Nonsingular quarkonium potential

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It is shown that the appearance of singular terms in the quasistatic potential models of quarkonium can be avoided with the use of an improved quasistatic approximation, which yields a nonsingular quark-antiquark potential.

Although quasistatic potential models for quarkonia have met with considerable success, the validity of such models has often been questioned. It has also been suggested[1–3] that the singular terms appearing in the quarkonium potential be replaced by phenomenological nonsingular form factors. We shall give a justification for the use of a quasistatic potential for the treatment of quarkonia, and further show that a nonsingular quarkonium potential can be derived by means of an improved quasistatic approximation.

Recently a quark-confinement mechanism inspired by renormalization-group-improved perturbative quantum chromodynamics was proposed,[4] and it was shown that quark confinement can be understood as a consequence of the fact that quarks and antiquarks can exchange only hard gluons. It was further argued[5] that this confinement mechanism helps to explain why a quasistatic quark-antiquark potential yields good results for quarkonium spectra even when the quark and antiquark possess appreciably large momenta. Let us consider the scattering of a quark and an antiquark in the center-of-mass system, and let $\mathbf{p}$ and $\mathbf{p}'$ be the initial and the final momenta of the quark. Since

$$\mathbf{p}^2=\tfrac{1}{4}\mathbf{k}^2+\tfrac{1}{4}\mathbf{s}^2 , \tag{1}$$

where

$$\mathbf{k}=\mathbf{p}'-\mathbf{p},\quad \mathbf{s}=\mathbf{p}'+\mathbf{p} , \tag{2}$$

it follows that if $\mathbf{k}^2$ is allowed to take only large values, $\mathbf{s}^2$ can be treated as small. This provides a justification for the quasistatic approximation in which terms of second and higher orders in $\mathbf{s}$ are ignored.

We shall now derive a singularity-free[6] quarkonium potential by an improvement of the usual treatment. In order to obtain the potential from the scattering operator,[7] let us carry out a nonrelativistic approximation of the quark-antiquark scattering matrix element by treating $\mathbf{p}^2/p_0{}^2$ as small. Since $\mathbf{p}^2/p_0{}^2$ is smaller than $\mathbf{p}^2/m^2$, this approximation is an improvement over the usual practice of treating $\mathbf{p}^2/m^2$ as small, and it can be expected to give an improved result for the potential at very short distances, which correspond to large momentum transfers. We thus obtain, for the Fourier transform of the second-order perturbative potential,

$$\mathscr{V}_2(\mathbf{k})=-\frac{16\pi\alpha_s}{3}\left[\frac{1}{\mathbf{k}^2}+\frac{\mathbf{p}^2}{p_0{}^2\mathbf{k}^2}-\frac{1}{2p_0{}^2}+\frac{\boldsymbol{\sigma}_1\cdot\mathbf{k}\boldsymbol{\sigma}_2\cdot\mathbf{k}-\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2\mathbf{k}^2}{4p_0{}^2\mathbf{k}^2}+\frac{3i(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\mathbf{k}\times\mathbf{p}}{4p_0{}^2\mathbf{k}^2}\right],$$

and since in the quasistatic approximation we can set $\mathbf{p}^2=\frac{1}{4}\mathbf{k}^2$ and $p_0{}^2=m^2+\frac{1}{4}\mathbf{k}^2$,

$$\mathscr{V}_2(\mathbf{r})=-\frac{4\alpha_s}{3}\left[\frac{1}{r}-\frac{4e^{-2mr}}{3r}(\mathbf{S}^2-\tfrac{3}{4})-\frac{3f_1}{2r}\mathbf{S}\cdot\mathbf{L}-\frac{f_2}{4r}S_{12}\right] \tag{3}$$

with

$$\mathbf{S}=\tfrac{1}{2}(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2),\quad S_{12}=3\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}-\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2 , \tag{4}$$

and

$$\begin{aligned} f_1&=[1-(1+2mr)e^{-2mr}]/m^2r^2 ,\\ f_2&=[1-(1+2mr+\tfrac{4}{3}m^2r^2)e^{-2mr}]/m^2r^2 . \end{aligned} \tag{5}$$

We observe that $\mathscr{V}_2(\mathbf{r})$ behaves as $1/r$ at the origin, and application of the same treatment shows that $\mathscr{V}_4(\mathbf{r})$ behaves as $\ln r/r$ at the origin.

It is also interesting to note that, according to the improved quasistatic approximation, Fourier transforms of the scalar-exchange and vector-exchange linear confining potentials are

$$\mathscr{V}_S(\mathbf{k})=-8\pi A\left[\frac{1}{\mathbf{k}^4}-\frac{i(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\mathbf{k}\times\mathbf{p}}{\mathbf{k}^4(4m^2+\mathbf{k}^2)}\right],$$

$$\mathscr{V}_V(\mathbf{k})=-8\pi A\left[\frac{1}{\mathbf{k}^4}-\frac{1}{\mathbf{k}^2(4m^2+\mathbf{k}^2)}+\frac{\boldsymbol{\sigma}_1\cdot\mathbf{k}\boldsymbol{\sigma}_2\cdot\mathbf{k}-\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2\mathbf{k}^2}{\mathbf{k}^4(4m^2+\mathbf{k}^2)}+\frac{3i(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\mathbf{k}\times\mathbf{p}}{\mathbf{k}^4(4m^2+\mathbf{k}^2)}\right],$$

so that

$$\mathscr{V}_S(\mathbf{r}) = Ar - \frac{A}{2m^2 r}(1 - \tfrac{1}{2}f_1)\mathbf{L}\cdot\mathbf{S} , \tag{6}$$

$$\mathscr{V}_V(\mathbf{r}) = Ar + \frac{2A}{3m^2 r}(1 - e^{-2mr})(\mathbf{S}^2 - \tfrac{3}{4}) + \frac{3A}{2m^2 r}(1 - \tfrac{1}{2}f_1)\mathbf{L}\cdot\mathbf{S} + \frac{A}{12m^2 r}(1 - \tfrac{3}{2}f_2)S_{12} , \tag{7}$$

which are finite at the origin.

As is well known, a singular potential has the disadvantage that the singular terms in the potential have to be treated as perturbation for the solution of the Schrödinger equation. A similar situation arises when the kinetic energy of a two-particle system is taken in the form

$$T = \mathbf{p}^2/m - \mathbf{p}^4/4m^3 ,$$

and it becomes necessary to treat the $\mathbf{p}^4$ term as a perturbation. With the use of the potentials derived here, it is possible to avoid altogether the use of perturbation for the calculation of energy levels and wave functions of quarkonium by expressing its Hamiltonian as[8]

$$\mathscr{H} = 2(\mathbf{p}^2 + m^2)^{1/2} + \mathscr{V}(\mathbf{r}) , \tag{8}$$

where $\mathscr{V}(\mathbf{r})$ represents the sum of nonsingular perturbative and confining potentials.

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