Supercurrents and superconformal symmetry

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A superspace variational formulation of supercurrents and their associated anomalies is presented, with emphasis on showing its foundation on superconformal symmetry. Parameters of superconformal transformations are embedded in a spinor superfield, in terms of which local-superconformal variations of superfields are defined. The one-loop supercurrent anomalies are calculated for the Wess-Zumino model and supersymmetric QED by the path-integral method.

I. INTRODUCTION

Superconformal symmetry[1–5] plays an important role in characterizing the space-time symmetry contents and short-distance structures of supersymmetric theories. All the superconformal currents are embedded in a single superfield, called the supercurrent.[6–8] The supercurrent possesses a quantum anomaly,[5–9] which arranges the anomalies of the R symmetry, special supersymmetry, and dilation symmetry in a supermultiplet.

The original derivation of the supercurrent, by Ferrara and Zumino,[6] made use of Noether's variational procedure for component fields. The direct superspace version of the Noether procedure, however, seems to be missing yet. As an alternative, super-Weyl field variations have been used to define supercurrents via the supergravity versions of flat-superspace theories.[5] Standard (and nonvariational) constructions[7,8] of supercurrents employ symmetry arguments based on the superconformal algebra.[2,3]

Fujikawa's path-integral method[10,11] is a useful means of formulating anomalies in field theory; known anomalies (chiral, conformal, and gravitational ones) are identified with Jacobian factors in a systematic manner. The knowledge of a superspace Noether theorem is needed for its extension to supercurrent anomalies.

The purpose of this paper is to present a superspace Noether theorem for supercurrents and to develop, by means of it, the path-integral formulation of supercurrent anomalies. Complementary to the super-Weyl approach[5] is our flat-superspace approach, where the foundation of supercurrents and superfield variations on superconformal symmetry is made manifest.

We shall extract the necessary superfield variations out of known supercurrents of some specific models and interpret them in terms of superconformal symmetry. Parameters of superconformal transformations turn out to be embedded in a spinor superfield, in terms of which (local) superconformal field variations are defined. In gauge models, these field variations are combined with gauge transformations to form gauge-covariant superfield variations.

A superspace variational formulation of supercurrents and their anomalies is developed in Sec. II. Local superconformal variations of chiral superfields and vector superfields are derived. Their interpretation in terms of superconformal symmetry is studied in Sec. III. The path-integral calculation of supercurrent anomalies is given for the Wess-Zumino model and supersymmetric quantum electrodynamics (SQED) in Sec. IV. Section V is devoted to concluding remarks; in particular, a connection of R symmetry and chiral symmetry in supersymmetric gauge models is discussed.

II. VARIATIONAL CHARACTERIZATION OF SUPERCURRENTS

In this section, we look for Noether's procedure for supercurrents. Let us first consider the Wess-Zumino model,[1] describing self-coupled chiral (and antichiral) superfields $\Phi(z)$ and $\overline{\Phi}(z)$, with the action

$$S[\Phi,\overline{\Phi}]=\int d^8z\,\overline{\Phi}\Phi+\left[\int d^6z\left(\tfrac{1}{2}m\Phi^2-\tfrac{1}{6}g\Phi^3\right)+\text{H.c.}\right], \tag{2.1}$$

where $d^8z=d^4x\,d^2\theta\,d^2\overline{\theta}$, $d^6z=d^4x\,d^2\theta$, etc.[12] Only the mass term breaks invariance under the (global) R transformation, $\Phi(x,\theta,\overline{\theta})\rightarrow e^{-in\beta}\Phi(x,e^{i\beta}\theta,e^{-i\beta}\overline{\theta})$ with a real parameter β, if we assign R weight $n=\frac{2}{3}$ to $\Phi(z)$.

The supercurrent contains the R-symmetry current as its lowest component.[6] Its variational formulation should therefore make use of a suitable superfield generalization of the *local-R* transformation. For the chiral superfield $\Phi(z)$ (of R weight $\frac{2}{3}$) and its antichiral partner $\overline{\Phi}(z)$, such field variations turn out to take the form

$$\begin{aligned}\delta\Phi(z)&=-\tfrac{1}{4}\overline{D}^2[\Omega^\alpha D_\alpha+\tfrac{1}{3}(D^\alpha\Omega_\alpha)]\Phi(z)\,,\\ \delta\overline{\Phi}(z)&=-\tfrac{1}{4}D^2[\overline{\Omega}_{\dot\alpha}\overline{D}^{\dot\alpha}+\tfrac{1}{3}(\overline{D}_{\dot\alpha}\overline{\Omega}^{\dot\alpha})]\overline{\Phi}(z)\,,\end{aligned} \tag{2.2}$$

where $\Omega^\alpha(z)$ is an arbitrary spinor superfield and $\overline{\Omega}^{\dot\alpha}(z)$ its Hermitian conjugate. The derivation of these formulas will soon become clear. Here we simply remark that, with Ω_α set equal to $i\theta_\alpha\overline{\theta}^2\beta$ (β is a real parameter), Eq. (2.2) gives the correct R transformation laws of Φ and $\overline{\Phi}$.

In response to the field variations (2.2), the interaction terms in the Wess-Zumino action change by total superspace divergences $\delta\Phi^3=-\frac{1}{4}\overline{D}^2D^\alpha(\Omega_\alpha\Phi^3)$ and $\delta\overline{\Phi}^3$

$=-\frac{1}{4}D^2\bar{D}_{\dot\alpha}(\bar\Omega^{\dot\alpha}\bar\Phi^3)$. At the same time, the action undergoes the change

$$\delta S=\int d^8z[\tfrac{1}{2}(D^\alpha\bar\Omega^{\dot\alpha}-\bar D^{\dot\alpha}\Omega^\alpha)R_{\alpha\dot\alpha} +\tfrac{1}{6}m(\Omega^\alpha D_\alpha\Phi^2+\bar\Omega_{\dot\alpha}\bar D^{\dot\alpha}\bar\Phi^2)]\ , \tag{2.3}$$

where $R_{\alpha\dot\alpha}$ is the supercurrent[6]

$$R_{\alpha\dot\alpha}=(D_\alpha\Phi)(\bar D_{\dot\alpha}\bar\Phi)-\tfrac{1}{3}[D_\alpha,\bar D_{\dot\alpha}](\bar\Phi\Phi)\ . \tag{2.4}$$

On the other hand, the variation δS is represented as

$$\delta S=\int d^6z(\delta\Phi)\delta S/\delta\Phi+\int d^6\bar z(\delta\bar\Phi)\delta S/\delta\bar\Phi\ , \tag{2.5}$$

which we rearrange in the form

$$\delta S=\int d^8z[\Omega^\alpha(\delta S/\delta\Omega^\alpha)+\bar\Omega_{\dot\alpha}(\delta S/\delta\bar\Omega_{\dot\alpha})]\ , \tag{2.6}$$

so that, e.g.,

$$\delta S/\delta\Omega^\alpha=(\delta/\delta\Omega^\alpha)\int d^6z(\delta\Phi)\delta S/\delta\Phi\ .$$

Then, varying Eq. (2.3) with respect to Ω^α and $\bar\Omega^{\dot\alpha}$ yields the local conservation laws of the supercurrent:

$$\begin{aligned}\bar D^{\dot\alpha}R_{\alpha\dot\alpha}-\tfrac{1}{3}mD_\alpha(\Phi^2)&=-2\delta S/\delta\Omega^\alpha\ ,\\ D^\alpha R_{\alpha\dot\alpha}-\tfrac{1}{3}m\bar D_{\dot\alpha}(\bar\Phi^2)&=2\delta S/\delta\bar\Omega^{\dot\alpha}\ .\end{aligned} \tag{2.7}$$

The right-hand side of Eqs. (2.7) may appear to vanish by virtue of the equations of motion ($\delta S/\delta\Phi=0$, etc.). This is not always the case. In the path-integral treatment[10] of anomalies, it is these equation-of-motion terms that are regularized to turn into (supercurrent) anomalies at the quantum level.

Note that Eqs. (2.3) and (2.7) are (operator) identities. Accordingly, if $R_{\alpha\dot\alpha}$ is known, they serve to determine $\delta S/\delta\Omega^\alpha$ and $\delta S/\delta\bar\Omega^{\dot\alpha}$, which in turn fix the field variations $\delta\Phi$ and $\delta\bar\Phi$. This is in fact the way we have obtained the superfield variations in Eq. (2.2).

We adopt the same strategy to derive the transformation law of the vector superfield. Let us consider supersymmetric quantum chromodynamics (SQCD), with the gauge-invariant action[12]

$$S=\int d^8z(\bar\Phi e^V\Phi+\chi e^{-V}\bar\chi)+(8g^2)^{-1}\mathrm{tr}\int d^6z\, W^\alpha W_\alpha\ , \tag{2.8}$$

where $\Phi(z)$ and $\chi(z)$ are a pair of SU(N)-fundamental matter chiral superfields of opposite color charge and $V(z)$ is the (color) vector superfield; $W_\alpha\equiv-\frac{1}{4}\bar D^2(e^{-V}D_\alpha e^V)$. For simplicity, matter fields are treated as massless. (Rescale $V\to 2gV$ and $W\to 2gW$ to recover the coupling constant g.) The massless SQCD action is invariant under global R transformations. Here, we do not take explicit account of the gauge-fixing term and the ghost sector, which are not needed in our present analysis.

The SQCD supercurrent is a gauge-invariant object (apart from the gauge-fixing complication). Correspondingly, we replace the matter-field variations in Eq. (2.2) by the gauge-covariant version:

$$\Delta\Phi=-\tfrac{1}{4}\bar D^2[\Omega^\alpha\nabla_\alpha\Phi+\tfrac{1}{3}(D\Omega)\Phi]\ , \tag{2.9}$$

$$\Delta\bar\chi=-\tfrac{1}{4}D^2[\bar\Omega_{\dot\alpha}\bar\nabla^{\dot\alpha}\bar\chi+\tfrac{1}{3}(\bar D\bar\Omega)\bar\chi]\ , \tag{2.10}$$

where $D\Omega\equiv D^\alpha\Omega_\alpha$ and $\bar D\bar\Omega\equiv\bar D_{\dot\alpha}\bar\Omega^{\dot\alpha}$. Here

$$\nabla_\alpha\equiv e^{-V}D_\alpha e^V=D_\alpha+Y_\alpha$$

and

$$\bar\nabla_{\dot\alpha}\equiv e^V\bar D_{\dot\alpha}e^{-V}=\bar D_{\dot\alpha}-\bar Y_{\dot\alpha}$$

are gauge-covariant spinor derivatives:

$$Y_\alpha\equiv e^{-V}(D_\alpha e^V),\quad \bar Y_{\dot\alpha}\equiv(Y_\alpha)^\dagger=(\bar D_{\dot\alpha}e^V)e^{-V}\ ,$$

and

$$(\nabla_\alpha)^\dagger=-\bar\nabla_{\dot\alpha}\ .$$

The $\Delta\bar\Phi$ is given by the Hermitian conjugate of Eq. (2.9):

$$\Delta\bar\Phi=-\tfrac{1}{4}D^2[\bar\Omega_{\dot\alpha}(\bar\nabla^{\dot\alpha}*\bar\Phi)+\tfrac{1}{3}(\bar D\bar\Omega)\bar\Phi]\ , \tag{2.11}$$

where $\bar\nabla^{\dot\alpha}*\bar\Phi\equiv\bar D^{\dot\alpha}\bar\Phi+\bar\Phi\bar Y^{\dot\alpha}$; similarly $\nabla_\alpha*\chi\equiv D_\alpha\chi-\chi Y_\alpha$ for $\Delta\chi$.

For the vector superfield $V(z)$, we try the simplest gauge-covariant field variation

$$\Delta e^V=-e^V\Omega^\alpha W_\alpha-\bar\Omega_{\dot\alpha}\bar W^{\dot\alpha}e^V\ , \tag{2.12}$$

where $W_\alpha=-\frac{1}{4}\bar D^2Y_\alpha$ and $\bar W_{\dot\alpha}=-\frac{1}{4}D^2\bar Y_{\dot\alpha}$. With Ω_α set equal to $i\theta_\alpha\bar\theta^2\beta$, Eq. (2.12) is seen to give the correct R-transformation laws of the component fields in the Wess-Zumino gauge:[12] $\delta\lambda_\alpha=-i\beta\lambda_\alpha$ and $\delta\bar\lambda_{\dot\alpha}=i\beta\bar\lambda_{\dot\alpha}$ for the gluino field and $\delta v_\mu=0$ for the gluon field.

Applying these field varations to the SQCD action, one obtains the gauge-invariant supercurrent

$$\begin{aligned}R^{\mathrm{SQCD}}_{\alpha\dot\alpha}=&-(\bar\nabla_{\dot\alpha}*\bar\Phi)e^V(\nabla_\alpha\Phi)+(\nabla_\alpha*\chi)e^{-V}(\bar\nabla_{\dot\alpha}\bar\chi)\\ &-\tfrac{1}{3}[D_\alpha,\bar D_{\dot\alpha}](\bar\Phi e^V\Phi+\chi e^{-V}\bar\chi)\\ &-(2g^2)^{-1}\mathrm{tr}(W_\alpha e^{-V}\bar W_{\dot\alpha}e^V)\ ,\end{aligned} \tag{2.13}$$

and the conservation laws in Eqs. (2.3) and (2.7) with m set equal to zero. From this supercurrent $R^{\mathrm{SQCD}}_{\alpha\dot\alpha}$, known in Ref. 8, one can derive the field transformation laws (2.9)—(2.12) in the reversed line of argument as given earlier. In the SQCD case, Eq. (2.5) reads

$$\delta S=\mathrm{tr}\int d^8z(\Delta e^V)\delta S/\delta e^V+\int d^6z(\Delta\Phi)\delta S/\delta\Phi+\cdots\ .$$

The $\Delta\Phi$ and $\delta\Phi$ differ by the gauge variation

$$\delta_G\Phi=-\tfrac{1}{4}(\bar D^2\Omega^\alpha Y_\alpha)\Phi\ . \tag{2.14}$$

Accordingly, we may subtract from Δe^V the corresponding gauge variation

$$\delta_G e^V=e^V\tfrac{1}{4}(\bar D^2\Omega^\alpha Y_\alpha)+\tfrac{1}{4}(D^2\bar\Omega_{\dot\alpha}\bar Y^{\dot\alpha})e^V \tag{2.15}$$

to define the new variation

$$\begin{aligned}\delta e^V&=\Delta e^V-\delta_G e^V\\ &=f^\alpha D_\alpha e^V+\bar f_{\dot\alpha}\bar D^{\dot\alpha}e^V-e^V\tfrac{1}{2}(\bar D^{\dot\alpha}\Omega^\alpha)\bar D_{\dot\alpha}Y_\alpha\\ &\quad-\tfrac{1}{2}(D^\alpha\bar\Omega^{\dot\alpha})(D_\alpha\bar Y_{\dot\alpha})e^V\ ,\end{aligned} \tag{2.16}$$

where $f^\alpha=-\frac{1}{4}\bar D^2\Omega^\alpha$ and $\bar f_{\dot\alpha}=-\frac{1}{4}D^2\bar\Omega_{\dot\alpha}$.

It is instructive to look into the Abelian (SQED) case, where Eqs. (2.12) and (2.16) are simplified to

$$\Delta V=-\Omega^\alpha W_\alpha-\bar{\Omega}_{\dot\alpha}\bar{W}^{\dot\alpha}\,, \tag{2.17}$$

$$\delta V=f^\alpha D_\alpha V+\bar{f}_{\dot\alpha}\bar{D}^{\dot\alpha}V-\tfrac{1}{2}(\bar{D}^{\dot\alpha}\Omega^\alpha)\bar{D}_{\dot\alpha}D_\alpha V -\tfrac{1}{2}(D^\alpha\bar{\Omega}^{\dot\alpha})D_\alpha\bar{D}_{\dot\alpha}V\,. \tag{2.18}$$

It is these $\delta\Phi$, δe^V, and δV that admit of a natural interpretation in terms of superconformal symmetry, as shown in Sec. III. Hence, the gauge-invariant supercurrent (2.13) of SQCD (and SQED) is characterized by the combined field variations of superconformal symmetry and the gauge symmetry. The combined transformations may be regarded as the superconformal analogue of covariant conformal transformations.[13]

III. SUPERCONFORMAL TRANSFORMATIONS

We shall now explore the physical meaning of the superfield variations derived in Sec. II. Let us first survey superconformal symmetry, whose algebra[2–4] consists of the conformal algebra [with generators $(P_\mu,M_{\mu\nu},D,K_\mu)$] combined with supertranslation $(Q_\alpha,\bar{Q}_{\dot\alpha})$, special supersymmetry $(S_\alpha,\bar{S}_{\dot\alpha})$ and R symmetry (R). We use the superspace notation $z^A=(\theta^\alpha,\bar\theta_{\dot\alpha},x^\mu)$ and $\partial_A=\partial/\partial z^A=(\partial_\alpha,\partial^{\dot\alpha},\partial_\mu)$. The exterior derivative $d=dz^A\partial_A$, when rewritten as $d=e^AD_A$ in terms of the covariant derivatives $D_A=(D_\alpha,\bar{D}^{\dot\alpha},\partial_\mu)$, defines the supertranslation-invariant basis $\{e^A\}$ of differentials:[12]

$$e^\alpha=d\theta^\alpha,\quad e_{\dot\alpha}=d\bar\theta_{\dot\alpha}\,,$$

and

$$e^\mu=dx^\mu+i\,d\theta\,\sigma^\mu\bar\theta+i\,d\bar\theta\,\bar\sigma^\mu\theta\,. \tag{3.1}$$

Following the standard treatment of conformal symmetry,[13] we shall set up the superconformal Killing equations. Consider the (infinitesimal) general coordinate transformations

$$z^A\to z'^A=z^A+f^A(z) \tag{3.2}$$

that scale the invariant length $ds^2=e^\mu e_\mu$ locally so that $e'^\mu e'_\mu=c(z)e^\mu e_\mu$. This condition amounts to the following set of constraints on $f^A(z)=(f^\alpha(z),\bar{f}_{\dot\alpha}(z),f^\mu(z))$:

$$\bar{D}^{\dot\alpha}h_\mu+2i(\bar\sigma_\mu f)^{\dot\alpha}=0\,, \tag{3.3}$$

$$D_\alpha h_\mu+2i(\sigma_\mu\bar{f})_\alpha=0\,, \tag{3.4}$$

$$\partial_\mu h_\nu+\partial_\nu h_\mu=\tfrac{1}{2}(\partial^\lambda h_\lambda)g_{\mu\nu}\,, \tag{3.5}$$

where

$$h_\mu(z)\equiv f_\mu(z)+if(z)\sigma_\mu\bar\theta+i\bar{f}(z)\bar\sigma_\mu\theta\,. \tag{3.6}$$

Of these only Eqs. (3.3) and (3.4) are independent, and Eq. (3.5) follows from them. The spin-$\frac{3}{2}$ parts of Eqs. (3.3) and (3.4) imply the relation $D^2h_\mu=\bar{D}^2h_\mu=0$, which is equivalent to the chirality constraint on f_α and $\bar{f}_{\dot\alpha}$:

$$\bar{D}_{\dot\gamma}f_\alpha=D_\gamma\bar{f}_{\dot\alpha}=0\,. \tag{3.7}$$

It is a simple exercise to show that Eqs. (3.3)—(3.7) (the superconformal Killing equations[14,15]) combine to define uniquely the (infinitesimal) superconformal coordinate transformations of $15+4+4+1=24$ real parameters; see Appendix A. In particular,

$$f_\alpha(z)=f_\alpha(x,\theta,\bar\theta)=f_\alpha(x-i\theta\sigma\bar\theta,\theta,0)$$

and

$$\bar{f}_{\dot\alpha}(z)=\bar{f}_{\dot\alpha}(x,\theta,\bar\theta)=\bar{f}_{\dot\alpha}(x+i\theta\sigma\bar\theta,0,\bar\theta)$$

have the parametrizations

$$\begin{aligned}f_\alpha(x,\theta,0)&=f_\alpha^{(0)}+(i\beta+\tfrac{1}{8}\partial^\mu f_\mu^{(0)})\theta_\alpha -\tfrac{1}{8}f_{\mu\nu}^{(0)}(\sigma^\mu\bar\sigma^\nu\theta)_\alpha-i\tfrac{1}{2}\theta^2(\partial\!\!\!/\bar{f}^{(0)})_\alpha\,,\\ \bar{f}^{\dot\alpha}(x,0,\bar\theta)&=\bar{f}^{(0)\dot\alpha}+(-i\beta+\tfrac{1}{8}\partial^\mu f_\mu^{(0)})\bar\theta^{\dot\alpha} -\tfrac{1}{8}f_{\mu\nu}^{(0)}(\bar\sigma^\mu\sigma^\nu\bar\theta)^{\dot\alpha}-i\tfrac{1}{2}\bar\theta^2(\bar\partial\!\!\!/ f^{(0)})^{\dot\alpha}\,,\end{aligned} \tag{3.8}$$

where $f_{\mu\nu}^{(0)}\equiv\partial_\mu f_\nu^{(0)}-\partial_\nu f_\mu^{(0)}$ and $\partial\!\!\!/\equiv\sigma^\mu\partial_\mu$. Here $f_\alpha^{(0)}(x)$, $\bar{f}_{\dot\alpha}^{(0)}(x)$, and $f_\mu^{(0)}(x)$ denote the lowest components of $f_\alpha(z)$, $\bar{f}_{\dot\alpha}(z)$, and $f_\mu(z)$, respectively. The $f_\mu^{(0)}(x)$ is nothing but the usual conformal Killing vector of 15 parameters. Supertranslation and special supersymmetry are parametrized by the Killing spinors[1] $f_\alpha^{(0)}(x)$ and $\bar{f}_{\dot\alpha}^{(0)}(x)$, which are most linear in x_μ; i.e., $f_\alpha^{(0)}(x)=\xi_\alpha+x^\mu(\sigma_\mu\bar\zeta)_\alpha$. A real constant β parametrizes the R transformation. The $f_\mu(z)$ is expressed as a sum of a chiral vector superfield and its conjugate; see Appendix A.

For the combination $\rho D+\beta R$ of dilation and R transformation, $f^\alpha=(\frac{1}{2}\rho+i\beta)\theta^\alpha$, $\bar{f}_{\dot\alpha}=(\frac{1}{2}\rho-i\beta)\bar\theta_{\dot\alpha}$, and $f^\mu=\rho x^\mu$. Consider a real scalar superfield $\Phi(z)$ of mass dimension d and R weight n. The superconformal field variation, which takes correct account of the dilation and R-transformation properties, is written as

$$\delta^f\Phi(z)=[f^A\partial_A+\tfrac{1}{2}d(\partial^Af_A)+\tfrac{1}{4}n(Df-\bar{D}\bar{f})]\Phi(z)\,, \tag{3.9}$$

where $Df\equiv D^\alpha f_\alpha$ and $\bar{D}\bar{f}\equiv\bar{D}_{\dot\alpha}\bar{f}^{\dot\alpha}$:

$$f^A\partial_A\equiv f^\alpha\partial_\alpha+\bar{f}_{\dot\alpha}\partial^{\dot\alpha}+f^\mu\partial_\mu=f^\alpha D_\alpha+\bar{f}_{\dot\alpha}\bar{D}^{\dot\alpha}+h^\mu\partial_\mu\,.$$

A chiral superfield $\Phi(z)$ must satisfy the relation $d=\frac{3}{2}n$ because of the chirality constraint $\bar{D}\delta^f\Phi=0$.

The Killing equations (3.3)—(3.7) characterize (global) superconformal transformations. The Noether currents, on the other hand, are constructed by use of local-symmetry transformations. Accordingly, the Killing equations have to be relaxed to admit suitable local transformations. To this end, we replace Eqs. (3.3) and (3.4) by

$$\bar{D}^{\dot\alpha}k_\mu+2i(\bar\sigma_\mu f)^{\dot\alpha}=0\,, \tag{3.10}$$

$$D_\alpha\bar{k}_\mu+2i(\sigma_\mu\bar{f})_\alpha=0\,. \tag{3.11}$$

Here $k_\mu(z)$ is a complexification of $h_\mu(z)$ such that $h_\mu(z)=\frac{1}{2}[k_\mu(z)+\bar{k}_\mu(z)]$; $\bar{k}_\mu(z)=[k_\mu(z)]^\dagger$. From Eqs. (3.10) and (3.11) follows again the chirality constraint (3.7), which is satisfied by setting

$$f_\alpha = -\tfrac{1}{4}\bar{D}^2\Omega_\alpha \text{ and } \bar{f}_{\dot\alpha} = -\tfrac{1}{4}D^2\bar{\Omega}_{\dot\alpha}\ , \tag{3.12}$$

where $\Omega_\alpha(z)$ and its conjugate $\bar{\Omega}_{\dot\alpha}(z)$ are unconstrained spinor superfields. With Ω_α and $\bar{\Omega}_{\dot\alpha}$, Eqs. (3.10) and (3.11) can be solved for k_μ and $\bar{k}_\mu$:

$$k_\mu = -i\bar{D}\bar{\sigma}_\mu\Omega\ , \tag{3.13}$$

$$\bar{k}_\mu = -iD\sigma_\mu\bar{\Omega}\ . \tag{3.14}$$

Conversely, Eqs. (3.12)–(3.14) combine to fix Ω_α in terms of f_α and k_μ:

$$\Omega_\alpha(z) = \psi_\alpha(y) - i\tfrac{1}{2}F^\mu(y,\theta)(\sigma_\mu\bar\theta)_\alpha + \bar\theta^2 f_\alpha(z)\ , \tag{3.15}$$

where

$$y_\mu \equiv x_\mu - i\theta\sigma_\mu\bar\theta$$

and

$$F_\mu(y,\theta) = k_\mu(z) + 2i\bar\theta\bar\sigma_\mu f(z)\ .$$

The chiral component $\psi_\alpha(y)$ of $\Omega_\alpha(z)$ is a kind of gauge degree of freedom. The chiral vector superfield $F_\mu(y,\theta)$ may be regarded as a complexification of $f_\mu(z)$ since

$$f_\mu(z) = \tfrac{1}{2}[F_\mu(y,\theta) + \bar{F}_\mu(y^\dagger,\bar\theta)]$$

holds

Global superconformal transformations correspond to the parametrization [see Eq. (A4) in Appendix A]

$$F_\mu(y,\theta) = f_\mu^{(0)}(y) - 2i\theta\sigma_\mu\bar{f}^{(0)}(y) \tag{3.16}$$

and $(f_\alpha(z), \bar{f}_{\dot\alpha}(z))$ given in Eq. (3.8). In this way, $\Omega_\alpha(z)$, when properly constrained, embodies the superconformal Killing vector and spinors.

Let us turn to field transformation laws. For a chiral superfield of R weight n (and $d = \frac{3}{2}n$), Eq. (3.9) reads

$$\delta^f\Phi(z) = [f^\alpha D_\alpha + h^\mu\partial_\mu + \tfrac{1}{2}n(\partial^\mu h_\mu + D^\alpha f_\alpha)]\Phi(z)\ , \tag{3.17}$$

where we have rewritten the R-weight term using the relation

$$Df + \bar{D}\bar{f} = -\partial^A f_A = -\tfrac{1}{2}\partial^\mu h_\mu\ , \tag{3.18}$$

which follows from the Killing equations. We generalize Eq. (3.17) by substituting $k^\mu(z)$ for $h^\mu(z)$ so that the new field variation is induced by $\Omega_\alpha(z)$. The result is cast in the compact form

$$\delta\Phi(z) = -\tfrac{1}{4}\bar{D}^2[\Omega^\alpha D_\alpha + \tfrac{1}{2}n(D^\alpha\Omega_\alpha)]\Phi(z)\ , \tag{3.19}$$

which is the local-superconformal field variation quoted in Eq. (2.2). In the same way, the superconformal variation of the antichiral superfield is promoted to the field variation [in Eq. (2.2)] generated by $\bar\Omega_{\dot\alpha}$.

A real scalar superfield has $n = d = 0$, with the global-superconformal transformation law

$$\delta^f V(z) = (f^\alpha D_\alpha + \bar{f}_{\dot\alpha}\bar{D}^{\dot\alpha} + h^\mu\partial_\mu)V(z)\ . \tag{3.20}$$

The photon and the Yang-Mills superfields (and e^V as well) obey this rule. Note that the substitution

$$h^\mu\partial_\mu = -\tfrac{1}{4}ih^\mu(D\sigma_\mu\bar{D} + \bar{D}\bar\sigma_\mu D)$$
$$\to -\tfrac{1}{4}i(k^\mu\bar{D}\bar\sigma_\mu D + \bar{k}^\mu D\sigma_\mu\bar{D})$$

immediately converts Eq. (3.20) into the local-superconformal variation (2.18) of the photon superfield. Likewise, on setting $k^\mu = \bar{k}^\mu = h^\mu$, the generalized transformation law (2.16) of the Yang-Mills superfield is reduced to the above global one. [Actually, Eqs. (2.16) and (2.18) differ only by nonlinear terms proportional to $k_\mu - \bar{k}_\mu$.]

Varying the actions with respect to unconstrained Ω_α and $\bar\Omega_{\dot\alpha}$, we have obtained the supercurrents [Eqs. (2.4) and (2.13)] and the associated conservation laws [Eq. (2.7)]; the latter may be called the generalized trace identities in view of their component structures.[7] Suppose now that we restrict Ω_α by setting

$$\Omega_\alpha(z) = iD_\alpha\Xi(z)\quad (\bar\Omega_{\dot\alpha} = -i\bar{D}_{\dot\alpha}\Xi)\ ,$$

where $\Xi(z)$ is a real scalar superfield. Then the variation of the action, Eq. (2.3), with respect to $\Xi(z)$ gives rise to the divergence of the supercurrent

$$\delta S = -2\int d^8z\,\Xi[\partial^\mu R_\mu + i\tfrac{1}{12}m(D^2\Phi^2 - \bar{D}^2\bar\Phi^2)]\ , \tag{3.21}$$

where $R_\mu \equiv \frac{1}{2}(\bar\sigma_\mu)^{\dot\alpha\alpha}R_{\alpha\dot\alpha}$. This variation, of course, carries less information than the original one.[16] Its physical content is made clear by comparing the component structure of $iD_\alpha\Xi$ with the (global) superconformal case [Eqs. (3.8), (3.15), and (3.16)]: It contains only translation (P_μ), supertranslation ($Q_\alpha, \bar{Q}_{\dot\alpha}$) and R transformation (R) of the superconformal algebra. Their respective parameters ($a_\mu, \xi_\alpha, \bar\xi_{\dot\alpha}, \beta$) are arranged in the form of an x-independent superfield:

$$\Xi \to -\tfrac{1}{2}\theta\sigma^\mu\bar\theta a_\mu - i\bar\theta^2\theta\xi + i\theta^2\bar\theta\,\bar\xi + \tfrac{1}{2}\theta^2\bar\theta^2\beta\ . \tag{3.22}$$

A consequence of this structure is that, when the supercurrent is conserved $\partial^\mu R_\mu = 0$, $\int d^3x R_0$ becomes a constant superfield representing the conserved charges[7] ($P_\mu, Q_\alpha, \bar{Q}_{\dot\alpha}, R$):

$$\int d^3x R_0 = R - i\theta Q + i\bar\theta\bar{Q} - 2\theta\sigma^\mu\bar\theta P_\mu\ . \tag{3.23}$$

IV. SUPERCURRENT ANOMALIES

In this section we study the anomalies of supercurrents. We avoid using Pauli-Villars regulator fields, use of which is sometimes problematic in supersymmetric calculations. Here we follow the algorithm of the path-integral approach[10] and regularize the variations $\delta S/\delta\Omega_\alpha$ and $\delta S/\delta\bar\Omega_{\dot\alpha}$ in Eqs. (2.7) to derive the anomalies. The basic program for the calculation is analogous to that of conformal anomalies.[17] It has been noted, in particular, that the algorithm of the path-integral derivation of anomalies has natural foundation within ζ-function regularization[18] of quantum theories.

We begin with the Wess-Zumino model. Eqs. (2.3), (2.5), and (2.7), being operator identities, are also valid as the Green's functions. The source term can be included into them by the replacement

$$S \to S^J = S + \int d^6z\, J\Phi + \int d^6\bar{z}\, \bar{J}\bar{\Phi} \ . \tag{4.1}$$

Then Eq. (2.3) is equivalent to the Ward-Takahashi identity of the Green's-function form

$$\langle \delta S + \int d^6z\, J\Phi + \int d^6\bar{z}\, \bar{J}\bar{\Phi} - \mathscr{A} \rangle = 0 \ . \tag{4.2}$$

Here δS stands for the right-hand side of Eq. (2.3) while $\mathscr{A}$ denotes the equation-of-motion representation of δS^J:

$$\mathscr{A} = \int d^6z\, \delta\Phi(\delta S^J/\delta\Phi) + \int d^6\bar{z}\, \delta\bar{\Phi}(\delta S^J/\delta\bar{\Phi}) \ . \tag{4.3}$$

It is this $\mathscr{A}$ that turns into the anomaly.

The quantum structure of the Wess-Zumino action (2.1) is extracted by use of the classical (ϕ) and quantum (η) splitting of the superfield $\Phi(z) = \phi(z) + \eta(z)$. In particular, the one-loop structure of the theory is described by the quadratic quantum action, written in the matrix form

$$S_2[\phi;\eta] = \tfrac{1}{2}(\eta,\bar{\eta}) \begin{bmatrix} (m - g\phi)1_- & 1_-1_+ \\ 1_+1_- & (m - g\bar{\phi})1_+ \end{bmatrix} \begin{bmatrix} \eta \\ \bar{\eta} \end{bmatrix} \tag{4.4}$$

$$\equiv \tfrac{1}{2} X^t \cdot \Gamma[\phi,\bar{\phi}] \cdot X \ , \tag{4.5}$$

where $X \equiv (\eta,\bar{\eta})^t$ and $\Gamma[\phi,\bar{\phi}]$ are matrices in the chiral and antichiral sectors of superspace. The dot implies a summation over superspace coordinate labels of appropriate chirality using d^6z or $d^6\bar{z}$. The operators $1_- \equiv -\frac{1}{4}\bar{D}^2$ and $1_+ \equiv -\frac{1}{4}D^2$ combine with the chiral measures to serve as δ functions;[12,19] e.g., $1_- \cdot 1_- = 1_-$ and $1_- \cdot \phi = \phi$.

The variation δ (on Φ), being a linear operation, acts on ϕ and η separately. Let us extract from $\mathscr{A}$ terms quadratic in the quantum fields η and $\bar{\eta}$. They, when suitably regularized, determine the one-loop anomaly. As a matter of fact, in the path-integral approach their effect is expressed as the Jacobian associated with the change of field variables $X \to X' = X + \delta X$ with $\delta X = (\delta\eta, \delta\bar{\eta})^t$. We shall evaluate this Jacobian. It is convenient to write δX in the matrix form $\delta X = B \cdot X$, where B is the diagonal matrix in the chiral sectors of superspace:

$$B = \mathrm{diag}(1_-[\Omega D + \tfrac{1}{3}(D\Omega)]1_- \ , \\ 1_+[\bar{\Omega}\bar{D} + \tfrac{1}{3}(\bar{D}\bar{\Omega})]1_+) \ . \tag{4.6}$$

Then the Jacobian is given by $\exp(\mathrm{Tr}B)$ and the anomaly $\langle \mathscr{A} \rangle$ is written as

$$\langle \mathscr{A} \rangle = i \langle \mathrm{Tr}B \rangle \ . \tag{4.7}$$

This Jacobian needs short-distance regularization, which may be carried out by setting

$$\mathrm{Tr}B^{\mathrm{reg}} = \mathrm{Tr}(B \cdot e^{\tau\Gamma[\phi,\bar{\phi}]^2}) \ \ (\tau \to 0_+) \ , \tag{4.8}$$

where a Wick rotation to Euclidean space has been assumed. Here $e^{\tau\Gamma^2}$, with $\Gamma^2 \equiv \Gamma \cdot \Gamma$, is defined in terms of the dot product:

$$e^{\tau\Gamma^2} = 1 + \tau\Gamma^2 + \tfrac{1}{2}\tau^2\Gamma^2 \cdot \Gamma^2 + \cdots \ ,$$

with $1 \equiv \mathrm{diag}(1_-, 1_+)$. As noted in Ref. 17, the above choice of regularization corresponds to ζ-function regularization[18] of the propagator $\langle X(z_1)X(z_2) \rangle$ using the heat kernel $\langle z_1 | e^{\tau\Gamma^2} | z_2 \rangle$; the regularized equation-of-motion term agrees with the regularized Jacobian. The regularized Jacobian is calculated in Appendix B. We quote the result for $\langle \delta S/\delta\Omega^\alpha \rangle \equiv \delta\langle \mathscr{A} \rangle / \delta\Omega^\alpha$:

$$(\delta S/\delta\Omega^\alpha)^{\mathrm{reg}} = \frac{1}{768\pi^2} D_\alpha \bar{D}^2 [2m - g(\phi + \bar{\phi})]^2 \ . \tag{4.9}$$

For $(\delta S/\delta\bar{\Omega}_{\dot{\alpha}})^{\mathrm{reg}}$, replace $D_\alpha \bar{D}^2$ by $\bar{D}^{\dot{\alpha}} D^2$. In Eq. (4.9) one may replace $(\phi,\bar{\phi})$ by $(\Phi,\bar{\Phi})$ to obtain the oparator form of the anomaly at the one-loop level.

Equation (4.9) introduces anomalies to both the generalized trace $\bar{D}^{\dot{\alpha}} R_{\alpha\dot{\alpha}}$ and the divergence $\partial^\mu R_\mu$. However, it is possible to redefine the supercurrent so that its divergence becomes anomaly-free. Indeed, if we define the new supercurrent

$$\hat{R}_{\alpha\dot{\alpha}} = R_{\alpha\dot{\alpha}} - \frac{i}{96\pi^2} (\sigma^\mu)_{\alpha\dot{\alpha}} \partial_\mu [N(\Phi) - N(\bar{\Phi})] \ , \tag{4.10}$$

where $N(x) \equiv g^2x^2 - 4mgx$, it obeys the conservation law

$$\bar{D}^{\dot{\alpha}} \hat{R}_{\alpha\dot{\alpha}} = \tfrac{1}{3} m D_\alpha \Phi^2 - (g^2/192\pi^2) D_\alpha \bar{D}^2 (\bar{\Phi}\Phi) \ ; \tag{4.11}$$

likewise, $D^\alpha \hat{R}_{\alpha\dot{\alpha}}$ has the anomaly $-(g^2/192\pi^2)\bar{D}_{\dot{\alpha}} D^2(\bar{\Phi}\Phi)$. It is easy to verify that $\partial^\mu \hat{R}_\mu$ possesses no anomaly. This form of the anomaly agrees with the one obtained by a different method in Ref. 9. (An adjustment of notations is needed for comparison.)

The form of the anomaly [such as Eq. (4.9)] depends on the way one regularizes the quantum theory. However, once the current is redefined so that, e.g., $\partial^\mu R_\mu$ is anomaly-free, the anomaly of $\bar{D}^{\dot{\alpha}} R_{\alpha\dot{\alpha}}$ is uniquely determined. This phenomenon, the trading of anomalies, has also been encountered in the path-integral formulation of conformal anomalies.[17]

The $\partial^\mu \hat{R}_\mu$ has no (one-loop) anomaly. Consequently, among the symmetries associated with $\hat{R}_\mu$, only R symmetry is broken by the mass term while translation (P_μ) and supertranslation ($Q_\alpha, \bar{Q}_{\dot{\alpha}}$) symmetries are kept exact, as seen from Eq. (3.21). Similarly, it can be shown that, with the supercurrent $\hat{R}_{\alpha\dot{\alpha}}$, the super-Poincaré part of superconformal symmetry becomes anomaly-free. This, in fact, is a general feature of expressing the generalized trace ($\bar{D}^{\dot{\alpha}} \hat{R}_{\alpha\dot{\alpha}}$) in terms of a chiral superfield.[7]

Let us next examine the SQED case. We shall here concentrate on the anomaly coming from the matter sector; then the analysis almost parallels that of the Wess-Zumino model. It is important, on physics grounds, to regularize the matter sector so that vector-gauge covariance is preserved. The matter sector of the SQED action can be written in the matrix form

$$S_{\mathrm{matter}} = (\chi, \bar{\Phi}) \begin{bmatrix} 0 & 1_- e^{-V} 1_+ \\ 1_+ e^V 1_- & 0 \end{bmatrix} \begin{bmatrix} \Phi \\ \bar{\chi} \end{bmatrix} \tag{4.12}$$

$$\equiv X^* \cdot \Gamma[V] \cdot X \ , \tag{4.13}$$

where V is now treated as an Abelian external superfield. Let us write the variation $\Delta X \equiv (\Delta\Phi, \Delta\bar{\chi})^t$ as $\Delta X = C \cdot X$ with

$$C=\text{diag}(1_-[\Omega\nabla+\tfrac{1}{3}(D\Omega)]1_-\,,\quad 1_+[\bar\Omega\bar\nabla+\tfrac{1}{3}(\bar D\bar\Omega)]1_+) \tag{4.14}$$

where $\Omega\nabla\equiv\Omega^\alpha\nabla_\alpha$ and $\bar\Omega\bar\nabla\equiv\bar\Omega_{\dot\alpha}\bar\nabla^{\dot\alpha}$. Analogously, the variation $\Delta X^*\equiv(\Delta\mathcal{X},\Delta\bar\Phi)$ is written as $\Delta X^*=X^*\cdot C'$ with

$$C'=\text{diag}(-1_-[\Omega\nabla+\tfrac{2}{3}(D\Omega)]1_-\,,\quad -1_+[\bar\Omega\bar\nabla+\tfrac{2}{3}(\bar D\bar\Omega)]1_+)\,. \tag{4.15}$$

The Jacobian associated with the change of variables $X\to X+\Delta X$ and $X^*\to X^*+\Delta X^*$ is given by $\exp[\text{Tr}(C+C')]$, which we regularize in such a way that

$$\text{Tr}(C+C')^{\text{reg}}=\text{Tr}[(C+C')\cdot e^{\tau\Gamma[V]^2}]\quad(\tau\to 0_+)\,. \tag{4.16}$$

with $\Gamma[V]^2=\Gamma[V]\cdot\Gamma[V]$. Substituting the explict forms of $\Gamma[V]$, C, and C' yields the following representation for the anomaly $\mathcal{A}=i\,\text{Tr}(C+C')^{\text{reg}}$:

$$\mathcal{A}=-i\tfrac{1}{3}\int d^8z[(D\Omega)\langle z\,|\,e^{\tau 1_- e^{-V}1_+e^{V}}1_-\,|\,z\rangle +(\bar D\bar\Omega)\langle z\,|\,e^{\tau 1_+e^{V}1_-e^{-V}}1_+\,|\,z\rangle]\,. \tag{4.17}$$

The evaluation of the matrix elements is standard, with the result

$$\lim_{\tau\to 0}\langle z\,|\exp(\tau\tfrac{1}{16}\bar D^2e^{-V}D^2e^{V})1_-\,|\,z\rangle=(i/64\pi^2)W^\alpha W_\alpha\,. \tag{4.18}$$

Consequently, the supercurrent obeys the anomalous conservation law (in the one-loop approximation):

$$\bar D^{\dot\alpha}R_{\alpha\dot\alpha}=(1/96\pi^2)D_\alpha W^2\,, \tag{4.19}$$

$$D^\alpha R_{\alpha\dot\alpha}=(1/96\pi^2)/\bar D_{\dot\alpha}\bar W^2\,. \tag{4.20}$$

This result for the anomalies confirms that obtained by different methods.[8,20] It will be clear that vector gauge covariance is manifest in each step of our treatment.

V. CONCLUDING REMARKS

In Secs. II and III, we have presented a variational formulation of supercurrents and their associated anomalies in such a manner that their foundation on superconformal symmetry becomes explicit.

In connection with the supercurrent anomalies calculated in Sec. IV, some remarks on R symmetry and chiral symmetry in gauge models will be useful. Chiral superfields have R weight $n=\frac{2}{3}$, as implied by the chirality constraint $n=\frac{2}{3}d$ which follows from the suprconformal algebra. This R-weight value is easily read from the Wess-Zumino action. On the other hand, the situation is less obvious in gauge models. The massless SQED (SQCD) action is chiral invariant as well as superconformal invariant. The chiral rotation of matter fields $(\Phi,\mathcal{X})\to e^{i\alpha}(\Phi,\mathcal{X})$ is contained in the chiral phase transformation $(\Phi,\mathcal{X})\to e^{\Lambda}(\Phi,\mathcal{X})$, where $\Lambda(z)$ is a chiral superfield. Because of the (global) chiral U(1) invariance of the action, the R weights of matter superfields may appear undetermined. Indeed, the R weight $\frac{2}{3}$ of $\Phi(z)$ in Eq. (2.9) can formally be changed to n if we add the chiral variation of $\Phi(z)$ with

$$\Lambda=(\tfrac{1}{2}n-\tfrac{1}{3})(-\tfrac{1}{4})\bar D^2D^\alpha\Omega_\alpha\,.$$

As verified easily, the one-loop anomalies in Eqs. (4.19) and (4.20) are multiplied by $3(1-n)$ if we assign R weight n to the chiral superfields. Correspondingly, the superconformal transformation of Φ (of $n=\frac{2}{3}$) combines with the chiral rotation of Φ with $\Lambda=-\frac{1}{24}\bar D^2D^\alpha\Omega_\alpha$ to become anomaly-free at the one-loop level. This, in particular, implies that the one-loop chiral anomaly is characterized by the same W^2 (or $\bar W^2$) as the supercurrent anomaly. The combined transformation, of course, is not the superconformal transformation any more.

Even the free-field action dictates the R weight n to be $\frac{2}{3}$ for chiral superfields: The current $R_\mu=\frac{1}{2}(\bar\sigma_\mu)^{\dot\alpha\alpha}R_{\alpha\dot\alpha}$ derived from the free-field action $\int d^8z\bar\Phi\Phi$ by the variational procedure is Hermitian only for $n=\frac{2}{3}$. Hence $n\neq\frac{2}{3}$ spoils the superconformal algebra; this, of course, is because $\Phi(z)$ has $d=1$.

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APPENDIX A

In this appendix we solve the superconformal Killing equations (3.3)—(3.7).

It is convenient to use the chiral coordinate $(y_\mu=x_\mu-i\theta\sigma_\mu\bar\theta,\theta,\bar\theta)$ in solving Eq. (3.3) for $h_\mu(z)$, with the result

$$h_\mu(x,\theta,\bar\theta)=h_\mu(y,\theta,0)-2i\bar\theta\bar\sigma_\mu f(z)\,. \tag{A1}$$

Equation (3.4) implies the Hermitian conjugate of (A1):

$$h_\mu(x,\theta,\bar\theta)=h_\mu(y^\dagger,0,\bar\theta)-2i\theta\sigma_\mu\bar f(z)\,, \tag{A2}$$

where $y^\dagger_\mu\equiv x_\mu+i\theta\sigma_\mu\bar\theta$. Take an average of Eqs. (A1) and (A2), recalling Eq. (3.6). The result is

$$f_\mu(z)=\tfrac{1}{2}[h_\mu(y,\theta,0)+h_\mu(y^\dagger,0,\bar\theta)]\,. \tag{A3}$$

Let us next equate (A1) with (A2) for each power of θ and $\bar\theta$. The $O(\theta)$ term fixes the structure of $h_\mu(y,\theta,0)$:

$$h_\mu(y,\theta,0)=f^{(0)}_\mu(y)-2i\theta\sigma_\mu\bar f^{(0)}(y)\,, \tag{A4}$$

which is quoted as $F_\mu(y,\theta)$ in Eq. (3.16). Comparing further powers of θ and $\bar\theta$ leads to the parametrizations of $f_\alpha(z)$ and $\bar f_{\dot\alpha}(z)$ quoted in Eq. (3.8).

APPENDIX B

In this appendix we evaluate the regularized Jacobian in Eq. (4.8).

In the usual notation, Eq. (4.8) reads

$$\text{Tr}B^{\text{reg}}=\int d^6z\langle z\,|\,1_-[\Omega D+\tfrac{1}{3}(D\Omega)]1_-\cdot I_{11}\,|\,z\rangle +\int d^6\bar z\langle z\,|\,1_+[\bar\Omega\bar D+\tfrac{1}{3}(\bar D\bar\Omega)]1_+\cdot I_{22}\,|\,z\rangle\,, \tag{B1}$$

where I_{11} and I_{22} stand for the diagonal elements of the 2×2 matrix $I\equiv e^{\tau\Gamma^2}$. Since I_{11} is chiral, $1_-\cdot I_{11}=I_{11}$. Note that the first 1_- in the matrix element is combined with d^6z to form d^8z. Hence we get

$$\mathrm{Tr}B^{\mathrm{reg}}=\int d^8z\langle z\,|\,[\Omega D+\tfrac{1}{3}(D\Omega)]I_{11}\,|\,z\rangle+(\overline{\Omega}\text{ part})\,. \tag{B2}$$

Now observe that I_{11} is symmetric: $I_{11}=I_{11}^t$; see Eq. (B5) below. This implies that $\langle z\,|\,\Omega^\alpha D_\alpha I_{11}\,|\,z\rangle=\frac{1}{2}\Omega^\alpha D_\alpha\langle z\,|\,I_{11}\,|\,z\rangle$. Consequently, Eq. (B2) gets further simplified:

$$\mathrm{Tr}B^{\mathrm{reg}}=-\tfrac{1}{6}\int d^8z[(D^\alpha\Omega_\alpha)\langle z\,|\,I_{11}\,|\,z\rangle+(\overline{D}_{\dot\alpha}\overline{\Omega}^{\dot\alpha})\langle z\,|\,I_{22}\,|\,z\rangle]\,. \tag{B3}$$

To calculate $\langle z\,|\,I_{11}\,|\,z\rangle$, let us divide $\Gamma^2=\Gamma\cdot\Gamma$ into two parts $\Gamma^2=H_0+H_1$:

$$H_0=\begin{bmatrix}p^2 1_- & 0\\ 0 & p^2 1_+\end{bmatrix},\quad H_1=\begin{bmatrix}M^2 1_- & 1_- L 1_+\\ 1_+ L 1_- & \overline{M}^2 1_+\end{bmatrix}, \tag{B4}$$

where $M=m-\phi$, $\overline{M}\equiv m-\overline{\phi}$, and $L\equiv 2m-\phi-\overline{\phi}$; we have used the relations[12,19] $1_-1_+1_-=p^2 1_-$ and $1_+1_-1_+=p^2 1_+$, where $p_\mu\equiv i\partial_\mu$.

We expand $\langle z\,|\,I_{11}\,|\,z\rangle$ in powers of H_1. The first-order term, which is proportional to $M^2 1_-$, vanishes since $\langle z\,|\,1_-\,|\,z\rangle=0$. The M^4 term in second order is vanishing for the same reason. Only the L^2 term survives the $\tau\to 0$ limit:

$$\langle z\,|\,I_{11}\,|\,z\rangle=\int_0^\tau ds\int_0^s du\,\langle z\,|\,e^{(\tau-s)p^2}1_-L1_+e^{(s-u)p^2}L1_-e^{up^2}\,|\,z\rangle \tag{B5}$$

$$=-\tfrac{1}{4}\overline{D}_z^2\int_0^\tau ds\int_0^s du\,\langle z\,|\,e^{(\tau-s)p^2}L1_+e^{(s-u)p^2}L1_-e^{up^2}\,|\,z\rangle\,. \tag{B6}$$

Recall[12,21] that one needs at least two D's and two $\overline{D}$'s to obtain a nonvanishing matrix element diagonal in θ and $\overline{\theta}$; in particular, $\langle z\,|\,1_+1_-\,|\,z\rangle=\langle x\,|\,1\,|\,x\rangle$. Hence, in the $\tau\to 0$ limit,

$$\langle z\,|\,I_{11}\,|\,z\rangle=-\tfrac{1}{4}\overline{D}^2(\tfrac{1}{2}\tau^2\langle x\,|\,e^{\tau p^2}\,|\,x\rangle L^2) \tag{B7}$$

$$=-i\frac{1}{128\pi^2}\overline{D}^2L^2\,. \tag{B8}$$

This leads to the expression for the anomaly in Eq. (4.9).

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