PHYSICAL REVIEW D VOLUME 35, NUMBER 6 15 MARCH 1987

Quantum mechanics of measurements distributed in time. II. Connections among formulations

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(Received 6 August 1986)

Measurements distributed in time provide information about a system at more than one time; they cannot be described in terms of the conventional language of a system quantum state evolving in time. This paper, the second in a series, explores connections among various ways of formulating a quantum-mechanical description of time-distributed measurements. The natural formulation, involving a "sum over histories," arises directly from Feynman's rules for combining probability amplitudes. One equivalent formulation uses a standard measurement model, in which the system is coupled to a set of "measuring apparatuses." A second equivalent formulation uses the language of "effects" and "operations." Still a third formulation attempts to create a new language of multiple-time states and multiple-time eigenstates.

I. INTRODUCTION

Quantum mechanics is a logic—a set of rules—for manipulating probability amplitudes.[1] The task of this logic—its only task—is to provide the joint statistics for successive observations of some system.[2] Often one implements the amplitude logic in terms of a wave function, or quantum state, for the system. Consider, for example, a sequence of instantaneous measurements of some quantity. One can calculate the required joint statistics from a system wave function, which evolves in time according to two rules—between measurements, unitary Schrödinger evolution, and at the time of each measurement, a sudden, nonunitary change called the "collapse of the wave function."[2-4] One is sometimes tempted to think that the wave function represents an objective "state of the system"—that it describes "physical reality" during the time between measurements—but quantum mechanics allows no such interpretation. The only reality is the network of observations we make; nothing can be said about a system between observations.[5] The wave function is merely a tool for implementing the amplitude logic of quantum mechanics.

Nowhere is this conclusion clearer than for measurements distributed in time. Nowhere clearer because the notion of a system quantum state loses its usefulness.[6,7] A measurement distributed in time[7] (or a multiple-time measurement[6,8-10]) provides information about a system not at a single time, but at many times—perhaps over a continuous time interval. During a sequence of measurements distributed in time, the measurements can overlap and interleave in complicated ways. There are in general no times when the system is undisturbed by measurements, no periods of unitary evolution. One can find no "time of each measurement" when there should be a sudden disturbance of the system, no set of times when there should be instantaneous collapses of the wave function. Ordinary unitary evolution and the disturbances produced by measurements become so entangled that there is no way to derive the joint statistics from a system quantum state evolving in time.

Despite these difficulties, one can still implement the amplitude logic of quantum mechanics for a sequence of measurements distributed in time. The natural formulation[6,7,9,10] involves a "sum over histories," from which one calculates the joint statistics. The effect of each measurement is to restrict the sum over histories. A sum-over-histories formulation is natural because each time-distributed measurement can restrict the sum—disturb the system—at many times, instead of at a single time.

This is the second in a series of papers on measurements distributed in time. The first paper in the series[7] developed a path-integral formulation for time-distributed measurements which provide information about the position of a system. Aharonov and Albert[6] and, more recently, D'Amato[9,10] have given a similar sum-over-histories formulation for multiple-time measurements involving spin.

In this paper I review in Sec. II the sum-over-histories formulation for measurements distributed in time, both for measurements involving position (Sec. II A) and for measurements involving spin (Sec. II B). The purpose of this paper is to make connections between the sum-over-histories formulation and other, equivalent formulations for measurements distributed in time. One equivalent formulation (Sec. III) recovers the notion of a quantum state by extending the quantum-mechanical analysis to include a set of "measuring apparatuses," each of which records and stores the result of one measurement.[6,7,9-11] A second equivalent formulation (Sec. IV) uses the language of "effects" and "operations,"[12] which have been used extensively to analyze measurements in quantum mechanics.[13-15] Still a third formulation (Sec. V) attempts to create a new language of multiple-time states and multiple-time eigenstates.[8-10] All of these formulations testify to the same conclusion: measurements distributed in time cannot be analyzed in terms of the conventional language of a system quantum state evolving in time. A final section (Sec. VI) offers concluding remarks.

II. SUM-OVER-HISTORIES FORMULATION

A sum over histories provides a natural language for describing measurements distributed in time—natural be-

cause a sum over histories arises directly from Feynman's rules[1] for combining probability amplitudes. Consider two quantities A and B. The amplitude of A given B times the amplitude of B is the joint amplitude of A and B. Depending on whether B is potentially observable, the probability of A is derived from the joint amplitude in one of two ways. If B is unobservable, then the joint amplitude, summed over all values of B, yields the amplitude of A, whose absolute square is the probability of A. If B is observable, then the absolute square of the joint amplitude is the joint probability of A and B, which, summed over B, yields the probability of A.

In this section I outline the sum-over-histories formulation, first for measurements involving position and then for measurements involving spin, and I interpret the formulation in terms of Feynman's rules.

A. Measurements involving position

Consider a one-dimensional, nonrelativistic, quantum-mechanical system with position x, momentum p ($[\hat{x},\hat{p}]=i\hbar$), Lagrangian $L(x,\dot{x};t)$, and Hamiltonian $\hat{H}$. (Throughout this paper Hilbert-space operators are distinguished by a caret.) Denote the δ-function normalized eigenstates of $\hat{x}$ by $|x\rangle$.

Consider further a sequence of Q measurements, each of which provides time-distributed information about $x(t)$. Each measurement can be regarded as sampling, at a particular time t, the value of a quantity $y(t)$, which is a functional of $x(t')$ for $t'\leq t$. Choose the functional form of $y(t)$ to be

$$y(t)\equiv\int_{-\infty}^{\infty}dt'\,Y_{tt'}(x(t'))=\int_{t-\Delta_t}^{t}dt'\,Y_{tt'}(x(t'))\,, \tag{2.1}$$

where $Y_{tt'}$ denotes a real function of position which depends on both t and t' and which vanishes identically if $t-t'<0$ or $t-t'>\Delta_t\geq 0$. Notice that a sampling of $y(t)$ yields information about $x(t')$ in the past, during the interval $[t-\Delta_t,t]$ of duration Δ_t. Finally, let the samplings of $y(t)$ occur at times $t_1,\ldots,t_Q$ ($t_1<t_2<\cdots<t_Q$); then the Q sampled quantities are given by

$$y(t_q)=\int_{t_q-\Delta_{t_q}}^{t_q}dt\,Y_{t_q t}(x(t))\,,\quad q=1,\ldots,Q\,. \tag{2.2}$$

After the samplings are completed, one has accumulated Q numbers, one result for each sampling. Label the result of the qth sampling by $\bar{y}_q$. The task of quantum mechanics is to provide a joint probability distribution $P(\bar{y}_1,\ldots,\bar{y}_Q)$ for the samplings to yield the sequence of results $\bar{y}_1,\ldots,\bar{y}_Q$. This joint probability distribution can be derived from a Feynman path integral,[1] but before writing the path-integral expression, two further ingredients are necessary.

First, one must take into account the irresolution or imprecision of the samplings. Such irresolution can be incorporated in a conditional probability amplitude $\Upsilon(\bar{y}-y)$, called the resolution amplitude;[7] $\Upsilon(\bar{y}-y)$ is the amplitude to obtain the result $\bar{y}$ in a sampling at time t, given that $y(t)$ has precisely the value y. (For simplicity, I assume here that all the samplings have the same resolution amplitude and that the resolution amplitude is only a function of the difference $\bar{y}-y$.) The form of the resolution amplitude must come ultimately from a detailed description of whatever apparatus is used to make the measurements, but quite generally the resolution amplitude means that even if $y(t)$ has a precise value, the result of a sampling at time t cannot be predicted with certainty.

Second, one must specify an initial state for the system at some time t_0 that precedes all times that contribute to the sampled quantities ($t_0<t_q-\Delta_{t_q}$ for $q=1,\ldots,Q$). Let $|\psi_0(t_0)\rangle$ be this initial state, with the corresponding wave function $\psi_0(x,t_0)=\langle x\,|\,\psi_0(t_0)\rangle$.

In hand now are the ingredients for deriving the joint probability distribution $P(\bar{y}_1,\ldots,\bar{y}_Q)$. Derive it from a joint probability amplitude $\Phi(\bar{y}_1,\ldots,\bar{y}_Q;x,t_Q)$, which is the amplitude that the samplings yield the results $\bar{y}_1,\ldots,\bar{y}_Q$ and that the system is at x at time t_Q. This fundamental joint probability amplitude has the following path-integral expression:[7]

$$\Phi(\bar{y}_1,\ldots,\bar{y}_Q;x,t_Q)=\int_{t_0}^{(x,t_Q)}\mathscr{D}x(t)\left[\prod_{j=1}^{Q}\Upsilon(\bar{y}_q-y(t_q))\right]e^{(i/\hbar)S[x(t)]}\psi_0(x(t_0),t_0)\,. \tag{2.3a}$$

In this expression the integral denotes a sum over all paths $x(t)$ on the interval $[t_0,t_Q]$ such that $x(t_Q)=x$ [initial positions $x(t_0)$ are summed over]; the sampled quantity $y(t_q)$ is evaluated from $x(t)$ using Eq. (2.2);[16] and

$$S[x(t)]=\int_{t_0}^{t_Q}dt\,L(x,\dot{x};t)$$

is the action for the path $x(t)$.

Interpretation of the path integral (2.3a) arises from Feynman's rules.[1] Select a path $x(t)$, and begin with the quantity $e^{(i/\hbar)S[x(t)]}$, the familiar quantum-mechanical amplitude for the path, conditioned on the initial value $x(t_0)$. Multiply by the initial wave function $\psi_0(x(t_0),t_0)$, the amplitude for the path's initial value $x(t_0)$; thereby obtain the unconditioned amplitude

$$e^{(i/\hbar)S[x(t)]}\psi_0(x(t_0),t_0)$$

for the path $x(t)$. For each $q=1,\ldots,Q$, multiply by a resolution amplitude $\Upsilon(\bar{y}_q-y(t_q))$, the amplitude to obtain the result $\bar{y}_q$ in the qth sampling, given the path's value for $y(t_q)$—or, more generally, given the path $x(t)$; thereby find the joint amplitude

$$\left[\prod_{q=1}^{Q}\Upsilon(\bar{y}_q-y(t_q))\right]e^{(i/\hbar)S[x(t)]}\psi_0(x(t_0),t_0)$$

for the sequence of results $\bar{y}_1,\ldots,\bar{y}_Q$ and for the path $x(t)$. Finally, compute the fundamental amplitude

$\Phi(\bar{y}_1,\ldots,\bar{y}_Q;x,t_Q)$ by summing over all paths such that $x(t_Q)=x$.

Why not sum over final values $x(t_Q)$ as well? Because the system's final position is potentially observable by an independent measurement. Hence, first take the absolute square of Φ, and then integrate over final positions x to obtain the joint probability distribution

$$P(\bar{y}_1,\ldots,\bar{y}_Q)=\int dx\ |\Phi(\bar{y}_1,\ldots,\bar{y}_Q;x,t_Q)|^2 . \tag{2.3b}$$

This joint probability distribution is normalized with respect to the integration measure $d\bar{y}_1\cdots d\bar{y}_Q$, i.e.,

$$1=\int d\bar{y}_1\cdots d\bar{y}_Q\,P(\bar{y}_1,\ldots,\bar{y}_Q) , \tag{2.4}$$

provided that the resolution amplitude $\Upsilon(\bar{y}-y)$ and the initial wave function $\psi_0(x,t_0)$ are normalized to unity [see Eq. (3.8)].

Equations (2.3) constitute the path-integral (or sum-over-histories) formulation for time-distributed measurements involving position. Mensky[17,18] has given a similar path-integral formulation for specific kinds of time-distributed measurements made on a harmonic oscillator. In particular, Mensky has analyzed measurements of the Fourier components of $x(t)$ when the system is a harmonic oscillator.[18] Schmid[19] has also recently considered a path-integral formulation for time-distributed measurements on dissipative linear systems.

Notice that if $y(t)=x(t)$, Eqs. (2.3) describe a sequence of instantaneous measurements (samplings) of position. In this situation the resolution amplitudes restrict the sum over paths only at the measurement times t_q. The periods of unrestricted sum over paths between measurements correspond to unitary evolution, and the restriction of the sum at each measurement time corresponds to instantaneous wave-function collapse. In contrast, for measurements distributed in time, the resolution amplitudes restrict the sum over paths at many times; there is, in general, no sharp demarcation between unitary evolution and the disturbances produced by measurements.

It is convenient to rewrite the fundamental amplitude (2.3a) in terms of a kernel which is independent of the system's initial wave function. There are two good choices for such a kernel. One choice is defined by

$$\kappa(y_1,\ldots,y_Q;x,t_Q\,|\,x_0,t_0)\equiv\int_{(x_0,t_0)}^{(x,t_Q)}\mathscr{D}x(t)\left[\prod_{q=1}^{Q}\delta(y_q-y(t_q))\right]e^{(i/\hbar)S[x(t)]} , \tag{2.5}$$

where the integral denotes a sum over all paths $x(t)$ on the interval $[t_0,t_Q]$ such that $x(t_0)=x_0$ and $x(t_Q)=x$. In Eq. (2.5) the δ functions restrict the sum over paths so that only paths which have the values $y_1,\ldots,y_Q$ for the sampled quantities contribute to the sum. The kernel κ can be interpreted as a conditional probability amplitude, the amplitude that the sampled quantities have precisely the values $y_1,\ldots,y_Q$ and that the system is at x at time t_Q, given that the system was at x_0 at time t_0. The second choice for a kernel, defined by

$$\begin{aligned}\mathscr{K}(\bar{y}_1,\ldots,\bar{y}_Q;x,t_Q\,|\,x_0,t_0)&\equiv\int\left[\prod_{q=1}^{Q}dy_q\,\Upsilon(\bar{y}_q-y_q)\right]\kappa(y_1,\ldots,y_Q;x,t_Q\,|\,x_0,t_0)\\&=\int_{(x_0,t_0)}^{(x,t_Q)}\mathscr{D}x(t)\left[\prod_{q=1}^{Q}\Upsilon(\bar{y}_q-y(t_q))\right]e^{(i/\hbar)S[x(t)]} ,\end{aligned} \tag{2.6}$$

takes into account the resolution of the samplings. It can be interpreted as the amplitude that the samplings yield the results $\bar{y}_1,\ldots,\bar{y}_Q$ and that the system is at x at time t_Q, given that the system was at x_0 at time t_0. The fundamental amplitude (2.3a) can now be written as

$$\Phi(\bar{y}_1,\ldots,\bar{y}_Q;x,t_Q)=\int dx_0\,\mathscr{K}(\bar{y}_1,\ldots,\bar{y}_Q;x,t_Q\,|\,x_0,t_0)\psi_0(x_0,t_0) . \tag{2.7}$$

In the absence of samplings, both κ and $\mathscr{K}$ reduce to the ordinary propagator

$$K(x,t_Q\,|\,x_0,t_0)\equiv\int_{(x_0,t_0)}^{(x,t_Q)}\mathscr{D}x(t)\,e^{(i/\hbar)S[x(t)]}=\langle x\,|\,\hat{U}(t_Q,t_0)\,|\,x_0\rangle , \tag{2.8}$$

the conditional amplitude that the system is at x at time t_Q, given that it was at x_0 at time t_0. In Eq. (2.8), $\hat{U}(t_Q,t_0)$ is the unitary evolution operator corresponding to the system Hamiltonian $\hat{H}$. Both kernels can be regarded as "modified propagators"—modified because of the measurements.

Important for later developments are the following Fourier-domain representations of the kernels:

$$\begin{aligned}\kappa(k_1,\ldots,k_Q;x,t_Q\,|\,x_0,t_0)&\equiv\int\left[\prod_{q=1}^{Q}dy_q e^{-ik_q y_q}\right]\kappa(y_1,\ldots,y_Q;x,t_Q\,|\,x_0,t_0)\\&=\int_{(x_0,t_0)}^{(x,t_Q)}\mathscr{D}x(t)\exp\left[\frac{i}{\hbar}\left[S[x(t)]-\sum_{q=1}^{Q}\hbar k_q y(t_q)\right]\right] ,\end{aligned} \tag{2.9}$$

$$\begin{aligned}\mathscr{K}(k_1,\ldots,k_Q;x,t_Q\,|\,x_0,t_0)&\equiv\int\left[\prod_{q=1}^{Q}\frac{d\bar{y}_q}{(2\pi)^{1/2}}e^{-ik_q\bar{y}_q}\right]\mathscr{K}(\bar{y}_1,\ldots,\bar{y}_Q;x,t_Q\,|\,x_0,t_0)\\&=\left[\prod_{q=1}^{Q}\Upsilon(k_q)\right]\kappa(k_1,\ldots,k_Q;x,t_Q\,|\,x_0,t_0) .\end{aligned} \tag{2.10}$$

Here $\Upsilon(k)$ is the Fourier transform of the resolution amplitude:

$$\Upsilon(k)\equiv\int\frac{d(\bar{y}-y)}{(2\pi)^{1/2}}\Upsilon(\bar{y}-y)e^{-ik(\bar{y}-y)}\ . \tag{2.11}$$

B. Measurements involving spin

Consider a spin-$\frac{1}{2}$ (two-level) system with spin $\mathbf{S}=\hbar\boldsymbol{\sigma}/2$ and Hamiltonian $\hat{H}$ (e.g., in the presence of a magnetic field $\mathbf{B}$, the Hamiltonian is $\hat{H}=-\mu\hat{\mathbf{S}}\cdot\mathbf{B}$, where μ is the system's gyromagnetic ratio). The components of the vector operator $\hat{\boldsymbol{\sigma}}$ can be represented by the Pauli matrices; one need only note here the commutator of the components of $\hat{\boldsymbol{\sigma}}$ along arbitrary unit vectors $\mathbf{e}_1$ and $\mathbf{e}_2$:

$$[\hat{\boldsymbol{\sigma}}\cdot\mathbf{e}_1,\hat{\boldsymbol{\sigma}}\cdot\mathbf{e}_2]=2i\hat{\boldsymbol{\sigma}}\cdot\mathbf{e}_1\times\mathbf{e}_2\ .$$

Let $\mathbf{e}$ be a unit vector, and denote the eigenstates of the spin component $\hat{\boldsymbol{\sigma}}\cdot\mathbf{e}$ by $|\sigma,\mathbf{e}\rangle$, where $\sigma=\pm1$ is the eigenvalue, i.e.,

$$\hat{\boldsymbol{\sigma}}\cdot\mathbf{e}\,|\,\sigma,\mathbf{e}\rangle=\sigma\,|\,\sigma,\mathbf{e}\rangle\ . \tag{2.12}$$

The task now is to develop a sum-over-histories formulation for a sequence of time-distributed measurements involving spin. The measurements can be regarded as a sequence of samplings of a quantity $y(t)$ at times $t_1,\ldots,t_Q$ ($t_1<t_2<\cdots<t_Q$). The qth sampled quantity $y(t_q)$ depends on previous spin values, in a way analogous to the qth sampled quantity for measurements involving position [Eq. (2.2)]. There are, however, two important differences. First, because spin is a discrete quantity, I find it convenient to let $y(t_q)$ be a sum of contributions from a set of discrete times t_{qj}, $j=1,\ldots,N_q$, where $t_{q1}<t_{q2}<\cdots<t_{qN_q}\le t_q$. The case where $y(t_q)$ is an integral over contributions from a continuous time interval could be treated by a limiting procedure. Second, I want to allow $y(t_q)$ to receive contributions from different spin components at different times. To deal with this possibility, introduce for each time t_{qj} a unit vector $\mathbf{e}_{qj}$, which specifies the spin component that contributes to $y(t_q)$ at time t_{qj}. These considerations lead to the following form for the qth sampled quantity:

$$y(t_q)\equiv\sum_{j=1}^{N_q}Y_{qj}(\boldsymbol{\sigma}(t_{qj})\cdot\mathbf{e}_{qj})\ . \tag{2.13}$$

Here Y_{qj} is a real-valued function of spin component (real-valued function defined on the domain consisting of $+1$ and -1).

Worth emphasizing are the instructions Eq. (2.13) gives for computing the qth sampled quantity: $y(t_q)$ is a sum of contributions from the times t_{qj}; to obtain the jth contribution, apply the function Y_{qj} to the value of the spin component $\boldsymbol{\sigma}\cdot\mathbf{e}_{qj}$ at time t_{qj}. For example, if $\mathbf{e}_{qj}=\mathbf{e}_z$ and $Y_{qj}(\sigma)=\sigma/N_q$, then

$$y(t_q)=\frac{1}{N_q}\sum_{j=1}^{N_q}\boldsymbol{\sigma}(t_{qj})\cdot\mathbf{e}_z$$

is the average of the z-spin values at the times t_{qj}.

Just as for measurements involving position, label the results of the samplings by $\bar{y}_1,\ldots,\bar{y}_Q$, and introduce a resolution amplitude $\Upsilon(\bar{y}-y)$ to account for the imprecision of the samplings. In addition, let $|\psi_0(t_0)\rangle$ be the initial state of the system at some time t_0 that precedes all the times t_{qj}; then $\langle\sigma_0,\mathbf{e}_0\,|\,\psi_0(t_0)\rangle$ is the amplitude that $\boldsymbol{\sigma}\cdot\mathbf{e}_0$ has the value σ_0 at time t_0.

Before formulating a sum over histories, two further preliminaries are necessary. First, one must order explicitly the times t_{qj}. To do so, introduce an index β which runs from 1 to

$$B\equiv\sum_{q=1}^{Q}N_q$$

(B is the total number of times t_{qj}). Further, define a map from the index β to the pair of indices qj:

$$\beta\rightarrow q_\beta j_\beta\equiv(\beta)\ . \tag{2.14}$$

Equation (2.14) introduces a shorthand notation: (β) stands for $q_\beta j_\beta$—i.e., $t_{(\beta)}=t_{q_\beta j_\beta}$ and $\mathbf{e}_{(\beta)}=\mathbf{e}_{q_\beta j_\beta}$. Finally, choose the map so that β orders the times t_{qj} (assume no two times coincide)—i.e.,

$$t_{(1)}<t_{(2)}<\cdots<t_{(B)}\ . \tag{2.15}$$

It is useful to extend the index β to $\beta=0$ so that the initial time is $t_0\equiv t_{(0)}<t_{(1)}$ and to $\beta=B+1$ so that $t_Q\equiv t_{(B+1)}\ge t_{(B)}$. One can choose initial and final spin directions $\mathbf{e}_0\equiv\mathbf{e}_{(0)}$ and $\mathbf{e}_Q\equiv\mathbf{e}_{(B+1)}$ for convenience, since these spin directions are not specified by the sampled quantities.

The second preliminary is to identify the analogue of $e^{(i/\hbar)S[x(t)]}$, the amplitude for a path $x(t)$, conditioned on the initial value $x(t_0)$. Let $\sigma_{(\beta)}=\pm1$ be a value for the spin component $\boldsymbol{\sigma}\cdot\mathbf{e}_{(\beta)}$ at time $t_{(\beta)}$; a sequence of spin values $\sigma_{(0)},\ldots,\sigma_{(B+1)}$ is a "history" for the system. The value of $y(t_q)$ for the history $\sigma_{(0)},\ldots,\sigma_{(B+1)}$ is obtained by applying the prescription (2.13):

$$y(t_q)=\sum_{\beta=1}^{B}\delta_{qq_\beta}Y_{(\beta)}(\sigma_{(\beta)})\ . \tag{2.16}$$

Notice that $\sigma_{(0)}\equiv\sigma_0$ and $\sigma_{(B+1)}\equiv\sigma$ do not contribute to the sampled quantities.

As the analogue of $e^{(i/\hbar)S[x(t)]}$, introduce a probability amplitude $\mathscr{A}(\sigma_{(1)},\ldots,\sigma_{(B+1)}\,|\,\sigma_{(0)})$, defined as the amplitude that $\boldsymbol{\sigma}\cdot\mathbf{e}_{(\beta)}$ has the value $\sigma_{(\beta)}$ at time $t_{(\beta)}$, for $\beta=1,\ldots,B+1$, given that $\boldsymbol{\sigma}\cdot\mathbf{e}_{(0)}$ had the value $\sigma_{(0)}$ at time $t_{(0)}$. Put another way, $\mathscr{A}(\sigma_{(1)},\ldots,\sigma_{(B+1)}\,|\,\sigma_{(0)})$ is the amplitude for the history $\sigma_{(0)},\ldots,\sigma_{(B+1)}$, conditioned on the initial value $\sigma_{(0)}$. One can write an explicit expression in terms of the evolution operator $\hat{U}(t,t')$ corresponding to the system Hamiltonian $\hat{H}$:

$$\mathscr{A}(\sigma_{(1)},\ldots,\sigma_{(B+1)}\,|\,\sigma_{(0)})=\prod_{\beta=1}^{B+1}\langle\sigma_{(\beta)},\mathbf{e}_{(\beta)}\,|\,\hat{U}(t_{(\beta)},t_{(\beta-1)})\,|\,\sigma_{(\beta-1)},\mathbf{e}_{(\beta-1)}\rangle\ . \tag{2.17}$$

The βth term in this product is the conditional amplitude that $\sigma\cdot\mathbf{e}_{(\beta)}$ has the value $\sigma_{(\beta)}$ at time $t_{(\beta)}$, given that $\sigma\cdot\mathbf{e}_{(\beta-1)}$ had the value $\sigma_{(\beta-1)}$ at time $t_{(\beta-1)}$.

Define now a joint probability amplitude $\Phi(\bar{y}_1,\ldots,\bar{y}_Q;\sigma,t_Q)$, the amplitude that the samplings yield the sequence of results $\bar{y}_1,\ldots,\bar{y}_Q$ and that $\sigma\cdot\mathbf{e}_Q\equiv\sigma\cdot\mathbf{e}_{(B+1)}$ has the value $\sigma\equiv\sigma_{(B+1)}$ at time t_Q. This fundamental amplitude can be written as a sum over histories:

$$\Phi(\bar{y}_1,\ldots,\bar{y}_Q;\sigma,t_Q)=\sum_{\sigma_{(0)},\ldots,\sigma_{(B)}}\left[\prod_{q=1}^{Q}\Upsilon(\bar{y}_q-y(t_q))\right]\mathscr{A}(\sigma_{(1)},\ldots,\sigma_{(B+1)}\,|\,\sigma_{(0)})\langle\sigma_{(0)},\mathbf{e}_{(0)}\,|\,\psi_0(t_0)\rangle\ . \tag{2.18a}$$

In this expression the sum includes all histories $\sigma_{(0)},\ldots,\sigma_{(B+1)}$ such that $\sigma_{(B+1)}=\sigma$, and $y(t_q)$ is evaluated from each history using Eq. (2.16).

The sum over histories (2.18a) is closely analogous to the path integral (2.3a), and the interpretations of the two are virtually identical. Select a history $\sigma_{(0)},\ldots,\sigma_{(B+1)}$, and begin with the amplitude $\mathscr{A}(\sigma_{(1)},\ldots,\sigma_{(B+1)}\,|\,\sigma_{(0)})$, the amplitude for the history, conditioned on the initial value $\sigma_{(0)}$. Multiply by the initial amplitude $\langle\sigma_{(0)},\mathbf{e}_{(0)}\,|\,\psi_0(t_0)\rangle$, the amplitude for $\sigma\cdot\mathbf{e}_{(0)}$ to have the value $\sigma_{(0)}$ at time $t_{(0)}$; thereby obtain the unconditioned amplitude

$$\mathscr{A}(\sigma_{(1)},\ldots,\sigma_{(B+1)}\,|\,\sigma_{(0)})\langle\sigma_{(0)},\mathbf{e}_{(0)}\,|\,\psi_0(t_0)\rangle$$

for the history $\sigma_{(0)},\ldots,\sigma_{(B+1)}$. For each $q=1,\ldots,Q$, multiply by a resolution amplitude $\Upsilon(\bar{y}_q-y(t_q))$, the amplitude to obtain $\bar{y}_q$ in the qth sampling, given the history's value for $y(t_q)$—or, more generally, given the history $\sigma_{(0)},\ldots,\sigma_{(B+1)}$; thereby find the joint amplitude

$$\left[\prod_{q=1}^{Q}\Upsilon(\bar{y}_q-y(t_q))\right]\mathscr{A}(\sigma_{(1)},\ldots,\sigma_{(B+1)}\,|\,\sigma_{(0)})\times\langle\sigma_{(0)},\mathbf{e}_{(0)}\,|\,\psi_0(t_0)\rangle$$

for the sequence of results $\bar{y}_1,\ldots,\bar{y}_Q$ and for the history $\sigma_{(0)},\ldots,\sigma_{(B+1)}$. Finally, compute the fundamental amplitude $\Phi(\bar{y}_1,\ldots,\bar{y}_Q;\sigma,t_Q)$ by summing over all histories such that $\sigma_{(B+1)}=\sigma$.

The joint probability distribution to obtain the results $\bar{y}_1,\ldots,\bar{y}_Q$ follows immediately as

$$P(\bar{y}_1,\ldots,\bar{y}_Q)=\sum_{\sigma}|\,\Phi(\bar{y}_1,\ldots,\bar{y}_Q;\sigma,t_Q)\,|^2\ . \tag{2.18b}$$

It is normalized with respect to the integration measure $d\bar{y}_1\cdots d\bar{y}_Q$, provided that $\Upsilon(\bar{y}-y)$ and $|\,\psi_0(t_0)\rangle$ are normalized [see Eq. (3.23)].

Equations (2.18) constitute the sum-over-histories formulation for time-distributed measurements involving spin. Aharonov and Albert[6] have outlined a similar sum-over-histories formulation, which has been developed in greater detail by D'Amato.[9,10]

The introduction of kernels follows closely the analogous development for measurements involving position [Eqs. (2.5)—(2.7)]. Here I merely list the appropriate definitions and relations:

$$\kappa(y_1,\ldots,y_Q;\sigma,t_Q\,|\,\sigma_0,t_0)\equiv\sum_{\sigma_{(1)},\ldots,\sigma_{(B)}}\left[\prod_{q=1}^{Q}\delta(y_q-y(t_q))\right]\mathscr{A}(\sigma_{(1)},\ldots,\sigma_{(B)},\sigma\,|\,\sigma_0)\ , \tag{2.19}$$

$$\begin{aligned}\mathscr{K}(\bar{y}_1,\ldots,\bar{y}_Q;\sigma,t_Q\,|\,\sigma_0,t_0)&\equiv\int\left[\prod_{q=1}^{Q}dy_q\,\Upsilon(\bar{y}_q-y_q)\right]\kappa(y_1,\ldots,y_Q;\sigma,t_Q\,|\,\sigma_0,t_0)\\&=\sum_{\sigma_{(1)},\ldots,\sigma_{(B)}}\left[\prod_{q=1}^{Q}\Upsilon(\bar{y}_q-y(t_q))\right]\mathscr{A}(\sigma_{(1)},\ldots,\sigma_{(B)},\sigma\,|\,\sigma_0)\ ,\end{aligned} \tag{2.20}$$

$$\Phi(\bar{y}_1,\ldots,\bar{y}_Q;\sigma,t_Q)=\sum_{\sigma_0}\mathscr{K}(\bar{y}_1,\ldots,\bar{y}_Q;\sigma,t_Q\,|\,\sigma_0,t_0)\langle\sigma_0,\mathbf{e}_0\,|\,\psi_0(t_0)\rangle\ . \tag{2.21}$$

In the absence of samplings, the kernels (2.19) and (2.20) reduce to the ordinary propagator

$$\sum_{\sigma_{(1)},\ldots,\sigma_{(B)}}\mathscr{A}(\sigma_{(1)},\ldots,\sigma_{(B)},\sigma\,|\,\sigma_0)=\langle\sigma,\mathbf{e}_Q\,|\,\hat{U}(t_Q,t_0)\,|\,\sigma_0,\mathbf{e}_0\rangle \tag{2.22}$$

[cf. Eq. (2.17)].

The Fourier transforms of the kernels (2.19) and (2.20) are defined exactly as in Eqs. (2.9) and (2.10). Indeed, with the replacements $x_0\to\sigma_0$ and $x\to\sigma$, Eqs. (2.9) and (2.10) can be taken over to the spin case as they stand, except for the latter equality in Eq. (2.9), which becomes

$$\kappa(k_1,\ldots,k_Q;\sigma,t_Q\,|\,\sigma_0,t_0)=\sum_{\sigma_{(1)},\ldots,\sigma_{(B)}}\mathscr{A}(\sigma_{(1)},\ldots,\sigma_{(B)},\sigma\,|\,\sigma_0)\exp\left[-i\sum_{q=1}^{Q}k_q y(t_q)\right]\ . \tag{2.23}$$

III. MEASUREMENT MODEL

In this section I show the equivalence of the sum-over-histories formulation to a model that uses the conventional language of nonrelativistic quantum mechanics—a quantum state evolving in time. The model is a standard measurement model, which involves enlarging the quantum-mechanical description to include "measuring apparatuses" that record and store the results of the samplings.

A. Measurements involving position

Begin with the Fourier-domain representation of the kernel κ [Eq. (2.9)], written in the form

$$\kappa(k_1,\ldots,k_Q;x,t_Q\,|\,x_0,t_0)=\int_{(x_0,t_0)}^{(x,t_Q)}\mathscr{D}x(t)\,e^{(i/\hbar)S_{\rm eff}[x(t)]}\,, \tag{3.1}$$

where the effective action $S_{\rm eff}[x(t)]$ is derived from a Lagrangian

$$L_{\rm eff}(x,\dot{x};t)\equiv L(x,\dot{x};t)-\sum_{q=1}^{Q}\hbar k_q\,Y_{t_q t}(x(t))\,. \tag{3.2}$$

The corresponding effective Hamiltonian is

$$\hat{H}_{\rm eff}=\hat{H}+\sum_{q=1}^{Q}\hbar k_q\,Y_{t_q t}(\hat{x})\,. \tag{3.3}$$

Equation (3.1) shows that $\kappa(k_1,\ldots,k_Q;x,t_Q\,|\,x_0,t_0)$ is the ordinary propagator for the effective action; hence, it can be written as

$$\kappa(k_1,\ldots,k_Q;x,t_Q\,|\,x_0,t_0)=\langle x\,|\,\hat{U}_{\rm eff}(t_Q,t_0)\,|\,x_0\rangle\,, \tag{3.4}$$

where $\hat{U}_{\rm eff}(t,t_0)$ is the unitary evolution operator for the effective Hamiltonian (3.3). Keep in mind that all the "effective" quantities just defined depend on $k_1,\ldots,k_Q$, although that dependence is not indicated explicitly.

Using Eq. (3.4), one can write the kernel $\mathscr{K}$ and the fundamental amplitude Φ as

$$\mathscr{K}(\bar{y}_1,\ldots,\bar{y}_Q;x,t_Q\,|\,x_0,t_0)=\int\left[\prod_{q=1}^{Q}\frac{dk_q}{(2\pi)^{1/2}}e^{ik_q\bar{y}_q}\Upsilon(k_q)\right]\langle x\,|\,\hat{U}_{\rm eff}(t_Q,t_0)\,|\,x_0\rangle\,, \tag{3.5}$$

$$\Phi(\bar{y}_1,\ldots,\bar{y}_Q;x,t_Q)=\int\left[\prod_{q=1}^{Q}\frac{dk_q}{(2\pi)^{1/2}}e^{ik_q\bar{y}_q}\Upsilon(k_q)\right]\langle x\,|\,\hat{U}_{\rm eff}(t_Q,t_0)\,|\,\psi_0(t_0)\rangle \tag{3.6}$$

[Eqs. (2.10) and (2.7)]. These relations can be used to demonstrate the following normalization conditions:

$$\int d\bar{y}_1\cdots d\bar{y}_Q dx\;\mathscr{K}^*(\bar{y}_1,\ldots,\bar{y}_Q;x,t_Q\,|\,x_0',t_0)\mathscr{K}(\bar{y}_1,\ldots,\bar{y}_Q;x,t_Q\,|\,x_0,t_0)=\left[\int dk\;|\,\Upsilon(k)\,|^2\right]^Q\delta(x_0-x_0')\,, \tag{3.7}$$

$$\int d\bar{y}_1\cdots d\bar{y}_Q dx\;|\,\Phi(\bar{y}_1,\ldots,\bar{y}_Q;x,t_Q)\,|^2=\left[\int dk\;|\,\Upsilon(k)\,|^2\right]^Q\langle\psi_0(t_0)\,|\,\psi_0(t_0)\rangle\,. \tag{3.8}$$

Equation (3.8) verifies the normalization of $P(\bar{y}_1,\ldots,\bar{y}_Q)$ given in Eq. (2.4), provided that $\Upsilon(k)$ and $|\,\psi_0(t_0)\rangle$ are normalized to unity, conditions which I assume henceforth.

Turn now to constructing the measurement model. Introduce for each sampling a "measuring apparatus," called a meter, which is a one-dimensional quantum-mechanical system with zero self-Hamiltonian. Let the qth meter have canonical coordinate $\bar{y}_q$ and canonical momentum $\bar{p}_q$ ($[\hat{\bar{y}}_q,\hat{\bar{p}}_r]=i\hbar\delta_{qr}$). Denote the δ-function normalized eigenstates of $\hat{\bar{y}}_q$ by $|\,\bar{y}_q\rangle$ and the δ-function normalized eigenstates of $\hat{\bar{p}}_q/\hbar$ by $|\,k_q\rangle$ [$\hat{\bar{p}}_q\,|\,k_q\rangle=\hbar k_q\,|\,k_q\rangle$; $\langle\bar{y}_q\,|\,k_q\rangle=(2\pi)^{-1/2}e^{ik_q\bar{y}_q}$]. Couple the meters to the system so that the total Hamiltonian is given by

$$\hat{H}_{\rm tot}=\hat{H}+\sum_{q=1}^{Q}\hat{\bar{p}}_q\,Y_{t_q t}(\hat{x})\,. \tag{3.9}$$

This total Hamiltonian leaves the meter momenta conserved, and it displaces the qth meter's coordinate by $y(t_q)$. Thus the coordinate of the qth meter stores the quantity one wants to measure.

Let the initial state of the qth meter be $|\,\Upsilon_q\rangle$, with wave function $\langle\bar{y}_q\,|\,\Upsilon_q\rangle=\Upsilon(\bar{y}_q)$ and with momentum-space wave function

$$\langle k_q\,|\,\Upsilon_q\rangle=\Upsilon(k_q)=\int\frac{d\bar{y}_q}{(2\pi)^{1/2}}\Upsilon(\bar{y}_q)e^{-ik_q\bar{y}_q}\,. \tag{3.10}$$

Notice that all the meters have the same initial wave function. The initial state for the total system is

$$|\,\Psi(t_0)\rangle=|\,\Upsilon_1\rangle\otimes\cdots\otimes|\,\Upsilon_Q\rangle\otimes|\,\psi_0(t_0)\rangle\,. \tag{3.11}$$

Now let $\hat{U}_{\rm tot}(t,t_0)$ be the unitary evolution operator for the total Hamiltonian (3.9). Using the property

$$\langle k_1,\ldots,k_Q\,|\,\hat{H}_{\rm tot}\,|\,k_1',\ldots,k_Q'\rangle=\hat{H}_{\rm eff}\left[\prod_{q=1}^{Q}\delta(k_q-k_q')\right] \tag{3.12}$$

and the evolution equations for $\hat{U}_{\rm tot}(t,t_0)$ and $\hat{U}_{\rm eff}(t,t_0)$, one can show that the (unnormalized) vector

$$\langle k_1,\ldots,k_Q\,|\,\hat{U}_{\rm tot}(t,t_0)\,|\,\Psi(t_0)\rangle\,,$$

which lies in the system Hilbert space, satisfies the same temporal differential equation as the (normalized) state vector

$$\hat{U}_{\rm eff}(t,t_0)\,|\,\psi_0(t_0)\rangle\,,$$

which also lies in the system Hilbert space. Comparison of initial conditions at $t=t_0$ shows that these vectors are

multiples of one another and thus establishes the relation

$$\langle k_1,\ldots,k_Q \mid \hat{U}_{\rm tot}(t_Q,t_0)\mid \Psi(t_0)\rangle = \left[\prod_{q=1}^{Q}\Upsilon(k_q)\right]\hat{U}_{\rm eff}(t_Q,t_0)\mid \psi_0(t_0)\rangle \ . \tag{3.13}$$

An equivalent relation in the meters' coordinate space reads

$$\langle \bar{y}_1,\ldots,\bar{y}_Q \mid \hat{U}_{\rm tot}(t_Q,t_0)\mid \Psi(t_0)\rangle = \int \left[\prod_{q=1}^{Q}\frac{dk_q}{(2\pi)^{1/2}}e^{ik_q\bar{y}_q}\Upsilon(k_q)\right]\hat{U}_{\rm eff}(t_Q,t_0)\mid \psi_0(t_0)\rangle \ . \tag{3.14}$$

One can now use Eq. (3.14) to show that the total wave function for the system and meters at time t_Q is simply the fundamental amplitude (3.6):

$$\langle \bar{y}_1,\ldots,\bar{y}_Q,x \mid \hat{U}_{\rm tot}(t_Q,t_0)\mid \Psi(t_0)\rangle = \Phi(\bar{y}_1,\ldots,\bar{y}_Q;x,t_Q) \ . \tag{3.15}$$

This same relation was obtained in Sec. III A of Ref. 7 in a less direct, but perhaps more physically illuminating fashion.

Equation (3.15) establishes the connection between the path-integral formulation and the measurement model. In terms of the model, the fundamental amplitude $\Phi(\bar{y}_1,\ldots,\bar{y}_Q;x,t_Q)$ is the total wave function for the system and meters, evolved to time t_Q using the Hamiltonian $\hat{H}_{\rm tot}$; it can be interpreted as the amplitude that the meter coordinates have values $\bar{y}_1,\ldots,\bar{y}_Q$ and that the system is at x at time t_Q. In the model the resolution amplitude $\Upsilon(\bar{y}_q-y_q)$ arises naturally as the initial wave function for the qth meter, displaced by a distance y_q; it can be interpreted as the conditional amplitude to find the qth meter at $\bar{y}_q$, given displacement by y_q. One can then reinterpret the kernel $\kappa(y_1,\ldots,y_Q;x,t_Q \mid x_0,t_0)$ [Eq. (2.5)] as the amplitude that the meter coordinates are displaced by the distances $y_1,\ldots,y_Q$ and that the system is at x at time t_Q, given that the system was at x_0 at time t_0. This makes clear the reason for integrating over $y_1,\ldots,y_Q$ at the amplitude level in Eq. (2.6): $y_1,\ldots,y_Q$ are unobservable displacements of the meter coordinates.

Two further aspects of the model deserve mention. First, the model reveals why the Fourier-domain representations of the kernels [Eqs. (2.9) and (2.10)] have such a simple form. The reason is that the meter momenta are conserved in the model. Second, one sees clearly why I restrict the sampled quantity $y(t_q)$ to have the functional form (2.2)—an integral over separate contributions from times t'. Were one to allow $y(t_q)$ to receive contributions that involve products of positions evaluated at different times, then the resulting effective action could not be derived from an effective Lagrangian which is local in time. The path-integral formulation could still be written down in exactly the same form, but whether it would be equivalent to some measurement model is not apparent.

B. Measurements involving spin

Begin again with the Fourier-domain representation of the kernel κ [Eq. (2.23)]. By using the relation

$$\sum_{q=1}^{Q}k_q y(t_q)=\sum_{\beta=1}^{B}k_{q_\beta}Y_{(\beta)}(\sigma_{(\beta)}) \tag{3.16}$$

[Eq. (2.16)] and by using the explicit form (2.17) for $\mathscr{A}(\sigma_{(1)},\ldots,\sigma_{(B)},\sigma\mid\sigma_0)$, one can write κ as an ordinary propagator

$$\kappa(k_1,\ldots,k_Q;\sigma,t_Q\mid\sigma_0,t_0)=\langle\sigma,\mathbf{e}_Q\mid\hat{U}_{\rm eff}(t_Q,t_0)\mid\sigma_0,\mathbf{e}_0\rangle \ , \tag{3.17}$$

where

$$\hat{U}_{\rm eff}(t_Q,t_0)=\hat{U}(t_Q,t_{(B)})\left[\prod_{\beta=1}^{B}\exp[-ik_{q_\beta}Y_{(\beta)}(\hat{\sigma}\cdot\mathbf{e}_{(\beta)})]\hat{U}(t_{(\beta)},t_{(\beta-1)})\right] \tag{3.18}$$

is the evolution operator for the effective Hamiltonian

$$\hat{H}_{\rm eff}\equiv\hat{H}+\sum_{\beta=1}^{B}\delta(t-t_{(\beta)})\hbar k_{q_\beta}Y_{(\beta)}(\hat{\sigma}\cdot\mathbf{e}_{(\beta)})=\hat{H}+\sum_{q=1}^{Q}\hbar k_q\sum_{j=1}^{N_q}\delta(t-t_{qj})Y_{qj}(\hat{\sigma}\cdot\mathbf{e}_{qj}) \ . \tag{3.19}$$

In Eq. (3.18) the product is time ordered—i.e., increasing values of β on the left.

Just as for measurements involving position, one can, using Eq. (3.17), write new expressions for the kernel $\mathscr{K}$ and the fundamental amplitude Φ:

$$\mathscr{K}(\bar{y}_1,\ldots,\bar{y}_Q;\sigma,t_Q\mid\sigma_0,t_0)=\int\left[\prod_{q=1}^{Q}\frac{dk_q}{(2\pi)^{1/2}}e^{ik_q\bar{y}_q}\Upsilon(k_q)\right]\langle\sigma,\mathbf{e}_Q\mid\hat{U}_{\rm eff}(t_Q,t_0)\mid\sigma_0,\mathbf{e}_0\rangle \ , \tag{3.20}$$

$$\Phi(\bar{y}_1,\ldots,\bar{y}_Q;\sigma,t_Q)=\int\left[\prod_{q=1}^{Q}\frac{dk_q}{(2\pi)^{1/2}}e^{ik_q\bar{y}_q}\Upsilon(k_q)\right]\langle\sigma,\mathbf{e}_Q\mid\hat{U}_{\rm eff}(t_Q,t_0)\mid\psi_0(t_0)\rangle \ . \tag{3.21}$$

These relations lead to the normalization conditions,

$$\sum_{\sigma}\int d\bar{y}_1\cdots d\bar{y}_Q\,\mathscr{K}^*(\bar{y}_1,\ldots,\bar{y}_Q;\sigma,t_Q\mid\sigma_0',t_0)\mathscr{K}(\bar{y}_1,\ldots,\bar{y}_Q;\sigma,t_Q\mid\sigma_0,t_0)=\left[\int dk\mid\Upsilon(k)\mid^2\right]^{Q}\delta_{\sigma_0\sigma_0'} \ , \tag{3.22}$$

$$\sum_{\sigma}\int d\bar{y}_1\cdots d\bar{y}_Q\mid\Phi(\bar{y}_1,\ldots,\bar{y}_Q;\sigma,t_Q)\mid^2=\left[\int dk\mid\Upsilon(k)\mid^2\right]^{Q}\langle\psi_0(t_0)\mid\psi_0(t_0)\rangle \ , \tag{3.23}$$

the latter of which verifies the normalization of $P(\bar{y}_1,\ldots,\bar{y}_Q)$ [Eq. (2.18b)].

Construction of a measurement model in this case proceeds just as for measurements involving position. One introduces the meters and their initial states. The form (3.18) for $\hat{U}_{\rm eff}(t_Q,t_0)$ can be used directly to establish Eq. (3.13) in this case, where $\hat{U}_{\rm tot}(t_Q,t_0)$ in Eq. (3.13) is the evolution operator corresponding to the total Hamiltonian

$$\hat{H}_{\rm tot}=\hat{H}+\sum_{q=1}^{Q}\hat{p}_q\sum_{j=1}^{N_q}\delta(t-t_{qj})Y_{qj}(\hat{\sigma}\cdot\mathbf{e}_{qj})$$
$$=\hat{H}+\sum_{\beta=1}^{B}\delta(t-t_{(\beta)})\hat{p}_{q_\beta}Y_{(\beta)}(\hat{\sigma}\cdot\mathbf{e}_{(\beta)}) \quad (3.24)$$

for the system and meters. One can then proceed to the desired result—that the fundamental amplitude $\Phi(\bar{y}_1,\ldots,\bar{y}_Q;\sigma,t_Q)$ [Eq. (3.21)] is the total wave function for the system and meters, evolved to time t_Q using the Hamiltonian $\hat{H}_{\rm tot}$.

Aharonov and Albert[6] and D'Amato[9,10] have used a slightly different, but equivalent measurement model for time-distributed measurements involving spin. Peres and Wootters[11] have analyzed in detail a special case of the measurement model specified by the Hamiltonian (3.24); the case they consider is time-distributed measurements involving only the z component of the spin.

IV. EFFECTS AND OPERATIONS

Effects and operations[12] provide still another mathematical language for describing measurements. The language is based on ordinary Hilbert-space operators, but it can do without the conventional notion of a quantum state evolving in time, because it is couched directly in terms of the results of measurements.

To illustrate how this language works, consider an instantaneous measurement of an observable $\hat{A}$, which has discrete eigenvalues A and eigenvectors $|A\rangle$. Label the possible results of the measurement by $\bar{A}$. For each $\bar{A}$, there is an *effect* $\hat{F}_{\bar{A}}$, a positive (self-adjoint) operator that is bounded from above by the unit operator:

$$\hat{0}\le\hat{F}_{\bar{A}}=\hat{F}_{\bar{A}}^{\dagger}\le\hat{1}\ .$$

The effect $\hat{F}_{\bar{A}}$ gives the probability $P(\bar{A})$ of obtaining the result $\bar{A}$ by

$$P(\bar{A})=\mathrm{tr}(\hat{\rho}\hat{F}_{\bar{A}})\ ,$$

where $\hat{\rho}$ is the density operator of the system. If the results $\bar{A}$ constitute a complete set of possible results, then $P(\bar{A})$ is normalized,

$$1=\sum_{\bar{A}}P(\bar{A})\ ,$$

which implies that the effects are complete:

$$\hat{1}=\sum_{\bar{A}}\hat{F}_{\bar{A}}\ .$$

For each result $\bar{A}$ there is also an *operation* $\mathscr{F}_{\bar{A}}$, a linear mapping of the space of trace-class operators into itself, which is positive,

$$\mathscr{F}_{\bar{A}}(\hat{X})\ge\hat{0}\ \text{ for all }\hat{X}\ge\hat{0}\ ,$$

and trace-decreasing,

$$\mathrm{tr}[\mathscr{F}_{\bar{A}}(\hat{X})]\le\mathrm{tr}(\hat{X})\ \text{ for all }\hat{X}\ge\hat{0}\ .$$

The operation determines the associated effect through the condition

$$\mathrm{tr}[\mathscr{F}_{\bar{A}}(\hat{\rho})]=\mathrm{tr}(\hat{\rho}\hat{F}_{\bar{A}})=P(\bar{A})\ .$$

Let $\hat{\rho}_{\bar{A}}$ denote the state of the system just after a measurement that yields the result $\bar{A}$. The operation $\mathscr{F}_{\bar{A}}$ relates the state before the measurement to the state afterward by

$$\hat{\rho}_{\bar{A}}=\mathscr{F}_{\bar{A}}(\hat{\rho})/P(\bar{A})\ .$$

Effects and operations generalize the usual description of measurements in terms of projection operators. The generalization provides a natural way to incorporate the irresolution or imprecision of a measurement, and it is essential when dealing with measurements of an observable with a continuous spectrum. One can recover the projection-operator description in the case where the possible results are the eigenvalues A of $\hat{A}$. One chooses operations defined by $\mathscr{F}_A(\hat{\rho})\equiv\hat{P}_A\hat{\rho}\hat{P}_A$, where $\hat{P}_A\equiv|A\rangle\langle A|$ projects onto the eigenstate $|A\rangle$, and one finds that $P(A)=\mathrm{tr}(\hat{\rho}\hat{P}_A)=\langle A|\hat{\rho}|A\rangle$, $\hat{F}_A=\hat{P}_A$, and $\hat{\rho}_A=\hat{P}_A$.

Effects and operations permit a complete description of a sequence of instantaneous measurements. The statistics of each measurement are determined by a set of effects, and the system state after each measurement is given by an operation, which generalizes the usual notion of wave-function collapse (state reduction). The only other ingredient is unitary evolution, which takes the system state from one measurement to another.

To see how effects and operations arise in the context of time-distributed measurements, consider measurements involving position (Sec. II A). Define first an operator

$$\hat{\Gamma}_{\bar{y}_1\cdots\bar{y}_Q}\equiv\int dx\,dx_0\,|x\rangle\mathscr{K}(\bar{y}_1,\ldots,\bar{y}_Q;x,t_Q|x_0,t_0)\langle x_0|$$
$$=\int\left[\prod_{q=1}^{Q}\frac{dk_q}{(2\pi)^{1/2}}e^{ik_q\bar{y}_q}\Upsilon(k_q)\right]\hat{U}_{\rm eff}(t_Q,t_0) \quad (4.1)$$

[Eq. (3.5)], whose matrix elements in the position basis yield the kernel $\mathscr{K}$ [Eq. (2.6)]:

$$\mathscr{K}(\bar{y}_1,\ldots,\bar{y}_Q;x,t_Q|x_0,t_0)=\langle x|\hat{\Gamma}_{\bar{y}_1\cdots\bar{y}_Q}|x_0\rangle\ . \quad (4.2)$$

Just as $\mathscr{K}$ can be viewed as a modified propagator—modified by the presence of measurements— so $\hat{\Gamma}_{\bar{y}_1\cdots\bar{y}_Q}$ can be viewed as a "modified evolution operator." The fundamental amplitude (2.7) can now be written as

$$\Phi(\bar{y}_1,\ldots,\bar{y}_Q;x,t_Q)=\langle x|\hat{\Gamma}_{\bar{y}_1\cdots\bar{y}_Q}|\psi_0(t_0)\rangle\ . \quad (4.3)$$

Notice the connection between $\hat{\Gamma}_{\bar{y}_1\cdots\bar{y}_Q}$ and the total evolution operator of the model in Sec. III A:

$$\hat{\Gamma}_{\bar{y}_1 \cdots \bar{y}_Q} \mid \psi_0(t_0)\rangle = \langle \bar{y}_1, \ldots, \bar{y}_Q \mid \hat{U}_{\text{tot}}(t_Q, t_0) \mid \Psi(t_0)\rangle \quad (4.4)$$

[Eqs. (3.14) and (4.1)].

Consider now the joint probability distribution $P(\bar{y}_1, \ldots, \bar{y}_Q)$ for obtaining the results $\bar{y}_1, \ldots, \bar{y}_Q$ [Eq. (2.3b)]. Using Eq. (4.3), one can write $P(\bar{y}_1, \ldots, \bar{y}_Q)$ as[20]

$$P(\bar{y}_1, \ldots, \bar{y}_Q) = \langle \psi_0(t_0) \mid \hat{F}_{\bar{y}_1 \cdots \bar{y}_Q} \mid \psi_0(t_0)\rangle$$
$$= \text{tr}[\hat{\rho}_0(t_0) \hat{F}_{\bar{y}_1 \cdots \bar{y}_Q}] , \quad (4.5)$$

where $\hat{\rho}_0(t_0) \equiv \mid \psi_0(t_0)\rangle\langle\psi_0(t_0) \mid$ is the system's initial density operator (which can now be generalized to be a mixed state), and

$$\hat{F}_{\bar{y}_1 \cdots \bar{y}_Q} \equiv \hat{\Gamma}^{\dagger}_{\bar{y}_1 \cdots \bar{y}_Q} \hat{\Gamma}_{\bar{y}_1 \cdots \bar{y}_Q} \quad (4.6)$$

is an *effect density*.[21] The measure

$$d\hat{F}_{\bar{y}_1 \cdots \bar{y}_Q} \equiv d\bar{y}_1 \cdots d\bar{y}_Q \, \hat{F}_{\bar{y}_1 \cdots \bar{y}_Q}$$

is an *effect-valued measure*,[21] normalized by

$$\hat{1} = \int d\bar{y}_1 \cdots d\bar{y}_Q \, \hat{F}_{\bar{y}_1 \cdots \bar{y}_Q} \quad (4.7)$$

[Eqs. (4.1) and (3.7)]. The lesson taught by Eqs. (4.5)—(4.7) is that the statistics of a sequence of samplings of $y(t)$ are derivable from an effect density.

The state of the system after a sequence of samplings of $y(t)$ is specified by an operation density. Let $\mid \psi_{\bar{y}_1 \cdots \bar{y}_Q}(t_Q)\rangle$ denote the state of the system at time t_Q, just after a sequence of samplings which yield results $\bar{y}_1, \ldots, \bar{y}_Q$. To determine this state, use a Bayesian logic applied to amplitudes.[7] Start with the fundamental amplitude $\Phi(\bar{y}_1, \ldots, \bar{y}_Q; x, t_Q)$, the joint amplitude for the results $\bar{y}_1, \ldots, \bar{y}_Q$ and for the system to be at x at time t_Q. Using Feynman's rules for combining probability amplitudes, write the fundamental amplitude as a product:

$$\Phi(\bar{y}_1, \ldots, \bar{y}_Q; x, t_Q) = \langle x \mid \psi_{\bar{y}_1 \cdots \bar{y}_Q}(t_Q)\rangle$$
$$\times [P(\bar{y}_1, \ldots, \bar{y}_Q)]^{1/2} . \quad (4.8)$$

The first term on the right is the system wave function at time t_Q—the amplitude that the system is at x, conditioned on the results $\bar{y}_1, \ldots, \bar{y}_Q$. The second term on the right is the "amplitude" that the samplings yield the results $\bar{y}_1, \ldots, \bar{y}_Q$. [There is clearly a phase ambiguity in obtaining this amplitude from $P(\bar{y}_1, \ldots, \bar{y}_Q)$, but the ambiguous phase has no effect on the subsequent behavior of the system.] Equations (4.3) and (4.8) show that

$$\mid \psi_{\bar{y}_1 \cdots \bar{y}_Q}(t_Q)\rangle = \frac{\hat{\Gamma}_{\bar{y}_1 \cdots \bar{y}_Q} \mid \psi_0(t_0)\rangle}{[P(\bar{y}_1, \ldots, \bar{y}_Q)]^{1/2}} , \quad (4.9)$$

which emphasizes again the role of $\hat{\Gamma}_{\bar{y}_1 \cdots \bar{y}_Q}$ as a modified evolution operator. In terms of the measurement model of Sec. III A, the state (4.9) is obtained by projecting the total state $\hat{U}_{\text{tot}}(t_Q, t_0) \mid \Psi(t_0)\rangle$ onto the results $\bar{y}_1, \ldots, \bar{y}_Q$ and then normalizing [cf. Eq. (4.4)].

Rewriting Eq. (4.9) in terms of the system density operator

$$\hat{\rho}_{\bar{y}_1 \cdots \bar{y}_Q}(t_Q) \equiv \mid \psi_{\bar{y}_1 \cdots \bar{y}_Q}(t_Q)\rangle\langle\psi_{\bar{y}_1 \cdots \bar{y}_Q}(t_Q) \mid , \quad (4.10)$$

one finds that[20]

$$\hat{\rho}_{\bar{y}_1 \cdots \bar{y}_Q}(t_Q) = \frac{\mathcal{F}_{\bar{y}_1 \cdots \bar{y}_Q}(\hat{\rho}_0(t_0))}{P(\bar{y}_1, \ldots, \bar{y}_Q)} \quad (4.11)$$

(once again, one can generalize to a mixed initial state), where $\mathcal{F}_{\bar{y}_1 \cdots \bar{y}_Q}$ is an *operation density*[21] defined by

$$\mathcal{F}_{\bar{y}_1 \cdots \bar{y}_Q}(\hat{\rho}) \equiv \hat{\Gamma}_{\bar{y}_1 \cdots \bar{y}_Q} \hat{\rho} \hat{\Gamma}^{\dagger}_{\bar{y}_1 \cdots \bar{y}_Q} . \quad (4.12)$$

The measure $d\bar{y}_1 \cdots d\bar{y}_Q \mathcal{F}_{\bar{y}_1 \cdots \bar{y}_Q}$ is an *operation-valued measure*.[21] The operation density (4.12) is a special kind, which maps pure states to (unnormalized) pure states. One can show immediately the relation between the operation and effect densities:

$$\text{tr}[\mathcal{F}_{\bar{y}_1 \cdots \bar{y}_Q}(\hat{\rho}_0(t_0))] = \text{tr}[\hat{\rho}_0(t_0) \hat{F}_{\bar{y}_1 \cdots \bar{y}_Q}]$$
$$= P(\bar{y}_1, \ldots, \bar{y}_Q) . \quad (4.13)$$

One has now translated the path-integral formulation into the language of effects and operations. The effects and operations, like the path integral, describe the sequence of samplings as a whole. The effect density $\hat{F}_{\bar{y}_1 \cdots \bar{y}_Q}$ yields the statistics for the entire sequence, and the operation density $\mathcal{F}_{\bar{y}_1 \cdots \bar{y}_Q}$ gives the system state after the entire sequence is completed. Is it really necessary to consider the entire sequence at once? Is there not some way to deal with one sampling at a time, finishing with one sampling before going on to the next? In general, the answer is no. The qth sampling gathers information about $x(t)$ during the time interval $[t_q - \Delta_{t_q}, t_q]$. In general, the intervals for different samplings overlap; before the interval for the qth sampling has ended, the $(q+1)$th sampling has already begun. Under these circumstances, there is no way to disentangle the samplings in order to deal with one at a time. Further discussion of this question can be found in Sec. III B of Ref. 7.

Consider now the case where it is possible to deal with one sampling at a time—the case where the samplings do not overlap, i.e., $t_{q-1} < t_q - \Delta_{t_q}$ for $q = 1, \ldots, Q$. In this case the kernel $\mathcal{K}$ can be decomposed as

$$\mathcal{K}(\bar{y}_1, \ldots, \bar{y}_Q; x, t_Q \mid x_0, t_0) = \int \mathcal{K}^{(Q)}(\bar{y}_Q; x, t_Q \mid x_{Q-1}, t_{Q-1}) \left[\prod_{q=1}^{Q-1} dx_q \mathcal{K}^{(q)}(\bar{y}_q; x_q, t_q \mid x_{q-1}, t_{q-1}) \right] , \quad (4.14)$$

where

$$\mathcal{K}^{(q)}(\bar{y}_q; x_q, t_q \mid x_{q-1}, t_{q-1}) \equiv \int_{(x_{q-1}, t_{q-1})}^{(x_q, t_q)} \mathcal{D}x(t) \, \Upsilon(\bar{y}_q - y(t_q)) e^{(i/\hbar) S[x(t)]} . \quad (4.15)$$

In Eq. (4.15) the sum over paths includes all paths on the interval $[t_{q-1},t_q]$ such that $x(t_{q-1})=x_{q-1}$ and $x(t_q)=x_q$. One recognizes $\mathscr{K}^{(q)}$ as a kernel for the qth sampling alone; it has the same interpretation and the same properties as the kernel $\mathscr{K}$ has when specialized to a single sampling. The decomposition (4.14) of $\mathscr{K}$ means that the modified evolution operator (4.1) factors into the form

$$\hat{\Gamma}_{\bar{y}_1\cdots\bar{y}_Q}=\prod_{q=1}^{Q}\hat{\Gamma}^{(q)}_{\bar{y}_q}\ , \tag{4.16}$$

where $\hat{\Gamma}^{(q)}_{\bar{y}_q}$ is a modified evolution operator for the qth sampling:

$$\hat{\Gamma}^{(q)}_{\bar{y}_q}\equiv\int dx_q dx_{q-1}\,|\,x_q\rangle \times\mathscr{K}^{(q)}(\bar{y}_q;x_q,t_q\,|\,x_{q-1},t_{q-1})\langle x_{q-1}|\ , \tag{4.17a}$$

$$\mathscr{K}^{(q)}(\bar{y}_q;x_q,t_q\,|\,x_{q-1},t_{q-1})=\langle x_q\,|\,\hat{\Gamma}^{(q)}_{\bar{y}_q}\,|\,x_{q-1}\rangle\ . \tag{4.17b}$$

In Eq. (4.16) the product is ordered with increasing values of q on the left. Properties of the operator $\hat{\Gamma}^{(q)}_{\bar{y}_q}$ follow from those of $\hat{\Gamma}_{\bar{y}_1\cdots\bar{y}_Q}$ specialized to a single sampling. For example, $\hat{\Gamma}^{(q)}_{\bar{y}_q}$ gives rise to an effect density

$$\hat{F}^{(q)}_{\bar{y}_q}\equiv\hat{\Gamma}^{(q)\dagger}_{\bar{y}_q}\hat{\Gamma}^{(q)}_{\bar{y}_q}\ , \tag{4.18}$$

normalized by

$$\int d\bar{y}_q\,\hat{F}^{(q)}_{\bar{y}_q}=\hat{1}\ . \tag{4.19}$$

The factorization (4.16) further allows one to decompose the operation density $\mathscr{F}_{\bar{y}_1\cdots\bar{y}_Q}$ [Eq. (4.12)] as

$$\mathscr{F}_{\bar{y}_1\cdots\bar{y}_Q}(\hat{\rho})=\mathscr{F}^{(Q)}_{\bar{y}_Q}[\cdots\mathscr{F}^{(2)}_{\bar{y}_2}(\mathscr{F}^{(1)}_{\bar{y}_1}(\hat{\rho}))\cdots]\ , \tag{4.20}$$

where $\mathscr{F}^{(q)}_{\bar{y}_q}$ is an operation density for the qth sampling, defined by

$$\mathscr{F}^{(q)}_{\bar{y}_q}(\hat{\rho})\equiv\hat{\Gamma}^{(q)}_{\bar{y}_q}\hat{\rho}\hat{\Gamma}^{(q)\dagger}_{\bar{y}_q}\ . \tag{4.21}$$

Notice that $\mathrm{tr}[\mathscr{F}^{(q)}_{\bar{y}_q}(\hat{\rho})]=\mathrm{tr}(\hat{\rho}\hat{F}^{(q)}_{\bar{y}_q})$. Physically, the factorization (4.16) means that the joint statistics of the first q samplings are independent of the existence of subsequent samplings; this property does not hold in the general case of overlapping samplings (for further discussion, see Sec. III B of Ref. 7).

One can now describe a sequence of nonoverlapping samplings in the following way. Define a system density operator $\hat{\rho}_{\bar{y}_1\cdots\bar{y}_{q-1}}(t_{q-1})$ just after the $(q-1)$th sampling (just after time t_{q-1}); this state is conditioned on the results of the first $q-1$ samplings. The probability distribution $P(\bar{y}_q\,|\,\bar{y}_1,\ldots,\bar{y}_{q-1})$ to obtain $\bar{y}_q$ as the result of the qth sampling, conditioned on the results of previous samplings, is determined by the effect density $\hat{F}^{(q)}_{\bar{y}_q}$:

$$P(\bar{y}_q\,|\,\bar{y}_1,\ldots,\bar{y}_{q-1})=\mathrm{tr}[\hat{\rho}_{\bar{y}_1\cdots\bar{y}_{q-1}}(t_{q-1})\hat{F}^{(q)}_{\bar{y}_q}]\ . \tag{4.22}$$

The density operator just after the qth sampling is given by the operation density $\mathscr{F}^{(q)}_{\bar{y}_q}$:

$$\hat{\rho}_{\bar{y}_1\cdots\bar{y}_q}(t_q)=\frac{\mathscr{F}^{(q)}_{\bar{y}_q}(\hat{\rho}_{\bar{y}_1\cdots\bar{y}_{q-1}}(t_{q-1}))}{P(\bar{y}_q\,|\,\bar{y}_1,\ldots,\bar{y}_{q-1})} =\frac{\hat{\Gamma}^{(q)}_{\bar{y}_q}\hat{\rho}_{\bar{y}_1\cdots\bar{y}_{q-1}}(t_{q-1})\hat{\Gamma}^{(q)\dagger}_{\bar{y}_q}}{P(\bar{y}_q\,|\,\bar{y}_1,\ldots,\bar{y}_{q-1})}\ . \tag{4.23}$$

Iterating Eqs. (4.22) and (4.23) leads to the joint probability distribution (4.5) and to the system state (4.11) after the Qth sampling.

Specialize now to a sequence of instantaneous measurements (samplings) of position $[y(t_q)=x(t_q)]$, for which the samplings certainly do not overlap. Let $\bar{x}_q$ label the result of the qth sampling. Then Eqs. (4.15), (4.17a), and (4.21) simplify to

$$\mathscr{K}^{(q)}(\bar{x}_q;x_q,t_q\,|\,x_{q-1},t_{q-1})=\Upsilon(\bar{x}_q-x_q)K(x_q,t_q\,|\,x_{q-1},t_{q-1})\ , \tag{4.24a}$$

$$\hat{\Gamma}^{(q)}_{\bar{x}_q}=\Upsilon(\bar{x}_q-\hat{x})\hat{U}(t_q,t_{q-1})\ , \tag{4.24b}$$

$$\mathscr{F}^{(q)}_{\bar{x}_q}(\hat{\rho})=\Upsilon(\bar{x}_q-\hat{x})\hat{U}(t_q,t_{q-1})\hat{\rho}\hat{U}^{\dagger}(t_q,t_{q-1})[\Upsilon(\bar{x}_q-\hat{x})]^{\dagger} \tag{4.24c}$$

[see Eq. (2.8)]. For instantaneous measurements of position, one can resurrect the notion of a system quantum state evolving in time. In view of Eq. (4.23), Eq. (4.24c) describes a system quantum state that undergoes unitary evolution between samplings [system evolution operator $\hat{U}(t_q,t_{q-1})$] and suffers an instantaneous wave-function collapse at the time of each sampling ["resolution operator" $\Upsilon(\bar{x}_q-\hat{x})$]. When one generalizes to nonoverlapping samplings of $y(t)$, these two kinds of evolution become inextricably entangled, as evidenced in the path integral for $\mathscr{K}^{(q)}$ [Eq. (4.15)]; indeed, it is not generally possible to define a system quantum state during the interval between the sampling times. Nonetheless, for nonoverlapping samplings it is still possible to define a system quantum state just after each sampling, which serves as initial state for the next sampling [cf. Eq. (4.23)]. When one generalizes further to overlapping samplings of $y(t)$, even this last possibility disappears; it is not possible, in general, to define a system quantum state at any time during a sequence of overlapping samplings of $y(t)$.[7,20] All that remains in the general case is the joint probability distribution for the results [Eq. (4.5)] and a system quantum state after all samplings are completed [Eq. (4.11)].

Barchielli, Lanz, and Prosperi[21–23] have developed a description of continuous position measurements (see also Refs. 24 and 25). They begin with a sequence of Q samplings (instantaneous measurements) of position, which occur at uniformly spaced times $t_q=t_0+q\tau$ in the interval $[t_0,t_f]$, where $t_f\equiv t_Q=t_0+Q\tau$. They choose the resolution amplitude to be a real Gaussian,

$$\Upsilon(\bar{x}-x)=(2\pi\sigma^2)^{-1/4}\exp[-(\bar{x}-x)^2/4\sigma^2]\ , \tag{4.25}$$

where σ^2, the variance of the associated conditional probability distribution $|\,\Upsilon(\bar{x}-x)\,|^2$, might be called the resolution of the samplings. They then let the sequence become continuous on the interval $[t_0,t_f]$ by taking the lim-

it $Q\to\infty$, $\tau\to 0$ ($Q\tau=t_f-t_0$); they simultaneously take the limit $\sigma^2\to\infty$ in such a way that

$$\sigma^2\tau\equiv 1/2\gamma=\text{const} . \tag{4.26}$$

By specializing the kernel $\mathscr{K}$ [Eq. (2.6)] to samplings of position [$\bar{y}_q=\bar{x}_q$; $y(t_q)=x(t_q)$] and by taking the above limit, one can derive a kernel which is a functional of $\bar{x}(t)$, the continuous sequence of results:

$$\mathscr{K}[\bar{x}(t);x,t_f\,|\,x_0,t_0]=\int_{(x_0,t_0)}^{(x,t_f)}\mathscr{D}x(t)\,\exp\left[-\tfrac{1}{2}\gamma\int_{t_0}^{t_f}dt\,[\bar{x}(t)-x(t)]^2\right]e^{(i/\hbar)S[x(t)]} \tag{4.27}$$

[cf. Eq. (3.11) of Ref. 22]. The kernel $\mathscr{K}[\bar{x}(t);x,t_f\,|\,x_0,t_0]$ is the amplitude that the continuous measurement of position yields the trajectory of results $\bar{x}(t)$ and that the system is at x at time t_f, given that it was at x_0 at time t_0. From the kernel (4.27) comes the fundamental amplitude,

$$\Phi[\bar{x}(t);x,t_f]=\int dx_0\,\mathscr{K}[\bar{x}(t);x,t_f\,|\,x_0,t_0]\psi_0(x_0,t_0) \tag{4.28}$$

[cf. Eq. (2.7)], and a functional probability distribution for the results $\bar{x}(t)$,

$$P[\bar{x}(t)]=\int dx\;|\,\Phi[\bar{x}(t);x,t_f]\,|^2 \tag{4.29}$$

[cf. Eq. (3.15) of Ref. 22]. This functional probability distribution is normalized with respect to the integration measure

$$\mathscr{d}\bar{x}(t)\equiv\lim_{Q\to\infty}(\gamma\tau/\pi)^{Q/2}\prod_{q=1}^{Q}d\bar{x}_q , \tag{4.30}$$

i.e.,

$$1=\int\mathscr{d}\bar{x}(t)\,P[\bar{x}(t)] . \tag{4.31}$$

It should be emphasized that $P[\bar{x}(t)]$ is not a functional probability distribution for system paths $x(t)$. Quantum mechanics assigns to a path $x(t)$ a probability *amplitude* $e^{(i/\hbar)S[x(t)]}\psi(x(t_0),t_0)$—not a probability. Feynman's rules do not instruct one to square the probability amplitude to obtain a functional probability distribution for the path; rather, they instruct one to sum the amplitude over all paths with a particular final value *before* squaring to obtain a probability distribution for the final value. Thus there is no sensible way in quantum mechanics to define a functional probability distribution for system paths $x(t)$. What then is $P[\bar{x}(t)]$? The overbar on $\bar{x}(t)$ signifies results of measurements; $P[\bar{x}(t)]$ is the functional probability distribution that the continuous measurement of position yields the trajectory of results $\bar{x}(t)$. This is made particularly clear by the measurement model of Sec. III A, which can easily be applied to the case of continuous measurements. In the model $P[\bar{x}(t)]$ is the functional probability distribution that the coordinates of an infinite set of meters have the continuous sequence of values $\bar{x}(t)$.

Attention should also be drawn to the distinction between continuous position measurements and measurements distributed in time. Given $P[\bar{x}(t)]$, one can calculate in principle a joint probability distribution for any set of functionals of $\bar{x}(t)$; one could, for example, consider functionals that are obtained from $\bar{x}(t)$ in the same way as the sampled quantities $y(t_q)$ [Eq. (2.2)] are obtained from $x(t)$. Would the resulting probability distribution be the same as that obtained for time-distributed measurements from the path-integral formulation? No. The difference lies in how one goes from amplitudes to probabilities. In the continuous measurement approach, one calculates first an amplitude $\Phi[\bar{x}(t);x,t_f]$ for a continuous measurement, squares and integrates to obtain a functional probability distribution $P[\bar{x}(t)]$, and then averages over $P[\bar{x}(t)]$ to get a probability distribution for the functionals of interest. Operationally, this corresponds to gathering arbitrarily wideband data about the position of the system and then processing this data to obtain the functionals of interest. In the case of time-distributed measurements, one calculates directly a probability amplitude $\Phi(\bar{y}_1,\ldots,\bar{y}_Q;x,t_Q)$ for the functionals of interest and then squares and integrates to obtain a probability distribution $P(\bar{y}_1,\ldots,\bar{y}_Q)$. Operationally, this corresponds to gathering only data about the functionals of interest.

V. MULTIPLE-TIME EIGENSTATES

The conventional language of a system quantum state evolving in time is inadequate for describing measurements distributed in time. That is the message delivered by the preceding sections. The concepts of multiple-time states and multiple-time eigenstates[8–10] constitute an attempt to generalize the conventional language to a higher-level language that can be applied to time-distributed measurements (or multiple-time measurements[6,8–10]). In this section I explore the usefulness of these concepts.

Consider measurements involving spin, as do Refs. 8–10 (referred to hereafter as AAD). For that purpose, recall the sum-over-histories formulation developed in Sec. II B. To facilitate comparison with AAD, make a simplification: choose $Y_{qj}(\sigma)=\sigma$, for all q and j. Thus, throughout this section, the sampled quantities (2.13) are simply sums of spin components in various directions at different times:

$$y(t_q)=\sum_{j=1}^{N_q}\boldsymbol{\sigma}(t_{qj})\cdot\mathbf{e}_{qj} . \tag{5.1}$$

This assumption implies that $y(t_q)$ can take on only discrete values from the sequence

$$-N_q,\,-N_q+2,\ldots,N_q-2,N_q . \tag{5.2}$$

As a consequence, it is useful to introduce a new kernel

$$\kappa(n_1,\ldots,n_Q;\sigma,t_Q\,|\,\sigma_0,t_0)\equiv\sum_{\sigma_{(1)},\ldots,\sigma_{(B)}}\left[\prod_{q=1}^{Q}\delta_{n_q,y(t_q)}\right]\mathscr{A}(\sigma_{(1)},\ldots,\sigma_{(B)},\sigma\,|\,\sigma_0) \tag{5.3}$$

[cf. Eq. (2.17)], where $\delta_{n_q,y(t_q)}$ is the Kronecker delta and

$$y(t_q)=\sum_{j=1}^{N_q}\sigma_{qj}=\sum_{\beta=1}^{B}\delta_{qq_\beta}\sigma_{(\beta)} \tag{5.4}$$

[cf. Eq. (2.16)]. The kernel (5.3) is the discrete analogue of the kernel (2.19). It can clearly be interpreted as the amplitude that the sampled quantities have the discrete values $n_1,\ldots,n_Q$ and that $\boldsymbol{\sigma}\cdot\mathbf{e}_Q$ has the value σ at time t_Q, given that $\boldsymbol{\sigma}\cdot\mathbf{e}_0$ had the value σ_0 at time t_0.

From the discrete kernel (5.3), one can derive the kernels (2.19) and (2.20):

$$\kappa(y_1,\ldots,y_Q;\sigma,t_Q\,|\,\sigma_0,t_0)=\sum_{n_1,\ldots,n_Q}\left[\prod_{q=1}^{Q}\delta(y_q-n_q)\right]\kappa(n_1,\ldots,n_Q;\sigma,t_Q\,|\,\sigma_0,t_0)\,, \tag{5.5a}$$

$$\mathscr{K}(\bar{y}_1,\ldots,\bar{y}_Q;\sigma,t_Q\,|\,\sigma_0,t_0)=\sum_{n_1,\ldots,n_Q}\left[\prod_{q=1}^{Q}\Upsilon(\bar{y}_q-n_q)\right]\kappa(n_1,\ldots,n_Q;\sigma,t_Q\,|\,\sigma_0,t_0)\,. \tag{5.5b}$$

In Sec. II the kernel κ [Eq. (5.5a)] is interpreted as an amplitude for the *values* of the sampled quantities, whereas the kernel $\mathscr{K}$ [Eq. (5.5b)], which takes into account the resolution of the samplings, is interpreted as an amplitude for the *results* of the samplings. In this section no such distinction is made; the analysis is framed entirely in terms of the discrete kernel (5.3). Why can I ignore this distinction? Because of the discreteness of the sampled quantities (5.1), and because I assume throughout this section that the samplings have sufficiently good resolution to resolve the discrete values—i.e., to distinguish neighboring values in the sequence (5.2). Such resolution is achieved if the resolution amplitude satisfies

$$\Upsilon(\bar{y}_q)=0\quad\text{for}\quad|\bar{y}_q|\geq 1\,. \tag{5.6}$$

When the qth sampling yields a result $\bar{y}_q$, it is viewed as yielding the nearest discrete result n_q. With the good resolution (5.6), there is a one-to-one correspondence between the value of the sampled quantity and this discrete result of the sampling. Thus, in this section, the Q samplings are viewed as yielding a sequence $n_1,\ldots,n_Q$ of discrete results. The kernel (5.3) can be interpreted directly—no need for a resolution amplitude—as the amplitude that the samplings yield the sequence $n_1,\ldots,n_Q$ and that $\boldsymbol{\sigma}\cdot\mathbf{e}_Q$ has the value σ at time t_Q, given that $\boldsymbol{\sigma}\cdot\mathbf{e}_0$ had the value σ_0 at time t_0. The kernel (5.3) is normalized by

$$\delta_{\sigma_0\sigma_0'}=\sum_{\sigma}\sum_{n_1,\ldots,n_Q}\kappa^*(n_1,\ldots,n_Q;\sigma,t_Q\,|\,\sigma_0',t_0)\times\kappa(n_1,\ldots,n_Q;\sigma,t_Q\,|\,\sigma_0,t_0)\,. \tag{5.7}$$

This normalization follows directly from the normalization of the kernel (5.5b) [see Eq. (3.22)], applied to a resolution amplitude that satisfies Eq. (5.6).

Define now an operator

$$\hat{\Lambda}_{n_1\cdots n_Q}\equiv\sum_{\sigma_0,\sigma}|\,\sigma,\mathbf{e}_Q\rangle\kappa(n_1,\ldots,n_Q;\sigma,t_Q\,|\,\sigma_0,t_0)\langle\sigma_0,\mathbf{e}_0\,|$$
$$=\hat{U}(t_Q,t_{(B)})\left[\sum_{\sigma_{(1)},\ldots,\sigma_{(B)}}\left[\prod_{q=1}^{Q}\delta_{n_q,y(t_q)}\right]\left[\prod_{\beta=1}^{B}|\,\sigma_{(\beta)},\mathbf{e}_{(\beta)}\rangle\langle\sigma_{(\beta)},\mathbf{e}_{(\beta)}\,|\,\hat{U}(t_{(\beta)},t_{(\beta-1)})\right]\right] \tag{5.8}$$

[Eqs. (5.3) and (2.17)], which is the discrete analogue of the operator $\hat{\Gamma}_{\bar{y}_1\cdots\bar{y}_Q}$ introduced in Sec. IV [see Eq. (4.1)]. In the second equality of Eq. (5.8), the operator product is ordered with increasing values of β on the left. The matrix elements of $\hat{\Lambda}_{n_1\cdots n_Q}$ give the discrete kernel

$$\kappa(n_1,\ldots,n_Q;\sigma,t_Q\,|\,\sigma_0,t_0)=\langle\sigma,\mathbf{e}_Q\,|\,\hat{\Lambda}_{n_1\cdots n_Q}\,|\,\sigma_0,\mathbf{e}_0\rangle\,. \tag{5.9}$$

Just as $\kappa(n_1,\ldots,n_Q;\sigma,t_Q\,|\,\sigma_0,t_0)$ can be regarded as a modified propagator, so $\hat{\Lambda}_{n_1\cdots n_Q}$ plays the role of a modified evolution operator. This role is particularly apparent in the second equality of Eq. (5.8): without the product of Kronecker deltas, Eq. (5.8) would yield the system's unitary evolution operator $\hat{U}(t_Q,t_0)$; the Kronecker deltas restrict the sum over histories to those histories which satisfy $y(t_q)=n_q$, for $q=1,\ldots,Q$.

A fundamental amplitude can now be defined by

$$\Phi(n_1,\ldots,n_Q;\sigma,t_Q)\equiv\sum_{\sigma_0}\kappa(n_1,\ldots,n_Q;\sigma,t_Q\,|\,\sigma_0,t_0)\times\langle\sigma_0,\mathbf{e}_0\,|\,\psi_0(t_0)\rangle$$
$$=\langle\sigma,\mathbf{e}_Q\,|\,\hat{\Lambda}_{n_1\cdots n_Q}\,|\,\psi_0(t_0)\rangle\,; \tag{5.10a}$$

it is the joint amplitude that the samplings yield the sequence of results $n_1,\ldots,n_Q$ and that $\boldsymbol{\sigma}\cdot\mathbf{e}_Q$ has the value σ at time t_Q. It gives rise to the joint probability for the results $n_1,\ldots,n_Q$,

$$P(n_1,\ldots,n_Q)=\sum_{\sigma}|\Phi(n_1,\ldots,n_Q;\sigma,t_Q)|^2 =\langle\psi_0(t_0)|\hat{\Lambda}^{\dagger}_{n_1\cdots n_Q}\hat{\Lambda}_{n_1\cdots n_Q}|\psi_0(t_0)\rangle\ , \tag{5.10b}$$

which is normalized according to

$$1=\sum_{n_1,\ldots,n_Q}P(n_1,\ldots,n_Q) \tag{5.11}$$

[Eq. (5.7)]. Notice that the effect

$$\hat{F}_{n_1\cdots n_Q}\equiv\hat{\Lambda}^{\dagger}_{n_1\cdots n_Q}\hat{\Lambda}_{n_1\cdots n_Q} \tag{5.12}$$

generates the probability $P(n_1,\ldots,n_Q)$.

For comparison with AAD, one further change of interpretation is needed. The amplitude

$$\frac{\Phi(n_1,\ldots,n_Q;\sigma,t_Q)}{\left[\sum_{n_1,\ldots,n_Q}|\Phi(n_1,\ldots,n_Q;\sigma,t_Q)|^2\right]^{1/2}} \tag{5.13}$$

is the (normalized) conditional amplitude that the sampled quantities have values $n_1,\ldots,n_Q$, given that the spin component $\boldsymbol{\sigma}\cdot\mathbf{e}_Q$ has the value σ at time t_Q. Its absolute square is the corresponding conditional probability that the sampled quantities have values $n_1,\ldots,n_Q$, given the final spin value σ. Thus, one can regard

$$\Phi(n_1,\ldots,n_Q;\sigma,t_Q)=\langle\sigma,\mathbf{e}_Q|\hat{\Lambda}_{n_1\cdots n_Q}|\psi_0(t_0)\rangle \tag{5.14}$$

as a *relative* (unnormalized) conditional amplitude; its absolute square is a relative conditional probability. With this new interpretation comes a new idea: AAD refer to the expression

$$\langle\sigma,\mathbf{e}_Q|\cdots|\psi_0(t_0)\rangle \tag{5.15}$$

as a "two-time state," determined by the initial state $|\psi_0(t_0)\rangle$ at time t_0 and by the final spin value σ for $\boldsymbol{\sigma}\cdot\mathbf{e}_Q$ at time t_Q. [Actually, AAD write $|\sigma,\mathbf{e}_Q\rangle\cdots\langle\psi_0(t_0)|$ for the two-time state (5.15), but this interchange of bras and kets is of no consequence. I find the form (5.15) to be more appropriate for the subsequent discussion.] The three dots in the two-time state (5.15) invite one to insert an operator between the bra and the ket; one gets the conditional amplitude (5.14) by inserting the modified evolution operator $\hat{\Lambda}_{n_1\cdots n_Q}$.

Focus now on a single sampling ($Q=1$). The single sampled quantity is given by

$$y(t_1)=\sum_{j=1}^{N_1}\boldsymbol{\sigma}(t_{1j})\cdot\mathbf{e}_{1j} \tag{5.16}$$

[cf. Eq. (5.1)]. The time-ordering map (2.14) is trivial:

$$\beta\rightarrow q_\beta j_\beta=1\beta\equiv(\beta)\ ,\quad \beta=1,\ldots,B\ ,\quad B=N_1\ . \tag{5.17}$$

From Eq. (5.14) comes the relative conditional amplitude

$$\langle\sigma,\mathbf{e}_1|\hat{\Lambda}_{n_1}|\psi_0(t_0)\rangle=\Phi(n_1;\sigma,t_1) \tag{5.18}$$

that the sampled quantity has value n_1, given the spin value σ for $\boldsymbol{\sigma}\cdot\mathbf{e}_1$ at time t_1. By specializing Eq. (5.8) to a single sampling, one obtains

$$\hat{\Lambda}_{n_1}=\hat{U}(t_1,t_{1N_1})\left[\sum_{\sigma_{11},\ldots,\sigma_{1N_1}}\delta_{n_1,y(t_1)}\left[\prod_{j=1}^{N_1}|\sigma_{1j},\mathbf{e}_{1j}\rangle\langle\sigma_{1j},\mathbf{e}_{1j}|\hat{U}(t_{1j},t_{1j-1})\right]\right]\ , \tag{5.19}$$

where

$$y(t_1)=\sum_{j=1}^{N_1}\sigma_{1j}\ . \tag{5.20}$$

AAD call the operator $\hat{\Lambda}_{n_1}$ a "multiple-time eigenstate," with "eigenvalue" n_1, of the "multiple-time observable" $y(t_1)$ defined by Eq. (5.16). Actually, AAD define explicitly the notion of a multiple-time eigenstate only when two further simplifications are made: (i) the spin system has zero Hamiltonian ($\hat{H}=0$), in which case the evolution operators in Eq. (5.19) become the unit operator, and (ii) $y(t_1)$ obtains information from only two times ($N_1=2$; "two-time measurement"). Equation (5.19) is a natural generalization of the definition given by AAD.

Why refer to $\hat{\Lambda}_{n_1}$ as an eigenstate at all? Consider for comparison the amplitude $\langle\sigma,\mathbf{e}|\psi\rangle$ that the spin component $\boldsymbol{\sigma}\cdot\mathbf{e}$ has the value σ. This amplitude is obtained by projecting the system state $|\psi\rangle$ onto the eigenstate $|\sigma,\mathbf{e}\rangle$ corresponding to σ. Firmly associated with the notion of an eigenstate is this procedure for obtaining the amplitude for some value of an observable: project the system state onto the eigenstate corresponding to the value of interest. For a single multiple-time measurement, AAD find this procedure expressed in the relative conditional amplitude $\langle\sigma,\mathbf{e}_1|\hat{\Lambda}_{n_1}|\psi_0(t_0)\rangle$ that the sampled quantity has value n_1. They interpret this amplitude as coming from a "projection" of the two-time state $\langle\sigma,\mathbf{e}_1|\cdots|\psi_0(t_0)\rangle$ onto the multiple-time eigenstate $\hat{\Lambda}_{n_1}$ corresponding to n_1. Here "projection" means to insert $\hat{\Lambda}_{n_1}$ in place of the three dots in the two-time state.

A check that the idea of a multiple-time eigenstate makes sense comes from specializing to zero system Hamiltonian and to an ordinary "one-time" ($N_1=1$) sampled quantity

$$y(t_1)=\boldsymbol{\sigma}(t_{11})\cdot\mathbf{e}_{11}\ . \tag{5.21}$$

With these simplifications, $\hat{\Lambda}_{n_1}$ should be a "one-time

eigenstate" of the "one-time observable" $y(t_1)$, and indeed Eq. (5.19) shows that

$$\hat{\Lambda}_{n_1} = \sum_{\sigma_{11}} \delta_{n_1,\sigma_{11}} |\sigma_{11},\mathbf{e}_{11}\rangle\langle\sigma_{11},\mathbf{e}_{11}| = |n_1,\mathbf{e}_{11}\rangle\langle n_1,\mathbf{e}_{11}| \tag{5.22}$$

is the projection operator onto the ordinary eigenstate $|n_1,\mathbf{e}_{11}\rangle$. Thus, in this simplest of cases, $\hat{\Lambda}_{n_1}$ does reduce to something associated with the appropriate eigenstate.

The real test of the idea of multitime eigenstates comes when one tries to apply the idea to many multitime measurements. For that purpose, it is sufficient to consider two measurements ($Q=2$; $B=N_1+N_2$). Further, it is instructive initially to specialize to the case considered by AAD: zero system Hamiltonian ($\hat{H}=0$) and "two-time" ($N_1=N_2=2$) sampled quantities

$$y(t_1)=\boldsymbol{\sigma}(t_{11})\cdot\mathbf{e}_{11}+\boldsymbol{\sigma}(t_{12})\cdot\mathbf{e}_{12}\ , \tag{5.23a}$$

$$y(t_2)=\boldsymbol{\sigma}(t_{21})\cdot\mathbf{e}_{21}+\boldsymbol{\sigma}(t_{22})\cdot\mathbf{e}_{22}\ . \tag{5.23b}$$

The relative conditional amplitude to obtain results n_1 and n_2 is given by

$$\langle\sigma,\mathbf{e}_2|\hat{\Lambda}_{n_1n_2}|\psi_0(t_0)\rangle=\Phi(n_1,n_2;\sigma,t_2) \tag{5.24}$$

[Eq. (5.14)], where

$$\hat{\Lambda}_{n_1n_2}=\sum_{\sigma_{11},\sigma_{12},\sigma_{21},\sigma_{22}} \delta_{n_1,\sigma_{11}+\sigma_{12}}\delta_{n_2,\sigma_{21}+\sigma_{22}} \times\left[\prod_{\beta=1}^{4}|\sigma_{(\beta)},\mathbf{e}_{(\beta)}\rangle\langle\sigma_{(\beta)},\mathbf{e}_{(\beta)}|\right] \tag{5.25}$$

[Eq. (5.8)]. If the two measurements were performed separately, they would be described by the operators

$$\hat{\Lambda}_{n_1}=\sum_{\sigma_{11},\sigma_{12}} \delta_{n_1,\sigma_{11}+\sigma_{12}}|\sigma_{12},\mathbf{e}_{12}\rangle \times\langle\sigma_{12},\mathbf{e}_{12}|\sigma_{11},\mathbf{e}_{11}\rangle\langle\sigma_{11},\mathbf{e}_{11}|\ , \tag{5.26a}$$

$$\hat{\Lambda}_{n_2}=\sum_{\sigma_{21},\sigma_{22}} \delta_{n_2,\sigma_{21}+\sigma_{22}}|\sigma_{22},\mathbf{e}_{22}\rangle \times\langle\sigma_{22},\mathbf{e}_{22}|\sigma_{21},\mathbf{e}_{21}\rangle\langle\sigma_{21},\mathbf{e}_{21}| \tag{5.26b}$$

[cf. Eq. (5.19)]; AAD identify $\hat{\Lambda}_{n_1}$ and $\hat{\Lambda}_{n_2}$ as the "two-time eigenstates" of the "two-time observables" $y(t_1)$ and $y(t_2)$. The objective now is to investigate the usefulness of this identification.

Consider for comparison two consecutive, precise, one-time measurements, the first a measurement of the spin component $\boldsymbol{\sigma}\cdot\mathbf{e}_1$ and the second a measurement of the spin component $\boldsymbol{\sigma}\cdot\mathbf{e}_2$. If $|\psi\rangle$ is the initial state, then the amplitude to obtain σ_1 as the result of the first measurement is $\langle\sigma_1,\mathbf{e}_1|\psi\rangle$, with corresponding probability

$$P(\sigma_1)=|\langle\sigma_1,\mathbf{e}_1|\psi\rangle|^2\ .$$

This first measurement leaves the system in a new state

$$|\psi_{\sigma_1}\rangle\equiv|\sigma_1,\mathbf{e}_1\rangle\langle\sigma_1,\mathbf{e}_1|\psi\rangle/[P(\sigma_1)]^{1/2}\ ,$$

obtained by projecting the initial state onto the eigenstate $|\sigma_1,\mathbf{e}_1\rangle$ corresponding to the result σ_1 and then normalizing. Given result σ_1 for the first measurement, the amplitude to obtain σ_2 as the result of the second measurement is $\langle\sigma_2,\mathbf{e}_2|\psi_{\sigma_1}\rangle$, with corresponding conditional probability

$$P(\sigma_2|\sigma_1)=|\langle\sigma_2,\mathbf{e}_2|\psi_{\sigma_1}\rangle|^2 =\frac{|\langle\sigma_2,\mathbf{e}_2|\sigma_1,\mathbf{e}_1\rangle\langle\sigma_1,\mathbf{e}_1|\psi\rangle|^2}{P(\sigma_1)}\ .$$

Thus the joint probability for the two measurements is

$$P(\sigma_1,\sigma_2)=P(\sigma_2|\sigma_1)P(\sigma_1) =|\langle\sigma_2,\mathbf{e}_2|\sigma_1,\mathbf{e}_1\rangle\langle\sigma_1,\mathbf{e}_1|\psi\rangle|^2\ ,$$

which is a special case of Eq. (5.10b). The joint probability $P(\sigma_1,\sigma_2)$ is derived from a joint amplitude $\langle\sigma_2,\mathbf{e}_2|\sigma_1,\mathbf{e}_1\rangle\langle\sigma_1,\mathbf{e}_1|\psi\rangle$. Firmly associated with the notion of an eigenstate is the procedure for obtaining the system state after the first measurement: project the system into the eigenstate corresponding to the result of the measurement. A consequence of this procedure and of the properties of ordinary eigenstates is that the measurements are *reproducible:* if the two spin measurements measure the same spin component ($\mathbf{e}_1=\mathbf{e}_2$), then the result of the second measurement must be the same as the result of the first—i.e., $P(\sigma_1,\sigma_2)=\delta_{\sigma_2\sigma_1}P(\sigma_1)$.

Return now to the case of two two-time measurements. Suppose that the measurement labeled by 2 is "nested" inside the measurement labeled by 1, i.e.,

$$t_{11}<t_{21}<t_{22}<t_{12} \tag{5.27}$$

(recall that $t_{12}\le t_1<t_2$); one can think of measurement 1 as the "outside measurement" and of measurement 2 as the "inside measurement." The time ordering (5.27) implies that $\hat{\Lambda}_{n_1n_2}$ [Eq. (5.25)] takes the explicit form

$$\hat{\Lambda}_{n_1n_2}=\sum_{\sigma_{11},\sigma_{21},\sigma_{22},\sigma_{12}} \delta_{n_1,\sigma_{11}+\sigma_{12}}\delta_{n_2,\sigma_{21}+\sigma_{22}} \times|\sigma_{12},\mathbf{e}_{12}\rangle\langle\sigma_{12},\mathbf{e}_{12}|\sigma_{22},\mathbf{e}_{22}\rangle\langle\sigma_{22},\mathbf{e}_{22}|\sigma_{21},\mathbf{e}_{21}\rangle\langle\sigma_{21},\mathbf{e}_{21}|\sigma_{11},\mathbf{e}_{11}\rangle\langle\sigma_{11},\mathbf{e}_{11}|\ , \tag{5.28a}$$

and Eq. (5.26b) further shows that

$$\hat{\Lambda}_{n_1n_2}=\sum_{\sigma_{11},\sigma_{12}} \delta_{n_1,\sigma_{11}+\sigma_{12}}|\sigma_{12},\mathbf{e}_{12}\rangle\langle\sigma_{12},\mathbf{e}_{12}|\hat{\Lambda}_{n_2}|\sigma_{11},\mathbf{e}_{11}\rangle\langle\sigma_{11},\mathbf{e}_{11}|\ . \tag{5.28b}$$

AAD view the relative amplitude $\langle\sigma,\mathbf{e}_2|\hat{\Lambda}_{n_1n_2}|\psi_0(t_0)\rangle$ as arising in the following way. If the outside measurement were

performed alone, the relative amplitude to obtain result n_1 would be $\langle\sigma,\mathbf{e}_2\,|\,\hat{\Lambda}_{n_1}\,|\,\psi_0(t_0)\rangle$ [cf. Eq. (5.26a)]. The outside measurement "projects" the system into a new two-time state

$$\sum_{\sigma_{11},\sigma_{12}} \delta_{n_1,\sigma_{11}+\sigma_{12}}\langle\sigma,\mathbf{e}_2\,|\,\sigma_{12},\mathbf{e}_{12}\rangle\langle\sigma_{12},\mathbf{e}_{12}\,|\,\cdots\,|\,\sigma_{11},\mathbf{e}_{11}\rangle\langle\sigma_{11},\mathbf{e}_{11}\,|\,\psi_0(t_0)\rangle\ , \tag{5.29}$$

obtained by "pulling apart" $\hat{\Lambda}_{n_1}$ at its middle. The relative amplitude $\langle\sigma,\mathbf{e}_2\,|\,\hat{\Lambda}_{n_1n_2}\,|\,\psi_0(t_0)\rangle$ for the two measurements then comes from "projecting" the new two-time state (5.29) onto the two-time eigenstate $\hat{\Lambda}_{n_2}$. This procedure can obviously be extended to include as many nested two-time measurements as desired. Are these nested measurements reproducible? Indeed they are. Suppose the two sampled quantities (5.23) measure the same sum of spin components—i.e., $\mathbf{e}_{21}=\mathbf{e}_{11}$ and $\mathbf{e}_{22}=\mathbf{e}_{12}$; then one finds that $\hat{\Lambda}_{n_1n_2}=\delta_{n_2n_1}\hat{\Lambda}_{n_1}$, which shows that the result of the inside measurement must be the same as the result of the outside measurement.

The concept of multitime eigenstates seems to provide a reasonable description of nested two-time measurements. The first fly in the ointment shows up when one considers what AAD call "crossed" two-time measurements. Two crossed measurement overlap according to the time ordering

$$t_{11}<t_{21}<t_{12}<t_{22}\ , \tag{5.30}$$

which implies that

$$\hat{\Lambda}_{n_1n_2}=\sum_{\sigma_{11},\sigma_{21},\sigma_{12},\sigma_{22}} \delta_{n_1,\sigma_{11}+\sigma_{12}}\delta_{n_2,\sigma_{21}+\sigma_{22}}\,|\,\sigma_{22},\mathbf{e}_{22}\rangle\langle\sigma_{22},\mathbf{e}_{22}\,|\,\sigma_{12},\mathbf{e}_{12}\rangle\langle\sigma_{12},\mathbf{e}_{12}\,|\,\sigma_{21},\mathbf{e}_{21}\rangle\langle\sigma_{21},\mathbf{e}_{21}\,|\,\sigma_{11},\mathbf{e}_{11}\rangle\langle\sigma_{11},\mathbf{e}_{11}\,| \tag{5.31}$$

[Eq. (5.25)]. Neither $\hat{\Lambda}_{n_1}$ nor $\hat{\Lambda}_{n_2}$ appears unmolested in $\hat{\Lambda}_{n_1n_2}$ [cf. Eqs. (5.26)]. To construct $\hat{\Lambda}_{n_1n_2}$, both $\hat{\Lambda}_{n_1}$ and $\hat{\Lambda}_{n_2}$ must be "pulled apart" at the middle; the components are then interleaved according to Eq. (5.31). Neither $\hat{\Lambda}_{n_1}$ nor $\hat{\Lambda}_{n_2}$ can be profitably regarded as "projecting" the system into a new two-time state. The idea of multitime eigenstates seems not to apply to crossed two-time measurements.[9] Yet such measurements are reproducible: if the two sampled quantities (5.23) measure the same sum of spin components—i.e., $\mathbf{e}_{21}=\mathbf{e}_{11}$ and $\mathbf{e}_{22}=\mathbf{e}_{12}$—then Eq. (5.31) reduces to $\hat{\Lambda}_{n_1n_2}=\delta_{n_2n_1}\hat{\Lambda}_{n_1}$.

Given a sequence of one-time measurements, one can work through the sequence serially, projecting the system into a new one-time state at each measurement. Given a set of nested two-time measurements, one can work through the set from the "outside" to the "inside," "projecting" the system into a new two-time state at each measurement. For crossed two-time measurements, neither of these procedures works; the difficulty is that neither $\hat{\Lambda}_{n_1}$ nor $\hat{\Lambda}_{n_2}$ is an appropriate building block for constructing $\hat{\Lambda}_{n_1n_2}$.

This difficulty becomes more acute when one generalizes to multitime measurements that receive information from more than two times. Return to the general case of Q multitime measurements of the quantities (5.1); the samplings are described by the operator $\hat{\Lambda}_{n_1\cdots n_Q}$ [Eq. (5.8)]. If the samplings were performed separately, they would be described by operators $\hat{\Lambda}_{n_q}$ for single samplings; the ideas of AAD suggest calling $\hat{\Lambda}_{n_q}$ a multitime eigenstate, with eigenvalue n_q, of the multitime observable $y(t_q)$. The issue is not whether $\hat{\Lambda}_{n_q}$ deserves the designation "eigenstate"; the issue is really whether $\hat{\Lambda}_{n_q}$ deserves any designation at all. True, $\hat{\Lambda}_{n_q}$ would describe the qth sampling if it were performed alone. True, the operators $\hat{\Lambda}_{n_q}$ collectively contain all the ingredients that go into $\hat{\Lambda}_{n_1\cdots n_Q}$. But, to construct $\hat{\Lambda}_{n_1\cdots n_Q}$, one must in general pull apart the operators $\hat{\Lambda}_{n_q}$ at every joint and then reassemble the components according to the ordering of the times t_{qj}. There seems to be no way to think of any of the operators $\hat{\Lambda}_{n_q}$ as projecting the system into some new sort of state.

The message here was actually delivered in Sec. IV: during a sequence of overlapping samplings, there is in general no way to analyze each sampling separately and no way to make use of the description each sampling would have were it performed separately. The physics lies in the modified evolution operator $\hat{\Lambda}_{n_1\cdots n_Q}$, which describes the Q samplings as a whole; it gives rise to the effect $\hat{F}_{n_1\cdots n_Q}$, which in turn generates the statistics of the samplings [Eqs. (5.10b) and (5.12)].

The concept of multitime eigenstates flounders when generalized beyond nested two-time measurements. Nonetheless, as suggested by the case of crossed two-time measurements, the property of reproducibility survives. This property deserves to be spelled out clearly, because it is the important physical property, regardless of whether one attaches to it some notion of an eigenstate. For that purpose, specialize to two samplings ($Q=2$) that measure the same sum of spin components—i.e., $N_1=N_2=N$ and

$$\mathbf{e}_{2j}=\mathbf{e}_{1j}\ ,\quad j=1,\ldots,N \tag{5.32a}$$

[cf. Eq. (5.1)]. For one-time measurements, reproducibility is associated with immediate repetition of a measurement. Thus one might think to analyze the situation where the second sampling begins just after the first sampling ends (i.e., $t_{21}=t_{1N}+\epsilon$), but two such samplings clearly do not in general yield the same result. For multitime measurements, "immediate repetition" means something different. It means that the jth contributions to the two sampled quantities occur at times which are infini-

tesimally different, i.e.,

$$t_{2j}=t_{1j}\pm\epsilon\ ,\quad j=1,\ldots,N\ , \tag{5.32b}$$

where $\epsilon>0$ is an infinitesimal time, and the sign $\pm$ can be chosen independently for each value of j. Thus t_{2j} can either precede or succeed t_{1j} by an infinitesimal amount. With assumptions (5.32), one can easily show that

$$\lim_{\epsilon\to 0}\hat{\Lambda}_{n_1n_2}=\delta_{n_2n_1}\hat{\Lambda}_{n_1} \tag{5.33a}$$

[Eqs. (5.8) and (5.19)], which implies that

$$\lim_{\epsilon\to 0}P(n_1,n_2)=\delta_{n_2n_1}P(n_1) \tag{5.33b}$$

[Eq. (5.10b)]. Hence the two samplings must yield the same result. If the system has zero Hamiltonian, then one need not take the limit $\epsilon\to 0$ in Eq. (5.33a). The property $\hat{\Lambda}_{n_1n_2}=\delta_{n_2n_1}\hat{\Lambda}_{n_1}$ holds so long as the times t_{qj} satisfy

$$t_{11},t_{21}<t_{12},t_{22}<\cdots<t_{1N},t_{2N}\ ; \tag{5.34}$$

for each value of j, one can choose either $t_{1j}<t_{2j}$ or $t_{1j}>t_{2j}$.

In a comment on the work of Aharonov and Albert,[6] Cohen and Peres[26] criticize the idea of two-time measurements on the grounds that such measurements are not reproducible when the second measurement begins after the first has been completed. In their reply Aharonov and Albert[27] emphasize that reproducibility is achieved in the way just described.

VI. CONCLUSION

It is often said that the quantum state of a system—whether a wave function or a density matrix—provides a complete quantum-mechanical description of the system. Measurements distributed in time force one to reexamine this statement. A quantum state contains all information about a system *at a particular time*, but it contains no *multitime* information. To analyze time-distributed measurements, one requires new quantum-mechanical tools—new ways of addressing multitime questions. An ideal conceptual tool is provided by a sum over histories: the histories contain multitime information; quantum mechanics comes in through the rule for summing probability amplitudes over histories. Though ideal conceptually, a sum over histories is not always the most effective tool for attacking a particular problem. In this paper I have attempted to enlarge the analytical tool box by establishing connections between the sum-over-histories formulation and other ways of formulating a quantum-mechanical description of measurements distributed in time.

ACKNOWLEDGMENTS

This work was supported in part by a Precision Measurement Grant from the National Bureau of Standards (Grant No. NB83-NADA-4038).

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