Mixing of quark flavors

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A specific framework to describe the weak-interaction mixing of an arbitrary number of quark generations is investigated. In particular the mixing matrices for four and five flavors are studied. In the case of three flavors a mixing matrix results which differs in a crucial way from the standard Kobayashi-Maskawa form.

The mixing of quark flavors in the interaction of quarks and W bosons is a yet-unexplained feature of the weak interactions, which must be related to the mechanism of mass generation for the quarks. If all quark masses were zero, the phenomenon of weak-interaction mixing would not exist. Thus the mixing parameters can be viewed as elements of the quark mass matrix. In case of three generations of quarks the weak-interaction eigenstates d', s', and b' are related to the mass eigenstates d, s, and b by the 3×3 unitary mixing matrix V (Ref. 1):

$$\begin{bmatrix} d' \\ s' \\ b' \end{bmatrix} = V \begin{bmatrix} d \\ s \\ b \end{bmatrix}, \quad V = \begin{bmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{bmatrix} . \tag{1}$$

Taking into account the experimental constraints and the constraints imposed by unitarity, one finds for the absolute values of the mixing elements (see Ref. 2, for a revision of the V_{ub} matrix element see, e.g., Ref. 3):

$$\begin{aligned} &|V_{ud}| = 0.9723-0.9737, \quad |V_{us}| = 0.228-0.234, \quad |V_{ub}| = 0.000-0.008 , \\ &|V_{cd}| = 0.228-0.234, \quad |V_{cs}| = 0.974-0.9727, \quad |V_{cb}| = 0.039-0.051 , \\ &|V_{td}| = 0.005-0.015, \quad |V_{ts}| = 0.038-0.050, \quad |V_{tb}| = 0.9987-0.9993 . \end{aligned} \tag{2}$$

Thus nature seems to prefer the mixing of nearest neighbors; e.g., a particular flavor is predominantly mixed with the quarks close by in the mass spectrum. This suggests possible relationships between quark masses and mixing angles, e.g., those discussed in Refs. 4 and 5.

It is not excluded that the mixing element V_{ub} is zero. However the unitarity constraints require V_{td} to be nonzero. In the limit $V_{ub}=0$ the phases of the complex matrix elements of V can be rotated away, and one is left with a real rotation matrix which can be parametrized by two angles. A third angle and a complex phase can be introduced as a slight perturbation. As a result one obtains the following representation of V discussed in Ref. 6:

$$\begin{aligned} V &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & C_{23} & S_{23} \\ 0 & -S_{23} & C_{23} \end{bmatrix} \begin{bmatrix} C_{13} & 0 & S_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -S_{13}e^{i\delta} & 0 & C_{13} \end{bmatrix} \begin{bmatrix} C_{12} & S_{12} & 0 \\ -S_{12} & C_{12} & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} C_{12}C_{13} & S_{12}C_{13} & S_{13}e^{-i\delta} \\ -C_{23}S_{12}-C_{12}S_{23}S_{13}e^{i\delta} & C_{12}C_{23}-S_{12}S_{23}S_{13}e^{i\delta} & C_{13}S_{23} \\ S_{12}S_{23}-C_{12}C_{23}S_{13}e^{i\delta} & -C_{12}S_{23}-C_{23}S_{12}S_{13}e^{i\delta} & C_{13}C_{23} \end{bmatrix} \simeq \begin{bmatrix} 1 & S_{12} & S_{13}e^{-i\delta} \\ -S_{12} & 1 & S_{23} \\ -S_{12}S_{23}-S_{13}e^{i\delta} & -S_{23} & 1 \end{bmatrix} . \end{aligned} \tag{3}$$

Here S and C denote $\sin\theta$ and $\cos\theta$, respectively, where θ_{ij} is the mixing angle describing the mixing of generation i with j. The Cabibbo angle θ_C is given by θ_{12}.

In this paper we should like to discuss a simple generalization of the representation (3) for an arbitrary number of generations. Especially it is our aim to parametrize each nondiagonal matrix element $V_{ij}(i\neq j)$ by a special angle θ_{ij}. A very simple generalization of Eq. (3) is found which in retrospect supports the claim made in Ref. 6 that the representation (3) is the most suitable one to parametrize the experimental data.

The mixing among n generations is described by an

$n\times n$ unitary matrix V as

$$\begin{pmatrix} q_1' \\ q_2' \\ \vdots \\ q_n' \end{pmatrix} = \begin{pmatrix} V_{11} & V_{12} & \cdots & V_{1n} \\ V_{21} & V_{22} & \cdots & V_{2n} \\ \vdots & \vdots & & \vdots \\ V_{n1} & V_{n2} & \cdots & V_{nn} \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{pmatrix}, \qquad (4)$$

where q is the quark-mass eigenstate, q' the weak-interaction eigenstate, $(q_1,q_2,q_3,\ldots)=(d,s,b,\ldots)$.

In order to arrive at a simple parametrization of V we suppose, in accordance with observation in the case of three generations, that the mixing elements can be expanded in a small parameter μ. The diagonal elements are of order 1 ($V_{ii}\approx 1$), the elements next to the main diagonal are of order μ ($V_{ij}\approx c_{ij}\mu$, $|i-j|=1$, c_{ij} are constants of order 1), etc. In case of three flavors such an expansion is similar to the one discussed by Wolfenstein.[7]

The structure of the mixing matrix which results is

$$(V_{ij})\simeq(c_{ij}\mu^{|i-j|}) = \begin{pmatrix} 1 & \sim\mu & \sim\mu^2 & \cdots & \sim\mu^n \\ \sim\mu & 1 & \sim\mu & \cdots & \sim\mu^{n-1} \\ \sim\mu^2 & \sim\mu & 1 & \cdots & \sim\mu^{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sim\mu^n & \sim\mu^{n-1} & \sim\mu^{n-2} & \cdots & 1 \end{pmatrix} \qquad (5)$$

(the symbol $\sim$ denotes the constants c_{ij}, which we have omitted).

The mixing of n generations is described in general by the n^2 parameters of V. We can adjust the $2n-1$ relative phases of the $2n$ quark fields such that $2n-1$ elements of V can be made real, for example, all diagonal elements V_{ii} and all elements above and next to the main diagonal V_{ii+1} (altogether $2n-1$ elements). The remaining $(n-1)^2$ parameters are $\frac{1}{2}n(n-1)$ angles and $\frac{1}{2}(n-1)(n-2)$ phases (for a general parametrization see Ref. 8).

In zeroth order of μ the mixing matrix is the unit matrix. Next we treat the first order of μ and consider the case where all elements V_{ii+1} are different from zero, but all elements above the latter vanish:

$$(V_{ij})\simeq \begin{pmatrix} 1 & \theta_{12} & 0 & \cdots & 0 \\ & 1 & \theta_{23} & \cdots & 0 \\ & & \ddots & \ddots & \vdots \\ & & & & \theta_{n-1,n} \\ & & & & 1 \end{pmatrix}. \qquad (6)$$

The exact form of the matrix can be easily obtained if we introduce the (complex) rotation matrices $\tilde{R}_{ij}$ as

$$\tilde{R}_{ij}= \begin{array}{c} \begin{array}{ccccccccc} 1\cdots & & i\cdots & & & & j\cdots & & n \end{array} \\ \begin{pmatrix} 1 & 0 & \cdots & & & & & & 0 \\ 0 & 1 & & & & & & & \vdots \\ \vdots & & \ddots & & & & & & \\ 0 & \cdots & c_{ij} & 0 & \cdots & 0 & s_{ij}e^{-i\delta_{ij}} & \cdots & 0 \\ \vdots & & & & & & & & \vdots \\ & & 0 & 1 & & & 0 & & \\ & & \vdots & & \ddots & & \vdots & & \\ & & 0 & & & 1 & 0 & & \\ 0 & \cdots & -s_{ij}e^{i\delta_{ij}} & 0 & \cdots & 0 & c_{ij} & \cdots & 0 \\ \vdots & & & & & & & 1 & \vdots \\ & & & & & & & & \ddots \\ 0 & \cdots & & & & & & & 1 \end{pmatrix} \end{array} \begin{array}{c} 1 \\ \vdots \\ \\ i \\ \vdots \\ \\ \\ \\ j \\ \vdots \\ \\ n \end{array}. \qquad (7)$$

First we shall use such rotation matrices with $\delta_{ij}=0$, in which case the $\sim$ symbol is left out: $R_{ij}=\tilde{R}_{ij}$. The matrix V is given by the (real) product

$$V=R_{n-1,n}R_{n-2,n-1}\cdots R_{23}R_{12}\,. \qquad (8)$$

In this case the mixing of flavors proceeds sequentially. First one has a mixing between d and s, followed by a mixing between s and b, etc. Only nearest neighbors mix. No phases are present.

We observe that the number of zeros above the main diagonal is equal to $\frac{1}{2}(n-1)(n-2)$, i.e., it corresponds exactly to the number of independent phases. This suggests, analogous to Eq. (6), the following parametrization (in lowest order of the angles):

$$(V_{ij})\approx \begin{pmatrix} 1 & \theta_{12} & \tilde{\theta}_{13} & \tilde{\theta}_{14} & \cdots & \tilde{\theta}_{1n} \\ & 1 & \theta_{23} & \tilde{\theta}_{24} & \cdots & \tilde{\theta}_{2n} \\ & & 1 & \theta_{34} & \cdots & \tilde{\theta}_{3n} \\ & & & \ddots & \ddots & \vdots \\ & & & & & \theta_{n-1,n} \\ & & & & & 1 \end{pmatrix}. \qquad (9)$$

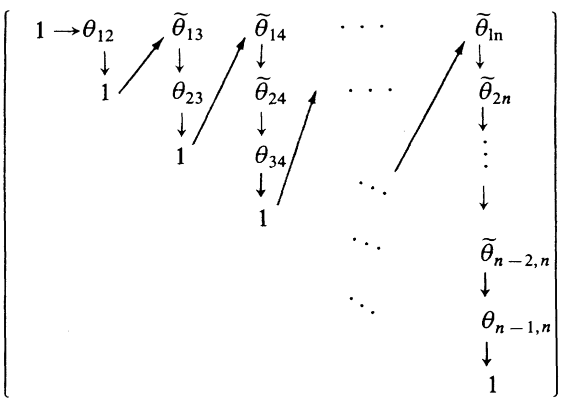
Here $\tilde{\theta}_{13}$ denotes $\theta_{13}e^{-i\delta_{13}}$, etc. Thus each matrix element V_{ij} $(j>i)$ is described by *one* angle θ_{ij} and *one* phase δ_{ij}. In the special case $j=i+1$ no phase appears, i.e., phases appear only if the two quarks involved in the mixing are not nearest neighbors.

An exact parametrization of the mixing matrix, which in lowest order of $^{(}\tilde{\theta}^{)}$ leads to Eq. (9), can be constructed analogous to Eq. (8). One finds

$$V=R_{n-1,n}\tilde{R}_{n-2,n}\cdots\tilde{R}_{2,n}\tilde{R}_{1,n}\cdots R_{k-1,k}\tilde{R}_{k-2,k}\times\cdots\tilde{R}_{2,k}\tilde{R}_{1,k}\cdots R_{23}\tilde{R}_{13}R_{12}\ . \qquad (10)$$

The matrix V is a product of (complex) rotation matrices. The order of these matrices given in Eq. (10) is not unique, due to the fact that two matrices $\tilde{R}_{ij}$ and $\tilde{R}_{kl}$ commute if $i\neq k$, $j\neq l$. For example, in case $n=4$ there is, besides the order given in Eq. (10), one additional possibility, while for $n=5$ eleven other possibilities exist.

The pattern of the rotation sequences given in Eq. (10) is denoted by the arrows in the scheme



. (11)

If we are only interested in the matrix V up to the second order in μ, one has in lowest order of $^{(}\tilde{\theta}^{)}$:

$$(V_{ij})=\begin{pmatrix} 1 & \theta_{12} & \tilde{\theta}_{13} & 0 & \cdots & & 0 \\ & 1 & \theta_{23} & \tilde{\theta}_{24} & \cdots & & 0 \\ & & 1 & & & & 0 \\ & & & \vdots & \vdots & \vdots & \vdots \\ & & & & 1 & \theta_{n-2,n-1} & \tilde{\theta}_{n-2,n} \\ & & & & & 1 & \theta_{n-1,n} \\ & & & & & & 1 \end{pmatrix}. \qquad (12)$$

The exact form of the mixing matrix is

$$V=R_{n-1,n}\tilde{R}_{n-2,n}R_{n-2,n-1}\cdots\tilde{R}_{24}R_{23}\tilde{R}_{13}R_{12}\ . \qquad (13)$$

The quark mixing matrix given in Eq. (10) is parametrized in terms of $\frac{1}{2}n(n-1)$ angles θ_{ij} and $\frac{1}{2}(n-1)(n-2)$ phases δ_{ij} $(j\geq i+2)$ and represents a very simple way to describe the phenomenon of weak-interaction mixing for the case of an arbitrary number of generations. Each matrix element $V_{ij}(j>i)$ is described by *one* angle θ_{ij} and *one* phase δ_{ij}. We believe that this is the most suitable generalization of the Cabibbo rotation matrix. Below we consider specific cases.

(a) $n=2$: The matrix V reduces to the real rotation matrix R_{12} (Cabibbo rotation).

(b) $n=3$: One arrives at the matrix Eq. (3), given in Ref. 6.

(c) $n=4$: The mixing matrix V is given by

$$(V_{ij})=R_{34}\tilde{R}_{24}\tilde{R}_{14}R_{23}\tilde{R}_{13}R_{12}$$

$$=\begin{pmatrix}
C_{12}C_{13}C_{14} & C_{13}C_{14}S_{12} & C_{14}S_{13}e^{-i\delta_{13}} & S_{14}e^{-i\delta_{14}} \\[2ex]
\begin{array}{l}-C_{23}C_{24}S_{12}-C_{12}C_{24}S_{13}S_{23}e^{i\delta_{13}}\\ -C_{12}C_{13}S_{14}S_{24}e^{i(\delta_{14}-\delta_{24})}\end{array} &
\begin{array}{l}C_{12}C_{23}C_{24}-C_{24}S_{12}S_{13}S_{23}e^{i\delta_{13}}\\ -C_{13}S_{12}S_{14}S_{24}e^{i(\delta_{14}-\delta_{24})}\end{array} &
\begin{array}{l}C_{13}C_{24}C_{23}\\ -S_{13}S_{14}S_{24}e^{-i(\delta_{13}+\delta_{24}-\delta_{14})}\end{array} &
C_{14}S_{24}e^{-i\delta_{24}} \\[2ex]
\begin{array}{l}-C_{12}C_{23}C_{34}S_{13}e^{i\delta_{13}}+C_{34}S_{12}S_{23}\\ -C_{12}C_{13}C_{24}S_{14}S_{34}e^{i\delta_{14}}\\ +C_{23}S_{12}S_{24}S_{34}e^{i\delta_{24}}\\ +C_{12}S_{13}S_{23}S_{24}S_{34}e^{i(\delta_{13}+\delta_{24})}\end{array} &
\begin{array}{l}-C_{12}C_{34}S_{23}-C_{23}C_{34}S_{12}S_{13}e^{i\delta_{13}}\\ -C_{12}C_{23}S_{24}S_{34}e^{i\delta_{24}}\\ -C_{13}C_{24}S_{12}S_{14}S_{34}e^{i\delta_{14}}\\ +S_{12}S_{13}S_{23}S_{24}S_{34}e^{i(\delta_{13}+\delta_{24})}\end{array} &
\begin{array}{l}C_{13}C_{23}C_{34}\\ -C_{13}S_{23}S_{24}S_{34}e^{i\delta_{24}}\\ -C_{24}S_{13}S_{14}S_{34}e^{i(\delta_{14}-\delta_{13})}\end{array} &
C_{14}C_{24}S_{34} \\[2ex]
\begin{array}{l}-C_{12}C_{13}C_{24}C_{34}S_{14}e^{i\delta_{14}}\\ +C_{12}C_{23}S_{13}S_{34}e^{i\delta_{13}}\\ +C_{23}C_{34}S_{12}S_{24}e^{i\delta_{24}}-S_{12}S_{23}S_{34}\\ +C_{12}C_{34}S_{13}S_{23}S_{24}e^{i(\delta_{13}+\delta_{24})}\end{array} &
\begin{array}{l}-C_{12}C_{23}C_{34}S_{24}e^{i\delta_{24}}+C_{12}S_{23}S_{34}\\ -C_{13}C_{24}C_{34}S_{12}S_{14}e^{i\delta_{14}}\\ +C_{23}S_{12}S_{13}S_{34}e^{i\delta_{13}}\\ +C_{34}S_{12}S_{13}S_{23}S_{24}e^{i(\delta_{13}+\delta_{24})}\end{array} &
\begin{array}{l}-C_{13}C_{23}S_{34}\\ -C_{13}C_{34}S_{23}S_{24}e^{i\delta_{24}}\\ -C_{24}C_{34}S_{13}S_{14}e^{i(\delta_{14}-\delta_{13})}\end{array} &
C_{14}C_{24}C_{34}
\end{pmatrix}. \qquad (14)$$

(d) $n=5$: One finds

$$V=R_{45}\tilde{R}_{35}\tilde{R}_{25}\tilde{R}_{15}R_{34}\tilde{R}_{24}\tilde{R}_{14}R_{23}\tilde{R}_{13}R_{12}\ . \qquad (15)$$

The full expression will not be given here. If we keep only the terms up to second order in μ, we obtain

$$V=R_{45}\tilde{R}_{35}R_{34}\tilde{R}_{24}R_{23}\tilde{R}_{13}R_{12}$$

$$=\begin{bmatrix} C_{12}C_{13} & C_{13}S_{12} & S_{13}e^{-i\delta_{13}} & 0 & 0 \\ \sim(-S_{12}-C_{12}S_{13}S_{23}e^{i\delta_{13}}) & C_{12}C_{23}C_{24}-C_{24}S_{12}S_{13}S_{23}e^{i\delta_{13}} & C_{13}C_{24}S_{23} & S_{24}e^{-i\delta_{24}} & 0 \\ \sim(-C_{12}S_{13}e^{i\delta_{13}}+S_{12}S_{23}) & \sim(-C_{12}S_{23}-S_{12}S_{13}e^{i\delta_{13}} -C_{12}S_{24}S_{34}e^{i\delta_{24}}) & C_{13}C_{23}C_{34}C_{35} -C_{13}C_{35}S_{23}S_{24}S_{34}e^{i\delta_{24}} & C_{24}C_{35}S_{34} & S_{35}e^{-i\delta_{35}} \\ \sim(C_{12}S_{13}S_{34}e^{i\delta_{13}}+S_{12}S_{24}e^{i\delta_{24}}) & \sim(-C_{12}S_{24}e^{i\delta_{24}}+S_{23}S_{34}) & \sim(-S_{34}-S_{23}S_{24}e^{i\delta_{24}} -S_{35}S_{45}e^{i\delta_{35}}) & C_{24}C_{34}C_{45} -C_{24}S_{34}S_{35}S_{45}e^{i\delta_{35}} & C_{35}S_{45} \\ \sim(C_{12}S_{13}S_{35}e^{i(\delta_{13}+\delta_{35})}) & \sim(C_{12}S_{24}S_{45}e^{i\delta_{24}}+C_{12}S_{23}S_{35}e^{i\delta_{35}}) & \sim(S_{34}S_{45}-S_{35}e^{i\delta_{35}}) & \sim(-S_{45}-S_{34}S_{35}e^{i\delta_{35}}) & C_{35}C_{45} \end{bmatrix} . \quad (16)$$

(In the elements below the main diagonal we have neglected all terms of the third or higher order in S_{ij}.)

In this paper we have presented a simple way to describe the mixing of quark flavors. A general pattern of the $n\times n$ mixing matrix was discussed. In the case $n=3$ this pattern reduces to the mixing matrix given in Eq. (3). This supports the idea that in the case of three flavors this matrix should be used in analyzing the experimental data, and all other proposals, including the one introduced in Ref. 1 should be abandoned. If nature should provide us with more than three generations of quarks, the matrices given in Eq. (14), Eq. (15), or, in general, in Eq. (10), should be used to describe the weak-interaction mixing. In case of four flavors our parametrization is very similar to the one discussed by Gronau, Johnson, and Schechter,[9] and coincides with the one used by Botella and Ling-Lie Chau[10] in their analysis of *CP* violation.

One of us (H.F.) would like to thank H. Harari for a discussion on the occasion of the DESY workshop in October 1985. After completion of this paper we were informed of a similar work of H. Harari and M. Leurer,[11] and of an interesting article written by Mignani[12] many years ago.

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