$ we have the energy-momentum "tensor" which is a true tensor in any case, and one may use it to discuss the covariant gravitational anomalies.

IV. AN ALTERNATIVE PROOF AND SOME TOPOLOGICAL CONSIDERATIONS

By studying the spinor effective action in *Euclidean* spacetime, one can gain a better understanding on anomalies. Concentrating on theories with minimal gauge interactions only, Alvarez-Gaumé and Witten[16] made an important observation that anomalies can arise only from the imaginary part of the Euclidean effective action, i.e., only the phase of the fermion integral matters for anomalies. Based on that, a topological understanding of gauge anomalies has been achieved in Ref. 8. In this section we shall extend these considerations by including various nonminimal spinor couplings to the theory. Indeed, these methods provide another way of understanding the fact that genuine spinor anomalies receive no contribution from various possible nonminimal couplings in the spinor Lagrangian. To obviate technical complications surrounding Euclidean curved space field theories and the topological study of gravitational anomalies,[27] we shall here only deal with gauge anomalies in general flat-space local field theories (considered in Sec. II).

Consider a general spinor Lagrangian of the form (2.1) in $2n$-dimensional Euclidean spacetime. The anomaly represents the failure of gauge invariance in the effective action of the theory. First, we wish to make it clear that if the gauge field matrix $B_\mu(x)$ in our theory (2.1) is restricted to be purely *vectorlike* [i.e., in Eq. (2.3), the axial-vector piece $A_\mu(x)$ vanishes], there will be no anomaly *irrespectively of the structure of various nonminimal couplings possible.* [Nonminimal couplings—terms involving $f(D_\mu,\Gamma)$ and $\mathcal{L}'$ in the action (2.1)—may involve γ_{2n+1} freely only if they make a gauge-invariant interaction Lagrangian.] The reason is simple: in that case, gauge-invariant Dirac mass terms of arbitrary magnitudes will be possible for all spinor fields and then one can have a well-defined and gauge-invariant spinor effective action by means of the usual Pauli-Villars regularization.[28]

When the gauge field $B_\mu(x)$ has a nonvanishing axial-vector part, the Pauli-Villars method cannot be used as a means to secure a gauge-invariant effective action and the theory can have anomalies. Even for this case, there will be in general no problem in preserving gauge invariance for the *real* part of the effective action.[16] We want a stronger result than that. When the gauge field takes a general form (2.3), let $W_E[B_\mu,\Gamma]$ denote the (Euclidean) effective action for the theory (2.1) and $W_{E,\min}[B_\mu]$ the effective action for the corresponding minimal theory (2.11). Then $W_E[B_\mu,\Gamma]$ may be divided as

$$\begin{aligned}W_E[B_\mu,\Gamma]&=i\,\mathrm{Im}\,W_{E,\min}[B_\mu]\\&\quad+\mathrm{Re}\,W_{E,\min}[B_\mu]\\&\quad+(W_E[B_\mu,\Gamma]-W_{E,\min}[B_\mu])\,.\end{aligned} \tag{4.1}$$

Our assertion is that there should not be any problem with maintaining gauge invariance for the quantity $\mathrm{Re}\,W_{E,\min}[B_\mu]$ or for the quantity

$$(W_E[B_\mu,\Gamma]-W_{E,\min}[B_\mu])\,,$$

viz., anomalies for our theory (2.1) may arise only from the failure of gauge invariance in the imaginary part of the effective action of the corresponding minimal theory. As will be explained below, this follows from a simple modification of the argument given in Ref. 16.

Note that if one wishes, the general spinor Lagrangian (2.1) can always be rewritten using left-handed Weyl spinors only, viz.,

$$\mathcal{L}(x)=\bar\chi_L(x)i\gamma^\mu D_\mu\psi_L(x)+\mathcal{L}_{\mathrm{nonmin}}(x;\psi_L,\bar\chi_L)\,, \tag{4.2}$$

where

$$D_\mu \equiv \partial_\mu - iB_\mu^a \lambda^a, \quad \frac{1-\gamma_{D+1}}{2}\psi_L = \psi_L ,$$

$$\bar{\chi}_L \frac{1+\gamma_{D+1}}{2} = \bar{\chi}_L ,$$

and

$$\mathscr{L}_{\text{nonmin}}(x;\psi_L,\bar{\chi}_L)$$

represents various nonminimal coupling of the theory (2.1). Following Ref. 16, we have used the notation $\bar{\chi}_L$ (instead of writing $\bar{\psi}_L$) to emphasize the fact that in Euclidean space $\bar{\chi}_L$ and ψ_L are not related by complex conjugation but correspond to independent variables. Using the Lagrangian (4.2), we may express the effective action $W_E[B_\mu,\Gamma]$ formally by

$$e^{-W_E[B_\mu,\Gamma]} = \int [\mathscr{D}\psi_L][\mathscr{D}\bar{\chi}_L]\exp\left[-\int d^{2n}x_E[\bar{\chi}_L(x)i\gamma^\mu(\partial_\mu - iB_\mu^a\lambda^a)\psi_L(x) + \mathscr{L}_{\text{nonmin}}(x;\psi_L,\bar{\chi}_L)]\right] . \tag{4.3}$$

If we set $\mathscr{L}_{\text{nonmin}}(x;\psi_L,\bar{\chi}_L)=0$ in the right-hand side of this equation, the resulting fermion integral will represent $e^{-W_{E,\min}[B_\mu]}$. We shall also need a simple path-integral expression for the quantity $e^{-W^*_{E,\min}[B_\mu]}$, with $W^*_{E,\min}[B_\mu]$ being the complex conjugate of the minimal theory effective action $W_{E,\min}[B_\mu]$. As noted already in Ref. 16, we can represent it via the fermion integral of the form

$$e^{-W^*_{E,\min}[B_\mu]} = \int [\mathscr{D}\chi'_L][\mathscr{D}\bar{\psi}'_L]\exp\left[-\int d^{2n}x_E\{\bar{\psi}'_L(x)i\gamma^\mu[\partial_\mu - iB_\mu^a(-\lambda^{*a})]\chi'_L\}\right] , \tag{4.4}$$

where $(\chi'_L,\bar{\psi}'_L)$ are dummy fermion integration variables obeying the conditions

$$\frac{1-\gamma_{D+1}}{2}\chi'_L = \chi'_L$$

and

$$\bar{\psi}'_L \frac{1+\gamma_{D+1}}{2} = \bar{\psi}'_L .$$

Note that, under the gauge group, $(\chi'_L,\bar{\psi}'_L)$ assume the complex-conjugate representations of those relevant for $(\psi_L,\bar{\chi}_L)$.

We may now combine the two theories defined through Eq. (4.3) and Eq. (4.4), respectively, into a single theory. Evidently, it should be possible to write

$$e^{-(W_E[B_\mu,\Gamma]+W^*_{E,\min}[B_\mu])} = \int [\mathscr{D}\psi_L][\mathscr{D}\bar{\chi}_L][\mathscr{D}\chi'_L][\mathscr{D}\bar{\psi}'_L] \times \exp\left[-\int d^{2n}x_E \mathscr{L}_s(x)\right] \tag{4.5}$$

with the Lagrangian

$$\mathscr{L}_s(x) = \bar{\chi}_L(x)i\gamma^\mu(\partial_\mu - iB_\mu^a\lambda^a)\psi_L(x) + \bar{\psi}'_L(x)i\gamma^\mu(\partial_\mu - iB_\mu^a(-\lambda^{*a}))\chi'_L(x) + \mathscr{L}_{\text{nonmin}}(x;\psi_L,\bar{\chi}_L) . \tag{4.6}$$

The point is that the Lagrangian $\mathscr{L}_s(x)$, while being a special case of the general form (4.2), really has a *vector-like* gauge interaction. In fact the second piece in the right-hand side of Eq. (4.6) may be rewritten using right-handed Weyl spinor variables $(\psi_R,\bar{\chi}_R)$, instead of the left-handed ones $(\bar{\psi}'_L,\chi'_L)$. That will give us the form $\bar{\chi}_R i\gamma^\mu(\partial_\mu - iB_\mu^a\lambda^a)\psi_R$, which can be combined with the first piece in the right-hand side of Eq. (4.6), $\bar{\chi}_L i\gamma^\mu(\partial_\mu - iB_\mu^a\lambda^a)\psi_L$, to yield the purely vectorial interaction $\bar{\chi}i\gamma^\mu(\partial_\mu - iB_\mu^a\lambda^a)\psi$ involving Dirac spinors $(\bar{\chi},\psi)$. Then the observation made at the beginning of this section will guarantee that there exists a well-defined and gauge-invariant definition for the corresponding effective action

$$W_{E,S}[B_\mu,\Gamma] = W_E[B_\mu,\Gamma] + W^*_{E,\min}[B_\mu] . \tag{4.7}$$

In the special case of $\mathscr{L}_{\text{nonmin}}(x;\psi_L,\bar{\chi}_L)=0$ (i.e., $W_E[B_\mu,\Gamma]=W_{E,\min}[B_\mu]$), $W_{E,S}$ reduces to $2\,\mathrm{Re}W_{E,\min}[B_\mu]$ and hence the existence of a gauge-invariant definition for $\mathrm{Re}\,W_{E,\min}[B_\mu]$ also follows from this consideration. This in turn leads to the conclusion that the quantity $(W_E[B_\mu,\Gamma]-W_{E,\min}[B_\mu])$, being equal to

$$(W_{E,S}[B_\mu,\Gamma] - 2\,\mathrm{Re}W_{E,\min}[B_\mu]) ,$$

should also admit a well-defined and gauge-invariant definition. Our assertion is now established.

According to what has been established above, we may set

$$W_E[B_\mu^g,\Gamma^g] - W_E[B_\mu,\Gamma] = i(\mathrm{Im}\,W_{E,\min}[B_\mu^g] - \mathrm{Im}\,W_{E,\min}[B_\mu]) , \tag{4.8}$$

where (B_μ^g,Γ^g) are related to (B_μ,Γ) by gauge transformation g:

$$B_\mu^g(x) = g^{-1}(x)B_\mu(x)g(x) + \frac{1}{i}g^{-1}(x)\partial_\mu g(x) , \tag{4.9a}$$

$$(\Gamma^i(x))^g = [D_g(\Gamma)]_{ij}(x)\Gamma^j(x) . \tag{4.9b}$$

In Eq. (4.8) we have a simple understanding of the relation (2.10), aside from that the Euclidean space language is being used here. Equation (4.8) has another implication—for the quantity $e^{-W_E[B_\mu,\Gamma]}$ [i.e., for the fermion integral shown in Eq. (4.3)], the failure of gauge invariance can be only in its *phase*. Working with the minimal theory (2.11), Alvarez-Gaumé and Ginsparg[8] (see also Ref. 7) have been able to relate the gauge anomaly to nontrivial topological properties of this U(1) phase when g is varied around a closed loop in the space of gauge transformations. In this light, the gauge anomaly becomes a manifestation of a *topological obstruction* to defining a gauge-invariant effective action. Then our Eq. (4.8) immediately suggests that various nonminimal couplings and non-gauge-type external fields in the theory have nothing to do with this topological obstruction (which forces the anomaly). Can one understand the fact through an argument of purely topological nature? The answer is in the affirmative, and we shall describe it below.

For a general theory (2.1) we shall take it for granted (in agreement with the above observation) that on the functional $e^{-W_E[B_\mu,\Gamma]}$, gauge transformations can be realized up to a 1-cocycle,[29] i.e.,

$$e^{-W_E[B_\mu^g,\Gamma^g]}=e^{iw(g;B_\mu,\Gamma)}e^{-W_E[B_\mu,\Gamma]} \quad (4.10)$$

with some phase factor $e^{iw(g;B_\mu,\Gamma)}$ [$w(g;B_\mu,\Gamma)$ is real] which can *a priori* depend on every detail of the given theory. Then, for a given semisimple gauge group G (and taking the Euclidean spacetime manifold to be the compactified S^{2n}), we consider a one-parameter family of gauge transformations[8]

$$\begin{aligned}&g(x)^\theta(\theta\in[0,2\pi]):S^1\times S^{2n}\to G\ ,\\&g(x)^{\theta=0}=g(x)^{\theta=2\pi}=1\ .\end{aligned} \quad (4.11)$$

It will be assumed that the map g^θ defines a nontrivial element of $\pi_1(\mathscr{G})$, where $\mathscr{G}$ denotes the space of gauge transformations. [Note that when $\pi_1(G)=\pi_{2n}(G)=0$, $\pi_1(\mathscr{G})$ is isomorphic to $\pi_{2n+1}(G)$.] Then, for some fixed reference configurations (B_μ,Γ), the phase $e^{iw(g^\theta;B_\mu,\Gamma)}$ as a function of the parameter θ will define a map $S^1\to S^1$. The corresponding winding number can be expressed as

$$m=\frac{1}{2\pi}\int_0^{2\pi}d\theta\frac{dw(g^\theta;B_\mu,\Gamma)}{d\theta} \quad (4.12)$$

with m restricted to integers. Here, because of Eq. (4.10), the integrand $[dw(g^\theta;B_\mu,\Gamma)/d\theta]$ can be related to the anomaly functional:

$$\begin{aligned}\frac{dw(g^\theta;B_\mu,\Gamma)}{d\theta}&=i\frac{dW_E[B_\mu^{g^\theta},\Gamma^{g^\theta}]}{d\theta}\\&\equiv i\mathscr{A}_E[B_\mu^{g^\theta},\Gamma^{g^\theta};\Lambda]\,|_{\Lambda=(g^\theta)^{-1}\partial_\theta g^\theta}\ .\end{aligned} \quad (4.13)$$

The winding number m is none other than the integrated anomaly along the one-parameter family of gauge transformations (4.11). If one finds circumstances in which the winding number m should be nonzero (by some other topological considerations), the effective action *must change* under a gauge transformation, thus demonstrating the existence of an anomaly as a consequence of a topological obstruction. With the minimal theory (2.11), Alvarez-Gaumé and Ginsparg[8] have implemented this program in an explicit manner by relating m to the index of an appropriate $(2n+2)$-dimensional Dirac operator; if the index is nonzero, we must have an anomaly. Before we discuss our general theory case, we shall first recapitulate the basic idea used in Ref. 8 below.

The authors of Ref. 8 extended $B_\mu^{g^\theta}$ to a two-parameter family:

$$B_\mu^{(\theta,t)}(x)=tB_\mu^{g^\theta}(x)\quad(t\in[0,1])\ . \quad (4.14)$$

The parameter θ and t can be viewed as polar coordinates of a disk D, and along its boundary we have $B_\mu^{(\theta,t=1)}=B_\mu^{g^\theta}$, i.e., $B_\mu^{(\theta,t)}$ reduces to the original one-parameter family of gauge fields. Then, for a fixed reference gauge field B_μ with zero Pontryagin number, one can easily relate the above winding number m to the (signed) sum of the number of points in the interior of D, where $e^{-W_{E,\min}[B_\mu^{(\theta,t)}]}$ vanishes. It is further shown in ref. 8 that these zeros of $e^{-W_{E,\min}[B_\mu^{(\theta,t)}]}$ can be put in one-to-one correspondence with zero modes of a specific $(2n+2)$-dimensional Dirac operator involving the background gauge field $B_\mu^{(\theta,t)}$. [Here the $(2n+2)$-dimensional space may be identified with $S^2\times S^{2n}$, with S^2 being an extension of the disk D (with D making the upper patch of the two-sphere) and S^{2n} the original spacetime.] Based on that it is possible to identify the winding number m with the index of the $(2n+2)$-dimensional Dirac operator, which can in turn be expressed in terms of the $(n+1)$th Chern character by the Atiyah-Singer index theorem.[11] The full anomaly functional can be extracted by using this $(n+1)$th Chern character representation of m together with the formulas (4.12) and (4.13). Note that these arguments provide a rationale for the mathematical manipulations used in Eqs. (2.22) and (2.23) to find the non-Abelian anomaly expression.

Turning to the case of the general theory (2.1), we shall now show that various nonminimal couplings give rise to no additional topological obstruction beyond that found with the corresponding minimal theory. For the purpose it would be sufficient to prove that

$$\frac{1}{2\pi}\int_0^{2\pi}d\theta\frac{dw(g^\theta,B_\mu,\Gamma)}{d\theta}=\frac{1}{2\pi}\int_0^{2\pi}d\theta\frac{dw_{\min}(g^\theta,B_\mu)}{d\theta}\ , \quad (4.15)$$

viz., the winding number m for the theory (2.1) cannot be different from the quantity defined for the corresponding minimal theory. [Note that $w_{\min}(g^\theta;B_\mu)$ is defined by Eq. (4.10) restricted to the minimal theory (2.11).] The proof is quite simple. When $\mathscr{L}_{\text{nonmin}}$ represents all nonminimal coupling terms in a given general local spinor Lagrangian (i.e., $\mathscr{L}=\mathscr{L}_{\min}+\mathscr{L}_{\text{nonmin}}$), we can here imagine a one-parameter family of theories described by the Lagrangian

$$\mathscr{L}_{[\lambda]}=\mathscr{L}_{\min}+\lambda\mathscr{L}_{\text{nonmin}}\quad(\lambda\in[0,1])\ . \quad (4.16)$$

[Gauge invariance of $\mathscr{L}_{\text{nonmin}}$ is necessary for the behavior (4.10) (Ref. 30).] With this one-parameter family of theories, we then ask how the quantity

$$\frac{1}{2\pi}\int_0^{2\pi}d\theta\frac{dw(g^\theta;B_\mu,\Gamma)}{d\theta}$$

will change as λ is varied. The answer is that it is *independent of* λ. The quantity in question is the winding number which can take only integer values, and hence its value cannot change under a continuous variation of λ. Also, during a continuous variation of λ in the interval [0,1], we do not expect the effective action functional to develop a singular behavior under gauge transformations.

Equation (4.15) has now been established with the left-hand side corresponding to the $\lambda=1$ expression and the right-hand side to the $\lambda=0$ expression of one λ-independent quantity.[31]

There is a comment to add. One might suggest turning off even the gauge field $B_\mu(x)$ (but not gauge symmetry) from the theory by considering the expression (4.12) for a one-parameter family of gauge fields $B_{\mu[\lambda']}(x) =\lambda' B_\mu(x)(\lambda'\in[0,1])$ and then by applying the same argument as above. The winding number m will not depend on the continuous parameter λ', and then it should be allowed to set $B_\mu(x)=0$ in Eqs. (4.12) and (4.13) without losing information on the topological obstruction responsible for non-Abelian anomalies. That is true: Eq. (4.13) in fact provides nontrivial information even with B_μ set to zero. The point is in contrast with non-gauge-type external fields $\{\Gamma^i\}$ which transform covariantly under gauge transformations [see Eq. (4.9b)], the connection field $B_\mu(x)$ acquires an *inhomogeneous piece* under gauge transformations [see Eq. (4.9a)]. Therefore, even when $B_\mu(x)$ is taken to vanish, we still have a nontrivial function $B_\mu^{g^\theta}(x)$ and it is this function that enters the anomaly functional in Eq. (4.13).

V. DISCUSSIONS

In this paper we have carried out the analysis of gauge and gravitational anomalies associated with quantized spin-$\frac{1}{2}$ matter fields in the context of a general $D=2n$ local field theory, renormalizable or not. It is established that *genuine* gauge or gravitational anomalies are only from usual minimal gauge and gravitational interactions of spin-$\frac{1}{2}$ fields; viz., in the genuine anomalies, we have no piece involving various possible nonminimal couplings or non-gauge-type external Bose fields in the theory. We have given our proof first by resorting to a suitable regularization scheme combined with a direct perturbation theoretic analysis (Secs. II and III). The finding has been supported by the argument based on certain general properties of the Euclidean space effective action and also by a topological consideration (Sec. IV). Our finding is consistent with the view that what matters for spinor anomalies under gauge symmetry (general covariance, local Lorentz symmetry) is not the details of the spinor Lagrangian but only the given *spinor field contents* under the gauge group. Although not dealt with explicitly in this paper, we expect some of our discussions to be also relevant for anomalies involving quantized spin-$\frac{3}{2}$ fields. That applies to so-called discrete anomalies[32] as well; in fact, we expect that the presence of additional nonminimal local couplings and non-gauge-type external fields will have no role on discrete anomalies, either. For supersymmetric field theories we do not know how our analysis will be affected by taking also supersymmetry transformation behaviors of the effective action into consideration.

What is the implication of our findings for low-energy effective theories? Note that, even for a low-energy phenomenological description of some theories, we may borrow the framework of a *local* Lagrangian quantum field theory and a gauge field theory in particular with effective dynamical degrees of freedom. [The standard model, a gauge theory based on the gauge group $SU(3)\times SU(2)\times U(1)$, may be regarded as a low-energy effective field theory from a certain unified model at more fundamental level.] An effective local Lagrangian will typically include renormalizable couplings[33] which play a dominant dynamical role, and nonrenormalizable couplings[34] (involving operators of dimension larger than four in the case of four spacetime dimensions) which may be less important dynamically but still crucial for some processes. There is a question which arises naturally in an effective low-energy (chiral or nonchiral) gauge theory with spinor fields. Will there not be certain restrictions to the structure of allowed higher dimensional local spinor couplings (besides naive gauge invariance of the forms) because of possible gauge anomaly problems? We can now give a definite answer to that—as long as the spinor field contents are such that the usual gauge anomaly cancellation condition[2] is satisfied, the effective gauge theory Lagrangian may include any gauge-invariant, renormalizable or nonrenormalizable, local spinor couplings without encountering gauge inconsistency by spinor loop effects.

The findings of this paper have also some relevance when the 't Hooft-type consistency conditions[35] are used to study the possible composite particle spectrum in a confining field theory. The 't Hooft consistency condition asserts that the anomaly produced by the fundamental fermions be identical to that of the composite massless fermions. The anomaly here refers to that in the vacuum expectation value of the three flavor (axial-)vector currents (or the anomalous piece in the effective action involving three spectator gauge fields). The flavor currents in question may be expressed either in terms of fundamental spinor fields of the preon theory, or in terms of spinor fields describing composite fermion degrees of freedom in the effective field theory; 't Hooft argued that the anomalies computed by both schemes should be equal, thus obtaining his consistency condition. (The Adler-Bardeen theorem[18] may be invoked to justify the use of the one-loop anomaly expression here.) In the original formulation 't Hooft modeled the effective field theory for composite particles also by a usual renormalizable field theory—a point with which many would not be satisfied. Our work shows the validity of the 't Hooft consistency condition even when the effective field theory is allowed to have various kinds of nonrenormalizable couplings (as might be more appropriate for an effective field theory which is not necessarily weakly coupled). This observation nicely complements the *S*-matrix-theoretic derivation[36] of the 't Hooft consistency condition. Our work also answers the following question: Can we obtain new kinds of 't Hooft-type consistency conditions by considering the anomalies in the spinor loop amplitudes involving (partly or wholly) local currents which are not vectors but tensors or something else? The answer is no. As far as internal-symmetry transformations are concerned, only the Ward-Takahashi relations involving certain numbers of usual (axial-)vector-type fermion bilinear currents may include genuine anomalous pieces. [This is a simple corollary of the fact that for genuine anomalies of our general theory (2.1), the corresponding minimal theory (2.11) is entirely responsible.]

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[17]One should remember here that anomalies can be understood strictly on the basis of long-wavelength physics (see Ref. 35). This in turn implies that anomalies do not really depend on possible modifications in short distance behaviors (as a cure for bad ultraviolet behaviors) of a given quantum field theory.

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[19]One may object to our using the Lagrangian path integral to define the effective action (in view of the fact that our Lagrangian may include *derivative* couplings). We hold the view that our Lagrangian (2.1) really corresponds to the spinor-dependent part of an *effective* Lagrangian. An effective Lagrangian of a theory is, by definition, the one with which one may use the Lagrangian path integral to describe the quantum theory, and in general it can be different from the classical Lagrangian. [In some cases one may account for the difference in the form of additional ghost interactions. See S. Coleman, in *Laws of Hadronic Matter,* edited by A. Zichichi (Academic, New York, 1975).]

[20](Mass) dimensions for various terms in the Lagrangian (2.12) can be assigned as follows: dimension 0 to all external Bose fields (and to M), dimension 1 for ∂_μ, and dimension $(D-1)/2$ for Fermi fields $(\psi,\bar{\psi})$. Also we limit ourselves to consideration of local Lagrangians with a finite upper bound in dimension.

[21]We will also have vertices associated with the B_μ- and Γ-independent bilinear terms in the Lagrangian (2.12):

$$\bar{\psi}(x)[a_0(0)+a_1(0)^\mu\partial_\mu+\cdots+a_r(0)^{\mu_1\cdots\mu_r}\partial_{\mu_1}\cdots\partial_{\mu_r}]\psi(x)\ .$$

Of course, allowing an infinite number of vertex insertions of this type on a free propagator is equivalent to simply introducing a new free propagator corresponding to the unperturbed Lagrangian given by the expression (2.13) plus the above terms. In the context of our partially regularized theory (2.12) (with large, but finite, M), this equivalence applies even for *internal* propagators of various Feynman diagrams contributing to $W_{\bar{R}}[B_\mu,\Gamma]$; our regularization is such that no infinity may arise as a result of vertex insertions of the above type.

[22]Note that, in the Lagrangian (2.1) or in the form (2.12), the terms $\bar{\psi}(x)f[D_\mu,\Gamma(x)]\psi(x)$ and $\mathscr{L}'(x)$ define gauge-invariant interactions separately.

[23]See, for instance, the first two references cited in Ref. 5.

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[25]Here it should be clear that as far as general covariance and local Lorentz symmetry are concerned, the interaction term involving gauge fields B_μ may be also regarded as a sort of nonminimal coupling.

[26]The well-known gravitational contribution to the axial-vector current anomaly in $D=4$ is a good example [here B_μ taken by a real or spectator $U_A(1)$ gauge field]. The axial anomaly in a gravitational background has been studied some time ago

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[27]For a topological understanding of pure gravitational anomalies, readers may consult Refs. 7 and 9. See also O. Alvarez, I. M. Singer, and B. Zumino, Commun. Math. Phys. **96**, 409 (1984).

[28]In general (especially with nonminimal couplings in the theory), more than one Pauli-Villars regulator fields will be necessary to secure a well-defined effective action.

[29]For a good review on these concepts, see R. Jackiw, in *Current Algebra and Anomalies,* edited by S. B. Treiman, R. Jackiw, B. Zumino, and E. Witten (World Scientific, Singapore, 1985), Sec. 6.

[30]Even with some very special, not manifestly gauge-invariant, couplings (which could be of a topological origin) the effective action may still follow the general behavior (4.10) under gauge transformations. But such couplings are not considered in this paper.

[31]A similar argument has been used by Witten to demonstrate the parameter independence of his index $\mathrm{Tr}(-1)^F$. See E. Witten, Nucl. Phys. **B202**, 253 (1982).

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