Nonstandard expansion techniques for the effective potential in $\lambda\phi^4$ quantum field theory

Anna Okopińska*
International Center for Theoretical Physics, Trieste, Italy
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The effective potential in scalar quantum field theory is calculated using different expansions of the path integral for the generating functional. The conventional loop expansion is compared with the interpolated, optimized, and mean-field expansions in the lowest orders. The optimized expansion is studied up to third order. In the space-time dimensions 0 and 1 the comparison with the "exact" effective potential calculated numerically shows that the only method which gives qualitative agreement in the whole range of Lagrangian parameters is the mean-field expansion. In 4 dimensions the mean-field method seems also to be most reliable and the theory to be noninteracting.

I. INTRODUCTION

The effective potential (EP) has been introduced[1] in quantum field theory in order to study the spontaneous symmetry breaking induced by scalar fields. Usually the effective potential as a function of the vacuum expectation values (VEV's) of the scalar fields is calculated in the loop expansion.[2] However, it was pointed out[3] that the loop expansion breaks down for the VEV of the fields in the region where the classical potential is nonconvex. Therefore the conventional loop expansion is not useful to investigate the shape of the EP when the symmetry is expected to be spontaneously broken.

In this work we will study the applicability of some nonstandard methods in this case. We limit ourselves to the theory of the real scalar field in n-dimensional Euclidean space with the classical action given by

$$S[\Phi]=\int d_nx[\tfrac{1}{2}\Phi(x)(-\partial^2+m^2)\Phi(x)+\lambda\Phi^4(x)] \ . \qquad (1.1)$$

The theory can be quantized by representing the generating functional for Green's functions as the path integral over the fields:

$$Z[J]=\int D\Phi \exp\left[-S[\Phi]+\int d_nx\, J(x)\Phi(x)\right] . \qquad (1.2)$$

We fix the normalization of the measure to have

$$\int D\Phi \exp\left[-\int d_nx\, d_ny\, \Phi(x)A(x,y)\Phi(y)\right]=\det{}^{-1/2}A \ . \qquad (1.3)$$

An effective method to calculate the generating functional (1.2) is the steepest-descent method, which generates a series of Gaussian functional integrals. As discussed in Sec. II, when the subintegral expression is represented in various forms the different expansions of the generating functional are obtained, leading to different expansions for the effective action and the effective potential.

The effective action, being the generating functional for one-particle-irreducible (1PI) Green's functions is defined as

$$\Gamma[\phi]=W[J]-\int d_nx\, J(x)\phi(x) \ , \qquad (1.4)$$

where $W[J]=\ln Z[J]$ is the generating functional for the connected Green's functions, and the background field

$$\phi(x)=\frac{\delta W}{\delta J(x)} =\int D\Phi\, \Phi \exp\left[-S[\Phi]+\int d_nx\, J(x)\Phi(x)\right] \qquad (1.5)$$

is the VEV of the scalar field. It can be proved[4] that

$$w(J)=-\frac{W[J]\,|_{J(x)=\text{const}}}{\int d_nx} \qquad (1.6)$$

is the vacuum energy density for the system interacting with the constant source J. The effective action, as the Legendre transition (1.4), satisfies

$$\frac{\delta\Gamma}{\delta\phi(x)}=-J(x) \ , \qquad (1.7)$$

and is stationary in the physical theory when the sources are absent. Only constant background fields can describe the true vacuum, since the translational invariance is not expected to be broken. Hence the true VEV can be found as the stationary point of the effective potential, defined as

$$V(\phi)=-\frac{\Gamma[\phi]\,|_{\phi(x)=\text{const}}}{\int d_nx} \ . \qquad (1.8)$$

The EP can be shown to be convex.[4] It can be interpreted as the work done by the external source J to displace the VEV from $\phi(J=0)$ to $\phi(J)$ in the unit volume. If the stationary point of the EP is at ϕ_{true} different from zero, the symmetry is spontaneously broken and the effective potential is not defined for $\phi<\phi_{\text{true}}$ (Refs. 4 and 5). The nonconvex part of the EP indicates the instability of the vacuum for corresponding ϕ (Ref. 5). This interval should be deleted from the domain of definition of the EP. However, every value of ϕ from this interval can be obtained for a nonhomogeneous vacuum taken as a linear combination of the states with VEV equal to the values at the inflection points of the EP. In analogy with the Maxwell construction in thermodynamics it gives the convex hull of the EP. The value of the EP at the minimum is equal to the vacuum energy density for vanishing source:

$$V(\phi_{\text{true}})=W(0)\ . \tag{1.9}$$

The 1PI vertices for vanishing external momenta can be obtained as derivatives of the effective potential. The renormalized mass and coupling constant (which are used to reparametrize the EP for scalar QFT to be finite in 4 dimensions) are usually defined as the second and fourth derivative at the stationary point of the EP.

In this work we will study the effective potential obtained in the (A) loop expansion (LE), (B) interpolated loop expansion (ILE), (C) optimized expansion (OE), and (D) mean-field (MF) expansion. These methods in arbitrary space-time dimensions are reviewed in Sec. II. The results are discussed in Sec. III. In the dimension $n=0$ and 1 the theory is finite and we can compare the results with the "exact" EP calculated numerically. In 4 dimensions two possibilities emerge in the comparison of different approximations for the renormalized QFT—triviality or precariousness.

II. FORMAL METHODS OF EVALUATION OF THE EFFECTIVE ACTION

A. Loop expansion

The conventional loop expansion is generated if the steepest-descent method is applied to the generating functional written in the form

$$Z[J]=\int D\Phi \exp\left[\frac{1}{\hbar}(-S[\Phi]+J\Phi)\right]. \tag{2.1}$$

As usual, for notational simplicity we suppress the space arguments and the integrations over them. The Planck constant, retained in the exponent, is the dimensionful quantity; therefore it cannot be thought of as a small parameter ($\hbar=1$ in natural units), but as a formal parameter of the expansion. Therefore, to each order, the result should be verified, showing that the higher-order contributions are really small.

Upon translating the integration variable Φ by ϕ_0 chosen to satisfy the classical equation of motion

$$\frac{\delta S}{\delta\phi_0}=-J\ , \tag{2.2}$$

and rescaling by $\hbar^{1/2}$, the exponential is expanded into a Taylor series, giving

$$\begin{aligned}Z[J]&=\exp\left[\frac{1}{\hbar}(-S[\phi_0]+J\phi_0)\right]\int D\Phi\exp\left[-\frac{1}{2}\frac{\delta^2 S}{\delta\Phi_0{}^2}\Phi^2\right]\\&\quad\times\left[1-\tfrac{1}{6}\hbar^{1/2}\frac{\delta^3 S}{\delta\phi_0{}^3}\Phi^3-\tfrac{1}{24}\hbar\frac{\delta^4 S}{\delta\phi_0{}^4}\Phi^4+\tfrac{1}{72}\hbar\left[\frac{\delta^3 S}{\delta\phi_0{}^2}\right]^2\Phi^6+\cdots\right]\\&\equiv\exp\left[\frac{1}{\hbar}(-S[\phi_0]+J\phi_0)\right]\\&\quad\times\text{Det}^{-1/2}\left[\frac{\delta^2 S}{\partial\phi_0{}^2}\right]\left[1-\tfrac{1}{8}\hbar\frac{\delta^4 S}{\delta\Phi_0{}^4}\left[\frac{\delta^2 S}{\partial\phi_0{}^2}\right]^{-2}+\tfrac{5}{24}\hbar\left[\frac{\delta^3 S}{\delta\Phi_0{}^3}\right]^2\left[\frac{\delta^2 S}{\delta\phi_0{}^2}\right]^{-3}+\cdots\right].\end{aligned} \tag{2.3}$$

The effective action, to each order in $\hbar$, can be obtained using the implicit definition (2.4) and (2.5). It can be proved[2] that $\Gamma[\phi]$ to the order k is a sum of k-loop 1PI vacuum Feynman diagrams in the theory with mass $(m^2+12\lambda\phi^2)^{1/2}$, quartic vertex 24λ, and cubic vertex $24\lambda\phi$. For the constant source, the energy density $w(J)$ and the effective potential $V(\phi)$ can be expressed in terms of the ordinary integrals in momentum space. We introduce the notation[6]

$$\begin{aligned}I_1(\Omega)&=\tfrac{1}{2}\int\frac{d_n p}{(2\pi)^n}\ln(p^2+\Omega^2)\ ,\\I_0(\Omega)&=\int\frac{d_n p}{(2\pi)^n}\frac{1}{p^2+\Omega^2}\ ,\\I_{-1}(\Omega)&=2\int\frac{d_n p}{(2\pi)^n}\frac{1}{(p^2+\Omega^2)^2}\ .\end{aligned} \tag{2.4}$$

The effective potential to the order $\hbar$ can be expressed as

$$V_1(\phi)=\frac{m^2}{2}\phi^2+\lambda\phi^4+\hbar I_1(m^2+12\lambda\phi^2)\ . \tag{2.5}$$

For $m^2<0$ the EP at $\phi=0$ is complex; therefore, the renormalized vertices are usually defined as the derivatives of the EP at the stationary point $\phi_1\neq 0$. To the first order we obtain the renormalized mass

$$m_R{}^2=m^2+12\lambda I_0(\Omega) \tag{2.6}$$

and the renormalized coupling constant

$$\begin{aligned}\lambda_R=\lambda[1&-18\lambda I_{-1}(\Omega)-864\lambda^2\phi_1{}^2 I'_{-1}(\Omega)\\&-3456\lambda^3\phi_1{}^4 I''_{-1}(\Omega)]\ ,\end{aligned} \tag{2.7}$$

where $\Omega^2=m^2+12\lambda\phi_1{}^2$ and a prime denotes a derivative with respect to Ω^2.

B. Interpolated loop expansion

As pointed out by Fujimoto, O'Raifeartaigh, and Parravicini,[3] the loop expansion is valid only if the Gaussian integrals in (2.3) converge, i.e., the operator

$$\frac{\delta^2 S}{\delta\phi(x)\delta\phi(y)}=[-\partial^2+m^2+12\lambda\phi^2(x)]\delta(x-y) \tag{2.8}$$

is positive definite, i.e., the classical potential is convex. In the case when we expect the reflection symmetry to be broken spontaneously ($m^2<0$), the classical potential is double-well (DW) shaped and the operator (2.8) is not positive definite for $\phi^2<-m^2/12\lambda$. The imaginary part of the effective potential, appearing in the order $\hbar$ in (2.5), is a signal of the failure of the loop expansion in this range of ϕ. In this case the interpolated loop expansion, which takes into account the contributions of all minima of $S[\phi]-J\phi$, has been proposed.[3] The range of integration in (2.1) is divided into two regions and the Taylor expansion is made about each minimum (ϕ_1 and ϕ_2) and the Gaussian integrations are performed over all ranges, arguing that the main contribution comes from the neighborhood of the minima. To the first order the energy density becomes

$$w(J)=\frac{1}{\int d_n x}\ln\left[\exp\left[-V_1(\phi_1)\int d_n x\right]+\exp\left[-V_1(\phi_2)\int d_n x\right]\right]. \tag{2.9}$$

The interpolated one-loop EP coincides with the conventional $V_1(\phi)$ in the region where the latter is convex and the region between its inflection points given by[3]

$$V(\phi)=\frac{\phi_1^0 V_1(\phi_2^0)-\phi_2^0 V_1(\phi_1^0)}{\phi_1^0-\phi_2^0}+\frac{V_1(\phi_1^0)-V_1(\phi_2^0)}{\phi_1^0-\phi_2^0}\phi+\frac{1}{\int d_n x}\left[\frac{\phi_1^0-\phi}{\phi_1^0-\phi_2^0}\ln\left[\frac{\phi_1^0-\phi}{\phi_1^0-\phi_2^0}\right]+\frac{\phi-\phi_2^0}{\phi_1^0-\phi_2^0}\ln\left[\frac{\phi-\phi_2^0}{\phi_1^0-\phi_2^0}\right]\right], \tag{2.10}$$

where ϕ_1^0 and ϕ_2^0 are the minima of the classical potential. The interpolated one-loop result is real and convex in agreement with the general proof[4] and lattice calculations,[7] and becomes a straight line (Maxwell construction) in the infinite-volume limit. The renormalized vertices are given by (2.6) and (2.7), as in the conventional LE.

The method is applicable only if the minima of the classical potential are far apart, and taking their contribution independently does not introduce double counting. If m^2 approaches zero, the interpolated loop expansion breaks down, as does the conventional loop expansion.

C. Optimized expansion

Nambu and Jona-Lasinio, in their study of the dynamical breaking of chiral invariance,[8] proposed to modify the bare mass in QFT to make the radiative corrections to the self-energy vanish. Using a similar trick in the calculation of the effective action, we can generate a series with all terms being converging Gaussian integrals in the whole range of ϕ even for $m^2<0$. We write the classical action as

$$\begin{aligned}S_\epsilon[\Phi]&=S^{(0)}[\Phi]+\epsilon S^{(1)}[\Phi]\\&=\tfrac{1}{2}\Phi(-\partial^2+\Omega^2)\Phi+\epsilon[\tfrac{1}{2}(m^2-\Omega^2)\Phi^2+\lambda\Phi^4],\end{aligned} \tag{2.11}$$

where the parameter ϵ has been introduced to identify the order of the perturbation and is set equal to one at the end. After choosing ϕ_0 to satisfy the classical equation of motion for the modified action

$$\frac{\delta S_\epsilon}{\delta\phi_0}=-J, \tag{2.12}$$

we apply the steepest-descent method to the functional $Z[J]$:

$$\begin{aligned}Z[J]&=\exp\{-\tfrac{1}{2}\phi_0(-\partial^2+\Omega^2)\phi_0-\epsilon[\tfrac{1}{2}(m^2-\Omega^2)\phi_0^2+\lambda\phi_0^4]+J\phi_0\}\\&\quad\times\int D\Phi\exp\left[-\tfrac{1}{2}\Phi(-\partial^2+\Omega^2)\Phi+\epsilon\left[-\frac{1}{2}\frac{\delta^2 S^{(1)}}{\delta\phi_0^2}\Phi^2-\frac{1}{6}\frac{\delta^3 S^{(1)}}{\delta\phi_0^3}\Phi^3-\frac{1}{24}\frac{\delta^4 S^{(1)}}{\delta\phi_0^4}\Phi^4\right]\right]\\&=\exp\{-\tfrac{1}{2}\phi_0(-\partial^2+\Omega^2)\phi_0-\epsilon[\tfrac{1}{2}(m^2-\Omega^2)\phi_0^2+\lambda\phi_0^4]+J\phi_0\}\\&\quad\times\int D\Phi\exp[-\tfrac{1}{2}\Phi(-\partial^2+\Omega^2)\Phi]\left[1-\epsilon\left[\frac{1}{2}\frac{\delta^2 S^{(1)}}{\delta\phi_0^2}\Phi^2-\frac{1}{6}\frac{\delta^3 S^{(1)}}{\delta\Phi_0^3}\Phi^3-\frac{1}{24}\frac{\delta^4 S^{(1)}}{\delta\phi_0^4}\Phi^4+\cdots\right]\right].\end{aligned} \tag{2.13}$$

The Gaussian integrals converge, as long as we choose $\Omega^2(x)$ to be positive everywhere, and we obtain the series

$$Z[J]=\exp\{-\tfrac{1}{2}\phi_0(-\partial^2+\Omega^2)\phi_0-\epsilon[\tfrac{1}{2}(m^2-\Omega^2)\phi_0{}^2+\lambda\phi_0{}^4]+J\phi_0\}\mathrm{Det}^{-1/2}(-\partial^2+\Omega^2)$$
$$\times\left[1-\epsilon\left[\frac{1}{2}\frac{\delta^2S^{(1)}}{\delta\phi_0{}^2}(-\partial^2+\Omega^2)^{-1}+\frac{1}{8}\frac{\delta^4S^{(1)}}{\delta\phi_0{}^4}(-\partial^2+\Omega^2)^{-2}\right]+\cdots\right]. \tag{2.14}$$

To the kth order in ϵ, $Z[J]$ can be found as the sum of all vacuum diagrams with k interactions in the theory with the mass $\Omega(x)$, quartic vertex 24λ, cubic vertex $24\lambda\phi_0$, and quadratic vertex $(\Omega^2-m^2-12\lambda\phi_0{}^2)$, where ϕ_0 is the classical VEV of the scalar field.

The auxiliary field $\Omega(x)$ has been introduced in such a way that $Z[J]$ does not depend on them. However, in the truncated series the dependence on $\Omega(x)$ appears. We can take advantage of the freedom of choosing $\Omega(x)$ to produce the optimized expansion (OE) (Ref. 9). Guided by Stevenson's principle of minimal sensitivity,[10] we require the kth-order approximant of the physical quantity $W[J]$ to be as insensitive as possible to the small variation of Ω, choosing Ω to satisfy

$$\frac{\delta W_k}{\delta\Omega}=0\ . \tag{2.15}$$

The effective action, as a Legendre transform, satisfies

$$\frac{\delta\Gamma_k}{\delta\Omega}=0\ , \tag{2.16}$$

and is a sum of 1PI vacuum diagrams with the mass $\Omega(x)$, quartic vertex 24λ, cubic vertex $24\lambda\phi$, and quadratic vertex $(\Omega^2-m^2-12\lambda\phi^2)$, where ϕ is the scalar VEV to the order k. If we limit ourselves to the constant Ω, the effective potential becomes

$$V_k(\phi,\Omega)=\frac{m^2\phi^2}{2}+\lambda\phi^4+I_1(\Omega)+\epsilon V^{(1)}(\phi,\Omega)+\cdots+\epsilon^kV^{(k)}(\phi,\Omega)\ , \tag{2.17}$$

where $V^{(j)}$ is a sum of 1PI vacuum diagrams with j interactions. The first three terms are calculated in the Appendix. To the given order, Ω can be found as a root of the equation

$$\frac{\partial V_k}{\partial\Omega}=0\ . \tag{2.18}$$

To the first order in the OE the effective potential is given by

$$V_1(\phi,\Omega)=\frac{m^2\phi^2}{2}+\lambda\phi^4+I_1(\Omega)+\epsilon\{\tfrac{1}{2}(m^2+12\lambda\phi^2-\Omega^2)I_0(\Omega)+3\lambda[I_0(\Omega)]^2\} \tag{2.19}$$

with Ω fixed by Eq. (2.18), which becomes

$$\Omega^2-m^2-12\lambda[\phi^2+I_0(\Omega)]=0\ . \tag{2.20}$$

The renormalized mass and coupling constant defined at $\phi=0$ are given by

$$m_R{}^2=m^2+12\lambda I_0(M_R)_0\ , \tag{2.21}$$

$$\lambda_R=\phi\frac{1-12\lambda I_{-1}(m_R)}{1+6\lambda I_{-1}(m_R)}\ . \tag{2.22}$$

The EP to the first order of the OE is exactly the same as the Gaussian effective potential (GEP). The GEP was obtained by solving the functional Schrödinger equation in QFT by variational method with Gaussian trial wave functional[11,12] and is equivalent to the Hartree approximation. It has been shown[12] that GEP is the sum of 1PI vacuum diagrams without overlapping divergences; therefore, it should be a much better approximation than the one-loop result, which sums only all tadpoles attached to this loop. The GEP has been recently studied by Stevenson[6] in 1, 2, 3 and 4 space-time dimensions. Our approach offers the possibility of systematically improving the variational result.

For $\phi=0$ Eq. (2.20) coincides with the Nambu—Jona-Lasinio "gap equation," which makes the radiative corrections to the self-energy in the scalar QFT vanish. This approach cannot be directly generalized to make the study of the spontaneous symmetry breaking possible. In the OE we can study dynamical and spontaneous symmetry breaking simultaneously.

D. Mean-field theory

The mean-field (MF) expansion, proposed by Cooper, Guralnik, and Kasdan,[13] can be formulated in the path-integral approach.[14] Using the Gaussian integral

$$\int D\theta\exp\left[-\frac{1}{16\lambda}[\theta-i(m^2+4\lambda\Phi^2)]^2\right]=\mathrm{Det}^{-1/2}(8\lambda I)\ , \tag{2.23}$$

where $I(x,y)=\delta(x-y)$, and the generating functional (1.2) can be expressed as

$$Z[J]=\int D\Phi\exp[-\tfrac{1}{2}\Phi(-\partial^2+m^2)\Phi-\lambda\Phi^4+J\Phi]\times\mathrm{Det}^{1/2}(8\lambda I)\times\int D\theta\exp\left[-\frac{1}{16\lambda}[\theta-i(m^2+4\lambda\Phi^2)]^2\right]. \tag{2.24}$$

Introducing the source $S(x)$ for the field θ

$$Z[J,S]=\int D\theta \exp\left[-\frac{1}{16\lambda}(\theta^2-2im^2\theta)+\frac{m^4}{16\lambda}+S\theta\right]$$
$$\times \mathrm{Det}^{1/2}(8\lambda I)$$
$$\times \int D\Phi \exp[-\tfrac{1}{2}\Phi(-\partial^2-i\theta)\Phi+J\Phi] , \tag{2.25}$$

and performing the integration over Φ gives

$$Z[J,S]=\mathrm{Det}^{1/2}(8\lambda I)\int D\theta \exp(-NF[\theta,J,S]) , \tag{2.26}$$

where

$$F[\theta,J,S]=-\frac{m^4}{16\lambda}+\frac{1}{16\lambda}\theta^2-\frac{1}{8\lambda}im^2\theta -\tfrac{1}{2}JGJ+\tfrac{1}{2}\mathrm{Tr}\ln G^{-1}-S\theta \tag{2.27}$$

and

$$G^{-1}(x,y)=[-\partial^2-i\theta(x)]\delta(x-y) . \tag{2.28}$$

The parameter N has been introduced only to identify the orders of the expansion and is set equal to one at the end. For the N component scalar field the number of components appears in (2.26) naturally, leading to the large-N expansion. Therefore, the mean-field result can be obtained by setting $N=1$ in the given order of the large-N expansion.

Translating the integration variable by θ_0 satisfying

$$\frac{\delta F}{\delta\theta_0}=0 , \tag{2.29}$$

rescaling by $N^{-1/2}$ and expanding the exponent gives

$$Z[J,S]=\mathrm{Det}^{1/2}(8\lambda I)\exp(-NF[\theta_0,J,S])\int D\theta \exp\left[-\frac{1}{2}\frac{\delta^2F}{\delta{\theta_0}^2}\theta^2\right]\left[1-\frac{1}{6N^{1/2}}\frac{\delta^3F}{\delta{\theta_0}^3}\theta^3-\frac{1}{24N}\frac{\delta^4F}{\delta{\theta_0}^4}\theta^4+\cdots\right] . \tag{2.30}$$

If the operator $\delta^2F/\delta{\theta_0}^2$ is positive definite, the Gaussian integrations can be performed, generating a series in $1/N$:

$$Z[J,S]=\exp\left[NF[\theta_0,J,S]-\tfrac{1}{2}\mathrm{Tr}\ln\left[\frac{1}{8\lambda}\frac{\delta^2F}{\delta{\theta_0}^2}\right]+\cdots\right] . \tag{2.31}$$

Keeping the first two terms only and replacing the sources by the background fields $\phi=\delta\ln Z/\delta J$ and

$$\Omega^2=-i\frac{\delta\ln Z}{\delta S} ,$$

with the aid of a Legendre transform, the effective potential $V(\phi,\theta)$ for constant background fields becomes

$$V(\phi,\Omega)=N\left[-\frac{m^4}{16\lambda}-\frac{1}{16\lambda}\Omega^4+\frac{1}{8\lambda}\Omega^2(m^2+4\lambda\phi^2)+\frac{1}{2}\int\frac{d_np}{(2\pi)^n}\ln(p^2+\Omega^2)\right]$$
$$+\frac{1}{2}\int\frac{d_np}{(2\pi)^n}\ln\left[1+\frac{8\lambda\phi^2}{p^2+\Omega^2}+4\lambda\int\frac{d_nq}{(2\pi)^n}\frac{1}{(q^2+\Omega^2)[(q+p)^2+\Omega^2]}\right] . \tag{2.32}$$

In the physical theory the sources are absent, and the EP satisfies

$$\frac{\partial V}{\partial\phi}=-J=0 \tag{2.33}$$

and

$$\frac{\partial V}{\partial\Omega^2}=-iS=0 . \tag{2.34}$$

To each order the auxiliary field Ω^2 can be eliminated with the aid of (2.34), and the consistency of the method requires $\Omega^2>0$. To the leading (zeroth) order, after setting $N=1$ we have

$$V(\phi,\Omega)=-\frac{m^4}{16\lambda}-\frac{1}{16\lambda}\Omega^4+\frac{1}{8\lambda}(m^2+4\lambda\phi^2)+I_1(\Omega) \tag{2.35}$$

with the VEV of the auxiliary field Ω^2 eliminated by the "gap equation" obtained from (2.34)

$$\Omega^2-m^2-4\lambda\phi^2-4\lambda I_0(\Omega)=0 . \tag{2.36}$$

The second and fourth derivative give the renormalized vertices

$$m_R{}^2=m^2+4\lambda I_0(m_R) , \tag{2.37}$$

$$\lambda_R=\frac{\lambda}{1+2I_{-1}(m_R)} . \tag{2.38}$$

To the next order, after setting $N=1$ we have

$$V(\phi,\Omega)=-\frac{m^4}{16\lambda}-\frac{1}{16\lambda}\Omega^4+\frac{1}{8\lambda}(m^2+4\lambda\phi^2)+\frac{1}{2}\int\frac{d_np}{(2\pi)^n}\ln(p^2+\Omega^2)$$
$$+\frac{1}{2}\int\frac{d_np}{(2\pi)^n}\ln\left[1+\frac{8\lambda\phi^2}{p^2+\Omega^2}+4\lambda\int\frac{d_nq}{(2\pi)^n}\frac{1}{(q^2+\Omega^2)[(q+p)^2+\Omega^2]}\right] \quad (2.39)$$

and the "gap equation" is the same as in the leading order (2.36).

After replacing λ by 3λ the MF "gap equation" (2.36) coincides with the first-order OE "gap equation" (2.20) and the EP in the leading order of the mean-field (MF0) approximation (2.35) coincides with the first-order OE result (2.19) after elimination of the auxiliary fields. The similarities of the MF and OE are due to the fact that in both methods some effects of the composite fields are included by means of the auxiliary field Ω^2. In the MF0 the same class of Feynman diagrams without overlapping divergences is summed as in OE1, only the coupling constant is different, as a reminescence of the fact that the diagrams with the coupling between different components of an O(N) multiplet dominate if N is large.

III. RESULTS AND CONCLUSIONS

The path-integral quantization enables study of the field theory in arbitrary space-time dimension. As the case $N=4$, which is believed to describe QFT in our world, is the most difficult, the theory in lower dimensions is frequently discussed. The large-order behavior of the perturbation theory has been derived first[15] in $n=1$ dimension and the divergence of the series was demonstrated. Afterward, the Borel summability has been rigorously proven for $m^2>0$ in $n<4$ dimensions.[16] For $m^2<0$ there are indications, that the perturbation series is not even Borel summable.[17]

We will study the applicability of the approximations discussed in the previous section to the evaluation of the energy density and the effective potential in different dimensions n. In the previous section we did not give the expressions for the energy density in the approximations studied, as up to the first order in these approximations it can be found as

$$w_1(J)=V_1(\phi_0)-J\phi_0 \quad (3.1)$$

with ϕ_0 given by (2.2) for the LE, (2.12) for the OE, and (2.29) for the MF. This is due to the fact that up to first order all the connected diagrams are 1PI. It is worthwhile to mention that the tree order is defined up to constant term in these approximations, applying the steepest-descent method to the path integral (1.2). If the constant factor is extracted from the classical action (i.e., by the change of the measure normalization) before introducing the formal parameter of expansion, the tree order EP will depend on them. In the first and higher orders the dependence cancels and the results are well defined. The tree-level results obtained with the measure normalized by (1.3) are shown in our figures (CL and MF0) just for illustration. However, the renormalized vertices in MF0 are meaningful, as the derivatives of the EP are well defined.

It will be convenient to get rid of one parameter by rescaling. The rescaled energy density $\hat{w}=w\lambda^{n/(n-4)}$, as a function of $\hat{J}=J\lambda^{[(n+2)/2]/(n-4)}$; and the rescaled effective potential $\hat{V}=V\lambda^{n/(n-4)}$, as a function of $\hat{\phi}=\phi\lambda^{(n-2)/2(n-4)}$, depend on only one dimensionless parameter: $z=\frac{1}{2}m^2\lambda^{2/(n-4)}$. When discussing the numerical results, we will use the rescaled quantities only, therefore we suppress the caret setting $\lambda=1$.

A. The simple integral

In zero dimensions the functional integral for the energy density becomes the simple one-dimensional integral

$$w[J]=-\ln\int_{-\infty}^{\infty}\frac{dx}{(2\pi)^{1/2}}\exp\left[-\frac{m^2}{2}x^2-\lambda x^4+Jx\right], \quad (3.2)$$

which can be calculated numerically, as well as all its derivatives and the Legendre transform. The EP in zero dimensions has been studied in the LE and MF (Ref. 18) and in the first-order OE (Ref. 9).

To the first order in the considered approximations the EP after rescaling ($z=\frac{1}{2}m^2\lambda^{-1/2}$, $V\to V-\frac{1}{4}\ln\lambda$, and $\phi\to\phi\lambda^{-1/4}$) becomes one of the following.

(a) A loop expansion

$$V(\phi)=z\phi^2+\phi^4+\tfrac{1}{2}\ln(2z+12\phi^2)\ . \quad (3.3)$$

(b) An interpolated loop expansion

$$V(\phi)=-\frac{z^2}{4}+\tfrac{1}{2}\ln(-2z)$$
$$+[\tfrac{1}{2}-\phi(-2z)^{-1/2}]\ln[\tfrac{1}{2}-\phi(-2z)^{-1/2}]$$
$$+[\tfrac{1}{2}+\phi(-2z)^{-1/2}]\ln[\tfrac{1}{2}+\phi(-2z)^{-1/2}]\ . \quad (3.4)$$

(c) An optimized expansion

$$V(\phi)=z\phi^2+\phi^4+\ln\Omega-\frac{\Omega^2-m^2-12\phi^2}{2\Omega^2}-\frac{3}{\Omega^4}\ , \quad (3.5)$$

where

$$\Omega^2=z+6\phi^2+[(z+6\phi^2)^2+12]^{1/2}\ . \quad (3.6)$$

(d) A mean-field expansion

$$V(\phi)=-\frac{z^2}{4}-\tfrac{1}{16}\Omega^4+\frac{z}{4}\Omega^2+\tfrac{1}{2}\Omega^2\phi^2$$
$$+\ln\Omega+\tfrac{1}{2}\ln\left[1+\frac{8\phi^2}{\Omega^2}+\frac{4}{\Omega^4}\right], \quad (3.7)$$

where

$$\Omega^2=\{z+2\phi^2+[(z+2\phi^2)^2+4]^{1/2}\} . \quad (3.8)$$

In the OE and MF the positive root of the "gap equation" is chosen to assure the consistence of the method.

The energy density can be obtained from Eqs. (3.3)–(3.8) with the aid of (3.1). In Fig. 1 we compare $w(0)$ as a function of the parameter z, calculated in the above approximations with "exact" result. For $z<0$ the LE and OE have two branches, depending on the choice of the symmetric $(x_0=0)$ or nonsymmetric $[x_0=\pm(-z/2)^{1/2}]$ solution of the classical equation of motion. For the LE we have plotted only the symmetric branch for $z>0$, when it is real. The other branch (not shown in the figure) is only higher by ln(2) than the ILE result, and diverges for $z=0$, as ILE does. For the optimized expansion we have plotted the symmetric (OES) as well as nonsymmetric (OEN) branch for $z<0$. For $z>-2.97$ OES, and for $z<-2.97$ OEN agrees better with the "exact" result. The MF result is reasonably good in the whole range of z.

Now we discuss the EP in the considered approximations. For $z<0$ the results of the LE, OE, and MF are very similar and agree with the "exact" EP; therefore, we discuss only $z<0$. For the LE and OE the agreement becomes less as z decreases and is very marked for negative values of z. In Fig. 2 we have plotted the results for $z=0,-1,-3,-5$. The discrepancy is particularly large in the case of the LE. The shape of the EP is very different from the "exact" one, being rather similar to the classical DW potential. The conventional one-loop EP is reliable only in the region of the convexity of the DW potential. It diverges at $\phi=-z/6$ and approaches the "exact" result only for large ϕ. For $z<-3$ the second minimum of finite depth appears at larger value of ϕ and the Maxwell construction can be done, giving the result similar to the ILE, the agreement with the "exact" EP improves for decreasing z. The mean-field results in the zeroth (MF0) and first (MF1) order are shown. The MF1 is in good agreement with the "exact" result. The same conclusion was drawn by Bender and Cooper[18] by comparison with the accurate EP for strong coupling in zero dimensions.

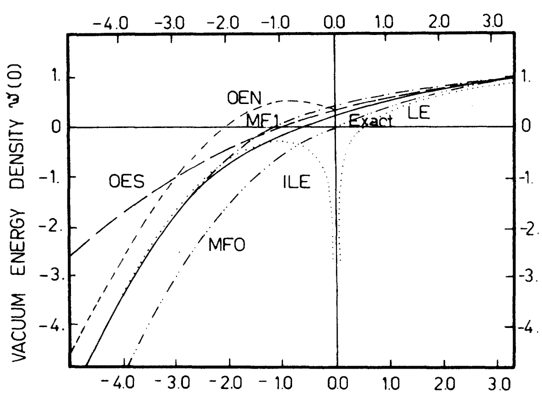

FIG. 1. The vacuum energy density $w(0)$ in zero-dimensional QFT is plotted vs $z=\frac{1}{2}m^2\lambda^{-1/2}$. The solid line is the result calculated numerically (Exact). The dotted lines are the one-loop (LE) and the interpolated one-loop (ILE) results. The dashed lines are the first-order optimized expansion results if the symmetric (OES) and the nonsymmetric (OEN) solutions of the classical equation of motion are chosen. The dashed-dotted lines give the mean field in zeroth (MF0) and first (MF1) order.

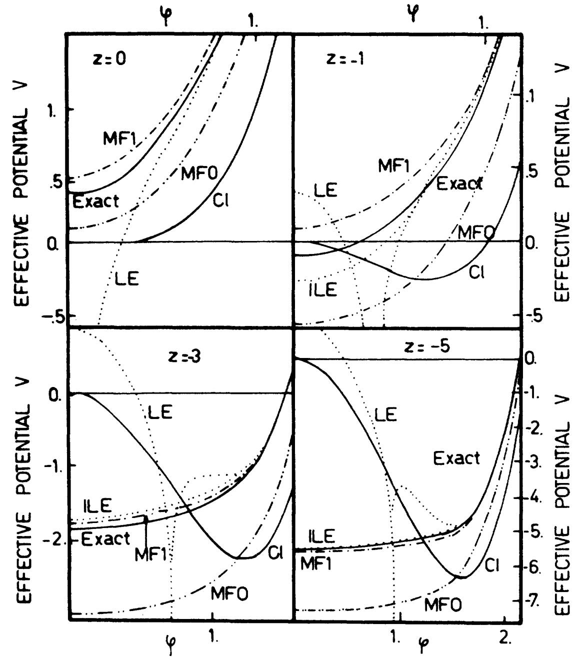

FIG. 2. The effective potential for $z=\frac{1}{2}m^2\lambda^{-1/2}=0,-1,-3,-5$ in zero-dimensional QFT. The classical (Cl) and numerical (Exact) effective potential are shown by solid lines. The dotted lines are the one-loop (LE) and the interpolated one-loop (ILE) results. The dashed-dotted lines show the zeroth- (MF0) and first- (MF1) order mean-field EP.

The EP in the first-order OE in zero dimensions[9] agrees quite well with the exact result for small $|z|$. For $z=z_{cr}\approx-2.45$ the second minimum appears at $\phi=0$ and for $z\approx-2.85$ it becomes deeper than the minimum at $\phi=0$. The shape of the EP differs considerably from the "exact" result, when $|z|$ increases, only the Maxwell construction make the agreement quite good. We can examine the change of the shape of the EP in higher orders of the OE. The "gap equation" to the second order obtained from Eq. (2.18), requiring the slope of the dependence of V_2 on Ω equal to vanish, cannot be satisfied by $\Omega^2>0$. We can choose Ω^2, as a complex root with real part positive. The real part of the resulting EP is shown in Fig. 3. We have checked that the imaginary part of the EP is small as long as $|z|<|z_{cr}|$, but for larger $|z|$ it becomes considerable in the region between the classical minima.

Alternatively, we can look for Ω^2 real and positive, which minimizes the slope of the dependence of V_2 on Ω, requiring the second (or higher) derivative to vanish. The resulting EP differs from the former one in the range where the imaginary part was large. Therefore, for $|z|$ large, the EP in the region between the classical minima

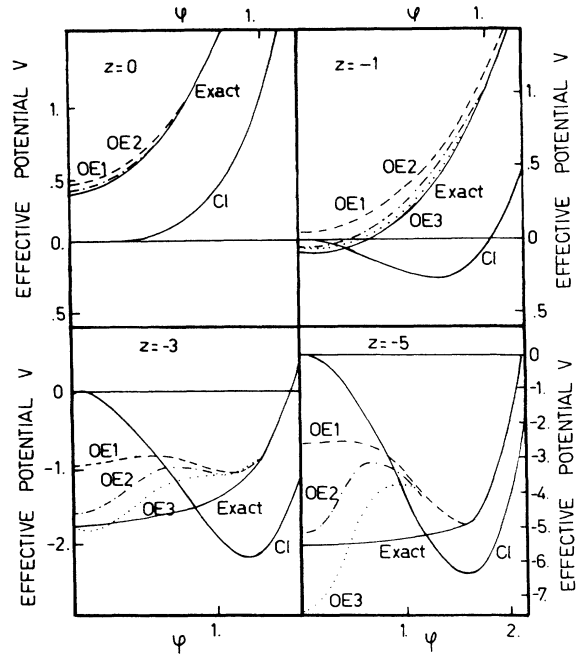

FIG. 3. As Fig. 2, but for the optimized expansion. The first order (OE1) is shown by the dashed line, the second order (OE2) by the dashed-dotted line, and the third order (OE3) by the dotted line.

depends heavily on the way of fixing Ω, and the second-order OE is not well defined. To the third (and all odd) order the situation is better, since Eq. (2.18) can be satisfied by real $\Omega^2>0$. The comparison in Fig. 3 shows, that the OE converge smoothly to the "exact" result only for $|z|<|z_{cr}|$. Above the "phase transition" the contributions of higher orders seem to increase with increasing $|z|$, especially for small ϕ. However, the Maxwell construction applied in each order improves the shape of the EP.

The energy density for vanishing source can be found as the value of the EP at the stationary point (1.8). In the MF approximation the EP has only one minimum at $\phi=0$, where the value of the EP equals $w(0)$. In the OE the value of the EP at $\phi=0$ coincides with the OES; however, for $z<-2.45$ according to the Maxwell construction the value of the EP at minimum at $\phi\neq0$ should be taken. For $z<-2.85$ this value is lower than OES and approaches OEN for large values of $|z|$, agreeing slightly better with the "exact" result than $w(0)$ calculated directly (even if the branch OEN for $z<-2.97$ is chosen). This result improves in higher orders, if the Maxwell construction is done, i.e., the value of the EP at the nonsymmetric minimum is taken, even if the minimum at $\phi=0$ becomes deeper. A similar situation appears already in the first order in the LE, as the EP develops an infinitely deep minimum. For $z<-3$, when the second minimum occurs the Maxwell construction skips the infinite minimum and gives the reasonable result. The vacuum energy density calculated as the value of the EP in the first-order LE, OE, and tree-level MF is shown in Fig. 4(a). In Figs. 4(b) and 4(c) we show in the same approximations the second and fourth derivative of the EP, reflecting the characteristic features of the EP. These derivatives are used to define the renormalized quantities in scalar QFT. In the LE the derivatives are taken at the nonsymmetric minimum, as the EP is complex at $\phi=0$. In the MF and OE they are taken at $\phi=0$ (taking the derivatives at nonsymmetric minimum in the OE gives the results similar to obtained in the LE). The only method which gives m_R and ϕ_R with reasonable accuracy for all values of z is the tree-level MF. To obtain the reasonable values of the EP, the first-order corrections have to be included (Fig. 2).

B. Quantum mechanics

The field theory in one dimension (time) is interesting on its own, as it is the quantum mechanics of the anharmonic oscillator (AO). With the measure normalization (1.3), the energy density

$$w(J)=\frac{1}{\int dt}\ln\int Dx\exp\left[\int dt\left[\frac{1}{2}x\frac{d^2x}{dt^2}-\frac{m^2}{2}x^2-\lambda x^4+Jx\right]\right], \quad (3.9)$$

is the ground-state energy of the AO interacting with the constant electric field J can be calculated as the lowest

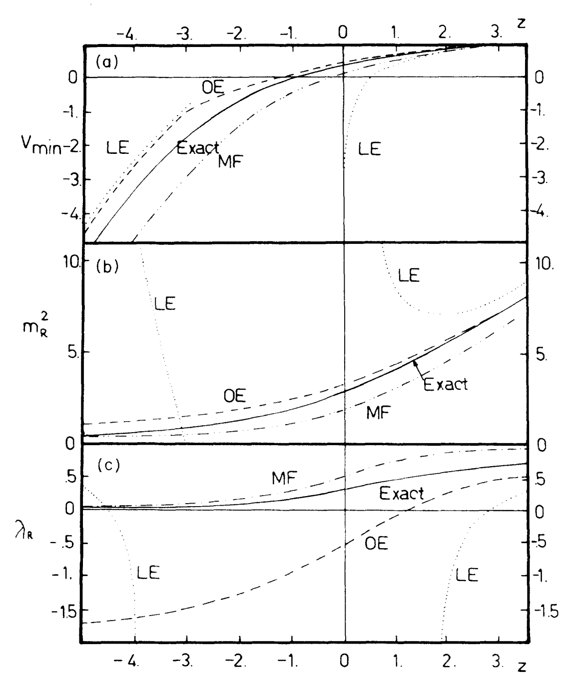

FIG. 4. (a) The value of the EP at the lowest minimum as compared with the numerical result. (b) The renormalized mass $m_R{}^2$ and (c) the renormalized coupling constant λ_R as the second and fourth derivatives of the EP in zero-dimensional QFT. In the LE the derivatives are taken at nonsymmetric and in the MF and OE at symmetric minimum.

eigenvalue of the Schrödinger equation

$$\left[-\frac{1}{2}\frac{d^2}{dx^2}+\frac{m^2}{2}x^2+\lambda x^4-Jx\right]\psi(x)=w(J)\psi(x) . \quad (3.10)$$

For $J=0$ this is the standard textbook example of perturbative calculations of bound-state energies in QM. After proving[15] that the conventional perturbation series diverges, much effort has been done to obtain the summation method. The reviews can be found in Refs. 19 and 20. Most of the methods consist in the partition of the Hamiltonian on the free and interaction part, different rather than standard. The optimized expansion is connected with the method proposed by Caswell[21] and Killingbeck[22] in QM. In the perturbative calculation with the Hamiltonian written as

$$H=\tfrac{1}{2}(p^2+\Omega^2x^2)+\epsilon[\lambda x^4+\tfrac{1}{2}(m^2-\Omega^2)x^2] \quad (3.11)$$

they obtained very good evaluation of bound-state energies. For $m^2>0$ the first order gives the ground-state energy with the error less then 2%. For DW potential it was necessary to sum more orders (for small values of λ even 20 orders were not enough). It was shown by Stevenson[23] that the value of the GEP at minimum gives the ground-state energy with the error less then 10%, for DW. The OE in QM is a generalization of the Caswell-Killingbeck perturbative approach for $J\neq 0$, which enables us to study the EP. It has been suggested by Stevenson[23] to improve the GEP in the perturbative calculation with the Hamiltonian written as

$$H=\tfrac{1}{2}p^2+\tfrac{1}{2}\Omega^2(x-\phi)^2+\tfrac{1}{2}m^2\phi^2+\lambda\phi^4 +\epsilon[\lambda x^4+\tfrac{1}{2}m^2x^2-\tfrac{1}{2}\Omega^2(x-\phi)^2-\tfrac{1}{2}m^2\phi^2-\lambda\phi^4] , \quad (3.12)$$

where apart from Ω an arbitrary translation ϕ of the position operator is introduced. To the first order, the GEP is obtained and the mean value of the position operator is equal to ϕ. In higher orders, ϕ has no longer such meaning, it can be treat only as an arbitrary parameter, which the calculated energy should not depend on. The quantity calculated in the method suggested by Stevenson is not the EP, as opposed to our OE method, which is constructed to maintain the interpretation of ϕ to all orders. Beyond the first order the results of both approaches differ.

To find the exact EP in QM, we have to solve Eq. (3.10). The accurate eigenvalues for $z<0$ have been obtained only for $J=0$ by variational methods.[24] To find the ground-state energy for $J\neq 0$, we have integrated the differential equation (3.10) numerically, requiring the wave function to vanish asymptotically and not to have any zero (for $J=0$ our results agree up to 8 digits with the variational calculations[24]). Afterward, to obtain the "exact" effective potential, the Legendre transform has been found numerically.

In QM the first order EP in the approximations studied in the previous section after rescaling ($z=\tfrac{1}{2}m^2\lambda^{-2/3}$, $V\to V\lambda^{1/3}$, $\phi\to\phi\lambda^{-1/6}$) become as follows.

(a) A loop expansion

$$V(\phi)=z\phi^2+\phi^4+\tfrac{1}{2}(2z+12\phi^2)^{-1/2} . \quad (3.13)$$

(b) An interpolated loop expansion (in the region between the classical minima)

$$V(\phi)=-\frac{z^2}{4}+\tfrac{1}{2}(-4z)^{-1/2} . \quad (3.14)$$

(c) An optimized expansion

$$V(\phi)=z\phi^2+\phi^4+\frac{\Omega}{2}+\frac{3}{4\Omega^2}-\frac{\Omega^2-m^2-6\lambda\phi^2}{4\Omega} , \quad (3.15)$$

with Ω satisfying the equation

$$\Omega^3-2(z+6\phi^2)\Omega-6=0 . \quad (3.16)$$

(d) A mean-field expansion

$$V(\phi)=-\frac{z^2}{4}-\frac{\Omega^4}{16}+\frac{\Omega^2}{4}(z+2\phi^2)-\Omega +\frac{1}{2\sqrt{2}}\left\{5\Omega^2+8\phi^2+\frac{4}{\Omega}+\left[\left[5\Omega^2+8\phi^2+\frac{4}{\Omega}\right]^2-16(\Omega^4+8\Omega^2\phi^2+\Omega)\right]^{1/2}\right\}^{1/2} +\frac{1}{2\sqrt{2}}\left\{5\Omega^2+8\phi^2+\frac{4}{\Omega}-\left[\left[5\Omega^2+8\phi^2+\frac{4}{\Omega}\right]^2-16(\Omega^4+8\Omega^2\phi^2+\Omega)\right]^{1/2}\right\}^{1/2} , \quad (3.17)$$

where Ω satisfies

$$\Omega^3-2(z+2\phi^2)\Omega-2=0 . \quad (3.18)$$

The results for the ground-state energy (Fig. 5) are similar, as in the zero-dimensional case, except there are no divergences in the LE (they appear only in higher orders). The "exact" effective potential (Fig. 6) has only one minimum at $\phi=0$, as there is no spontaneous symmetry breaking in QM. In the ILE it is flat bottomed. The shape of the EP in the OE is as in zero dimensions, only the numerical values are different. The first-order OE effective potential has an unsymmetric minimum for $z<-2.48$, which becomes deeper than the symmetric one for $z<-2.69$. It can be seen in Fig. 7 that the higher-order contributions at $\phi=0$ increase for increasing $|z|$, just the same as for $w(0)$, as it equals $V(0)$ in the OES.

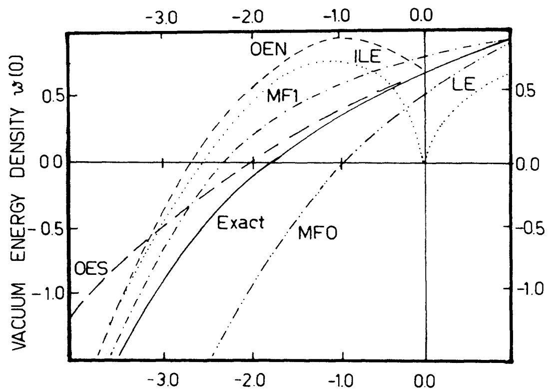

FIG. 5. The vacuum energy density $w(0)$ in one-dimensional QFT (i.e., the ground-state energy of the QM anharmonic oscillator) is plotted vs $z=\frac{1}{2}m^2\lambda^{-2/3}$. The denotation as in Fig. 1.

This is in agreement with Caswell's ground-state energy calculations,[21] where it was found that for the parameters corresponding to our $z\approx-4$ even summing 20 orders is not enough to obtain the convergence.

As in zero dimension, the value of the EP at minimum [Fig. 8(a)] is the same as $w(0)$ (MF), or agrees better (OE) with the "exact" ground-state energy. In the LE the unsymmetric minimum develops for $z<-3.25$ and the value of the EP gives quite good ground-state energy. Therefore, if the ground-state energy is calculated as a value of the EP at minimum, all the discussed methods are satisfactory. However, the only method which gives the qualitatively good dependence of the EP on ϕ in QM is the mean-field approximation, which is clearly marked in Figs. 8(b) and 8(c) where the derivatives of the EP are shown.

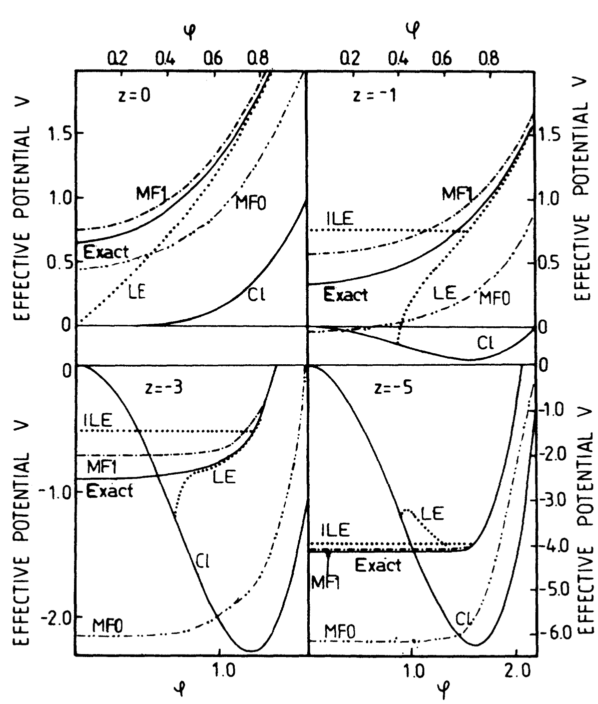

FIG. 6. The effective potential for $z=\frac{1}{2}m^2\lambda^{-2/3}=0,-1,-3,-5$ in one-dimensional QFT. The denotation as in Fig. 2.

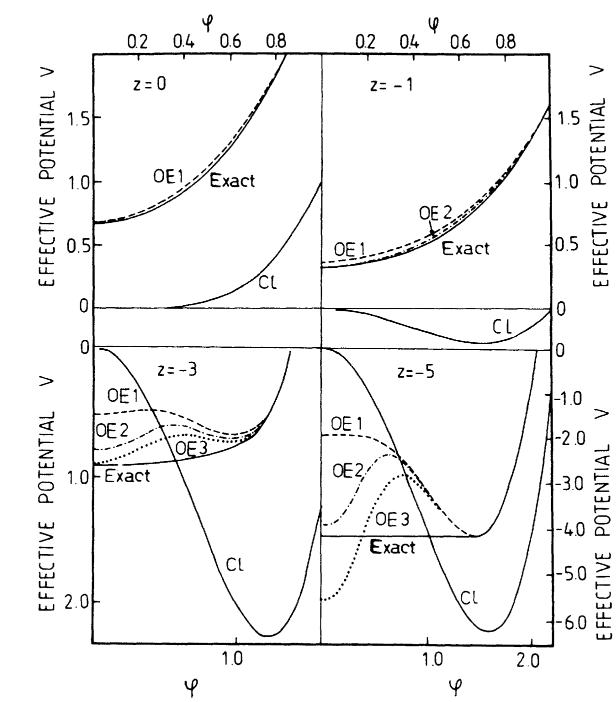

FIG. 7. As Fig. 6, but for the optimized expansion. The denotation as in Fig. 3.

C. Quantum field theory in 4 dimensions

If the dimension of the space-time is greater than one, the integrals in momentum space become divergent; therefore they should be considered as regularized. In the loop expansion the renormalization can be accomplished by expressing the effective action in terms of the physical mass and coupling constant if the VEV of the scalar field is equal to zero, as well as in the case when the symmetry is spontaneously broken.

The first-order OE is equal to the Gaussian EP, and its renormalization has been studied by Stevenson.[10] Defining the renormalized mass and coupling constant as (2.21) and (2.22), he has shown that there are two possibilities. If the positive bare coupling constant λ is kept fixed and positive, the EP for any finite cutoff is DW with the minima at the cutoff scale, but the renormalized coupling becomes -2λ and the renormalized EP

$$V(\phi)=\tfrac{1}{2}{m_R}^2\phi^2-2\lambda\phi^4 \tag{3.19}$$

becomes unstable. The other possibility is the "precarious theory,[10] if the bare coupling constant is taken to be infinitesimal and negative

$$\lambda=-\frac{1}{6I_{-1}(m_R)}\left[1+\frac{1}{2\lambda I_{-1}(m_R)}\right]. \tag{3.20}$$

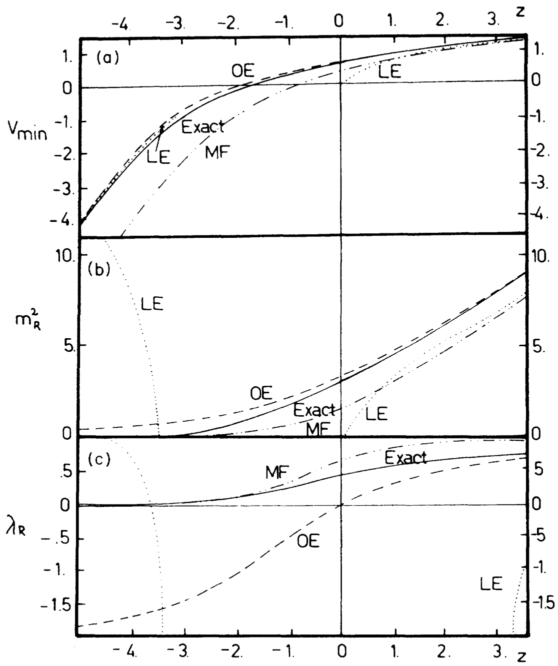

FIG. 8. As Fig. 4 but in one-dimensional QFT.

The "precarious" EP in the weak-coupling limit is different from the EP obtained with the perturbative procedure applied before sending the cutoff to infinity.[10]

In the MF the renormalization should be done at the tree level as the divergencies appear. The renormalized mass and coupling constant are given by (2.37) and (2.38). Keeping the bare coupling constant λ fixed and positive, the renormalized coupling constant approaches zero, and the renormalized EP

$$V(\phi)=\tfrac{1}{2}{m_R}^2\phi^2 \tag{3.21}$$

becomes trivial. This result has been obtained in large-N expansion.[25] If the bare coupling constant is infinitesimal and negative,

$$\lambda=-\frac{1}{2I_{-1}(m_R)}\left[1+\frac{1}{2\phi I_{-1}(m_R)}\right], \tag{3.22}$$

the same "precarious" EP as in the first-order OE is obtained. However, in the MF approximation the weak-coupling limit of the "precarious" EP agrees with the perturbative result. The different relation to the perturbative approach is due to the different relation between bare and renormalized coupling in OE (3.20) and MF (3.22). It was observed by Stevenson[10] that the large-N limit restores the connection between "precarious" and "perturbative" ϕ^4 theory. The loss of this connection can be the feature of the OE approximation, but not the ϕ^4 QFT.

The results of OE and MF for scalar QFT with positive bare coupling differ from the conventional, perturbatively renormalized theory. The MF result that the renormalized theory is trivial agrees with the rigorous results.[26] This indicates that the MF expansion is the most reliable of the considered methods, as it is in lower dimensional space-time. The unstable EP obtained in the OE, can be attributed to the breaking down of the method. In 4 dimensions the parameter $z=\frac{1}{2}m^2\lambda^{2/(n-4)}\rightarrow-\infty$, as the bare mass should be negative to cancel $I_0(m_R)$ contribution. In lower dimensions the OE method fails for $z\rightarrow-\infty$. The EP is double well, while the "exact" one is single-well shaped; therefore, the derivatives corresponding to the renormalized parameters are completely wrong (Figs. 4 and 8). If it persists in 4 dimensions, the reparametrization in terms of renormalized mass and coupling constant can introduce the unboundness.

ACKNOWLEDGMENTS

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APPENDIX

In this appendix we will calculate the effective potential $V(\phi,\Omega)$ in the optimized expansion up to the third order in ϵ. The first three terms in the expansion (2.18) are drawn in Fig. 9 by means of Feynman diagrams. It is straightforward to derive the corresponding momentum integrals:

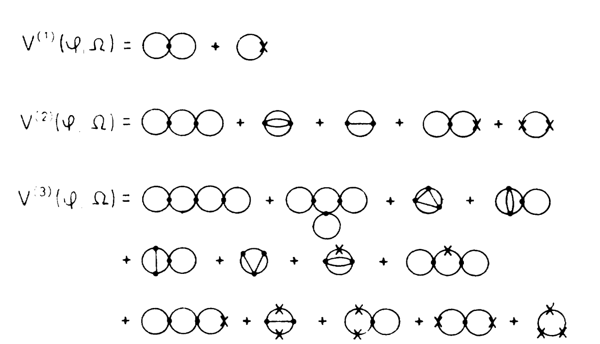

FIG. 9. The Feynman diagrams, which contribute to the EP in the first three orders of the OE. The solid line denotes the propagator $1/(k^2+\Omega^2)$, the dots denote the quartic (24λ) and cubic ($24\lambda\phi$) vertices, and the cross corresponds to the quadratic vertex ($\Omega^2-m^2-12\lambda\phi^2$).

$$V^{(1)}(\phi,\Omega)=\tfrac{1}{2}(m^2+12\lambda\phi^2-\Omega^2)\int\frac{d_np}{(2\pi)^n}\frac{1}{p^2+\Omega^2}+3\lambda\left[\int\frac{d_np}{(2\pi)^n}\frac{1}{p^2+\Omega^2}\right]^2, \tag{A1}$$

$$\begin{aligned}V^{(2)}(\phi,\Omega)=&-12\lambda^2\int\frac{d_np\,d_nq\,d_nr}{(2\pi)^{3n}}\frac{1}{(p^2+\Omega^2)(q^2+\Omega^2)(r^2+\Omega^2)[(p+q+r)^2+\Omega^2]}\\&-36\lambda^2\int\frac{d_np}{(2\pi)^n}\frac{1}{(p^2+\Omega^2)^2}\left[\int\frac{d_nq}{(2\pi)^n}\frac{1}{(q^2+\Omega^2)}\right]^2\\&-48\lambda^2\phi^2\int\frac{d_np\,d_nq}{(2\pi)^{2n}}\frac{1}{(p^2+\Omega^2)(q^2+\Omega^2)[(p+q)^2+\Omega^2]}\\&-6\lambda(m^2+12\lambda\phi^2-\Omega^2)\int\frac{d_np}{(2\pi)^n}\frac{1}{(p^2+\Omega^2)^2}\int\frac{d_nq}{(2\pi)^n}\frac{1}{(q^2+\Omega^2)}\\&-\tfrac{1}{4}(m^2+12\lambda\phi^2-\Omega^2)^2\int\frac{d_np}{(2\pi)^n}\frac{1}{(p^2+\Omega^2)^2},\end{aligned} \tag{A2}$$

and

$$\begin{aligned}V^{(3)}(\phi,\Omega)=&432\lambda^3\left[\int\frac{d_np}{(2\pi)^n}\frac{1}{(p^2+\Omega^2)^2}\right]^2\left[\int\frac{d_np}{(2\pi)^n}\frac{1}{(p^2+\Omega^2)}\right]^2\\&+288\lambda^3\int\frac{d_np\,d_nq\,d_nr\,d_ns}{(2\pi)^{4n}}\frac{1}{(p^2+\Omega^2)(q^2+\Omega^2)(r^2+\Omega^2)(s^2+\Omega^2)[(p+q+r)^2+\Omega^2][(p+q+s)^2+\Omega^2]}\\&+288\lambda^3\int\frac{d_np}{(2\pi)^n}\frac{1}{(p^2+\Omega^2)^3}\left[\int\frac{d_np}{(2\pi)^n}\frac{1}{(p^2+\Omega^2)}\right]^3\\&+576\lambda^3\int\frac{d_np}{(2\pi)^n}\frac{1}{(p^2+\Omega^2)}\int\frac{d_np\,d_nq\,d_nr}{(2\pi)^{3n}}\frac{1}{(p^2+\Omega^2)^2(q^2+\Omega^2)(r^2+\Omega^2)[(p+q+r)^2+\Omega^2]}\\&+1728\lambda^3\phi^2\int\frac{d_np\,d_nq}{(2\pi)^{2n}}\frac{1}{(p^2+\Omega^2)^2(q^2+\Omega^2)[(p+q)^2+\Omega^2]}\int\frac{d_np}{(2\pi)^n}\frac{1}{(p^2+\Omega^2)}\\&+1728\lambda^3\phi^2\int\frac{d_np\,d_nq\,d_nr}{(2\pi)^{3n}}\frac{1}{(p^2+\Omega^2)(q^2+\Omega^2)(r^2+\Omega^2)[(p+q)^2+\Omega^2][(p+r)^2+\Omega^2]}\\&+48\lambda^2(m^2+12\lambda\phi^2-\Omega^2)\int\frac{d_np\,d_nq\,d_nr}{(2\pi)^{3n}}\frac{1}{(p^2+\Omega^2)^2(q^2+\Omega^2)(r^2+\Omega^2)[(p+q+r)^2+\Omega^2]}\\&+72\lambda^2(m^2+12\lambda\phi^2-\Omega^2)\int\frac{d_np}{(2\pi)^n}\frac{1}{(p^2+\Omega^2)^3}\left[\int\frac{d_np}{(2\pi)^n}\frac{1}{(p^2+\Omega^2)}\right]^2\\&+72\lambda^2(m^2+12\lambda\phi^2-\Omega^2)\left[\int\frac{d_np}{(2\pi)^n}\frac{1}{(p^2+\Omega^2)^2}\right]^2\int\frac{d_np}{(2\pi)^n}\frac{1}{(p^2+\Omega^2)}\\&+144\lambda^2\phi^2(m^2+12\lambda\phi^2-\Omega^2)\int\frac{d_np\,d_nq}{(2\pi)^{2n}}\frac{1}{(p^2+\Omega^2)^2(q^2+\Omega^2)[(p+q)^2+\Omega^2]}\\&+6\lambda(m^2+12\lambda\phi^2-\Omega^2)^2\int\frac{d_np}{(2\pi)^n}\frac{1}{(p^2+\Omega^2)^3}\int\frac{d_np}{(2\pi)^n}\frac{1}{(p^2+\Omega^2)}\\&+3\lambda(m^2+12\lambda\phi^2-\Omega^2)^2\left[\int\frac{d_np}{(2\pi)^n}\frac{1}{(p^2+\Omega^2)^2}\right]^2\\&+\tfrac{1}{6}(m^2+12\lambda\phi^2-\Omega^2)^3\int\frac{d_np}{(2\pi)^n}\frac{1}{(p^2+\Omega^2)^3}.\end{aligned} \tag{A3}$$

These expressions are finite only if the dimension of the space-time is less than 2. In zero dimensions we have to perform "the integration in zero-dimensional momentum space," i.e., to drop all the integrals, setting the momenta equal to zero and we obtain

$$V(\phi,\Omega)=\tfrac{1}{2}m^2\phi^2+\lambda\phi^4+\ln\Omega+\epsilon\left[\frac{3\lambda}{\Omega^4}-\frac{\Omega^2-m^2-6\lambda\phi^2}{2\Omega^2}\right]$$
$$+\epsilon^2\left[-\frac{48\lambda^2}{\Omega^8}-\frac{48\lambda^2\phi^2}{\Omega^6}+\frac{6\lambda(\Omega^2-m^2-6\lambda\phi^2)}{\Omega^6}-\frac{(\Omega^2-m^2-6\lambda\phi^2)^2}{4\Omega^4}\right]$$
$$+\epsilon^3\left[\frac{1584\lambda^3}{\Omega^{12}}+\frac{3456\lambda^3\phi^2}{\Omega^{10}}-\frac{192\lambda^2(\Omega^2-m^2-6\lambda\phi^2)}{\Omega^{10}}-\frac{144\lambda^2\phi^2(\Omega^2-m^2-6\lambda\phi^2)}{\Omega^8}\right.$$
$$\left.+\frac{9\lambda(\Omega^2-m^2-6\lambda\phi^2)^2}{\Omega^8}-\frac{(\Omega^2-m^2-6\lambda\phi^2)^3}{6\Omega^6}\right]+\cdots\,. \tag{A4}$$

In one dimension the one-dimensional momentum integrals can be easily evaluated giving

$$V(\phi,\Omega)=\tfrac{1}{2}m^2\phi^2+\lambda\phi^4+\frac{\Omega}{2}+\epsilon\left[\frac{3\lambda}{4\Omega^2}-\frac{\Omega^2-m^2-6\lambda\phi^2}{4\Omega}\right]$$
$$+\epsilon^2\left[-\frac{21\lambda^2}{8\Omega^5}-\frac{4\lambda^2\phi^2}{\Omega^4}+\frac{3\lambda(\Omega^2-m^2-6\lambda\phi^2)}{4\Omega^4}-\frac{(\Omega^2-m^2-6\lambda\phi^2)^2}{16\Omega^3}\right]$$
$$+\epsilon^3\left[\frac{333\lambda^3}{16\Omega^8}+\frac{78\lambda^3\phi^2}{\Omega^7}-\frac{105\lambda^2(\Omega^2-m^2-6\lambda\phi^2)}{16\Omega^7}-\frac{8\lambda^2\phi^2(\Omega^2-m^2-6\lambda\phi^2)}{\Omega^6}\right.$$
$$\left.+\frac{3\lambda(\Omega^2-m^2-6\lambda\phi^2)^2}{4\Omega^6}-\frac{(\Omega^2-m^2-6\lambda\phi^2)^3}{32\Omega^5}\right]+\cdots\,. \tag{A5}$$

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*Permanent address: Institute of Physics, Warsaw University, Białystok Branch, 15-424 Białystok, Lipowa 41, Poland.

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