Birkhoff's theorem for electromagnetic fields in a new scalar-tensor theory

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A result having formal similarity to Birkhoff's theorem in general relativity is proved in the presence of electromagnetic fields in a new scalar-tensor theory of gravitation proposed by Schmidt *et al.* for the special case when the scalar field is massless and independent of time.

I. INTRODUCTION

This work is a continuation of our recent paper[1] in which it is shown that an analogue of Birkhoff's theorem exists for the special case when the scalar field is massless and independent of time in a new scalar-tensor theory of gravitation proposed by Schmidt *et al.*,[2] where the gravitation constant depends on a scalar field.

In this Brief Report we extend the result[1] from the purely gravitational case to the combined electromagnetic and gravitational case and show that an analogue of Birkhoff's theorem exists for electromagnetic fields in a new scalar-tensor theory proposed by Schmidt *et al.*,[2] for the special case when the mass term and the cosmological constant are each equal to zero.

II. THE BIRKHOFF THEOREM

The Birkhoff theorem guarantees that the outer field of any spherical source is static and it does not depend on its inner structure, but only on its Newtonian mass. This result holds in the presence of electromagnetic fields also.[3,4] The possibility of the theorem being valid in other theories of gravitation have been examined by several authors.[5–16]

The field equations in the scalar-tensor theory proposed by Schmidt *et al.*,[2] when the scalar field is coupled to an electromagnetic field, are given by

$$\left[\gamma-\frac{\beta}{12}\varphi^2\right]G_{ij}=-\tfrac{1}{2}T_{ij}-\tfrac{1}{2}[\varphi_{,i}\varphi_{,j}-\tfrac{1}{2}g_{ij}(\varphi_{,k}\varphi^{,k}-\mu^2\varphi^2)]+\frac{\beta}{12}[(\varphi^2)_{;ij}-g_{ij}(\varphi^2)_{;k}{}^{;k}]\ , \tag{1}$$

$$\Box\varphi+\left[\mu^2+\frac{\beta}{6}R\right]\varphi=0\ , \tag{2}$$

$$F^{ij}{}_{;j}=0,\quad F_{[ij,k]}=0\ . \tag{3}$$

Here μ is the mass of the scalar field, β is an arbitrary coupling constant, $\gamma=c^2/16\pi G$ is half of the inverse gravitational constant.

We consider the spherically symmetric metric in the form

$$ds^2=e^{\nu}dt^2-e^{\lambda}dr^2-r^2(d\theta^2+\sin^2\theta\, d\Phi^2)\ , \tag{4}$$

where $\lambda=\lambda(r,t)$, $\nu=\nu(r,t)$, with the scalar field $\varphi=\varphi(r,t)$. On account of spherical symmetry

$$F_{12}=F_{13}=F_{24}=F_{34}=0,\quad F_{14}\text{ and }F_{23}\neq 0\ . \tag{5}$$

In view of the metric (4) it follows from Eq. (3) that

$$F_{14}=\left[\frac{q}{r^2}\right]e^{(\lambda+\nu)/2}\ , \tag{6}$$

$$F_{23}=m\sin\theta\ , \tag{7}$$

where q and m are arbitrary constants and can be interpreted, respectively, as the charge and the magnetic pole strength of the point source. Now, for the metric (4) the field equations (1) and (2) are

$$\left[\gamma-\frac{\beta}{12}\varphi^2\right]\left[-e^{-\lambda}\left[\frac{\nu'}{r}+\frac{1}{r^2}\right]+\frac{1}{r^2}\right]=-\frac{q^2+m^2}{2r^4}+\tfrac{1}{4}e^{-\nu}\dot{\varphi}^2+\tfrac{1}{4}e^{-\lambda}\varphi'^2-\frac{\mu^2\varphi^2}{4}+\frac{\beta}{12}\left[e^{-\lambda}(\varphi^2)'\left[\frac{\nu'}{2}+\frac{2}{r}\right]+e^{-\nu}(\varphi^2)^{\cdot}\frac{\dot{\nu}}{2}-e^{-\nu}(\varphi^2)^{\cdot\cdot}\right]\ , \tag{8}$$

$$\left[\gamma-\frac{\beta}{12}\varphi^2\right]\left[-e^{-\lambda}\left[\frac{\nu''}{2}-\frac{\lambda'\nu'}{4}+\frac{\nu'^2}{4}+\frac{\nu'-\lambda'}{2r}\right]+e^{-\nu}\left[\frac{\ddot{\lambda}}{2}+\frac{\dot{\lambda}^2}{4}-\frac{\dot{\lambda}\dot{\nu}}{4}\right]\right]$$

$$=-\frac{q^2+m^2}{2r^4}+\tfrac{1}{4}(e^{-\nu}\dot{\varphi}^2-e^{-\lambda}\varphi'^2-\mu^2\varphi^2)$$

$$-\frac{\beta}{12}\left[e^{-\lambda}\left[\frac{(\varphi^2)'}{2}(\lambda'-\nu')-(\varphi^2)''-\frac{(\varphi^2)'}{r}\right]+e^{-\nu}\left[\frac{(\varphi^2)^{\cdot}}{2}(\dot{\lambda}-\dot{\nu})+(\varphi^2)^{\cdot\cdot}\right]\right], \tag{9}$$

$$\left[\gamma-\frac{\beta}{12}\varphi^2\right]\left[e^{-\lambda}\left[\frac{\lambda'}{r}-\frac{1}{r^2}\right]+\frac{1}{r^2}\right]=-\frac{q^2+m^2}{2r^4}-\tfrac{1}{4}(e^{-\nu}\dot{\varphi}^2+e^{-\lambda}\varphi'^2+\mu^2\varphi^2)$$

$$+\frac{\beta}{12}\left[-e^{-\lambda}\left[\frac{(\varphi^2)'\lambda'}{2}(\varphi^2)''-\frac{2(\varphi^2)'}{r}\right]e^{-\nu}\frac{(\varphi^2)^{\cdot\cdot}}{r}\dot{\lambda}\right], \tag{10}$$

$$-e^{-\lambda}\frac{\dot{\lambda}}{r}\left[\gamma-\frac{\beta}{12}\varphi^2\right]=\frac{e^{-\lambda}}{2}\varphi'\dot{\varphi}+\frac{\beta e^{-\lambda}}{12}\left[\frac{(\varphi^2)'\dot{\lambda}}{2}-(\varphi^2)^{\cdot\prime}+(\varphi^2)^{\cdot}\frac{\nu'}{2}\right], \tag{11}$$

$$e^{-\lambda}\left[\frac{(\varphi^2)'}{2}(\lambda'-\nu')-(\varphi^2)''-\frac{2(\varphi^2)'}{r}\right]+e^{-\nu}\left[\frac{(\varphi^2)^{\cdot}}{2}(\dot{\lambda}-\dot{\nu})+(\varphi^2)^{\cdot\cdot}\right]$$

$$=\frac{2\varphi\beta}{12\gamma+\beta(\beta-1)\varphi^2}\left[-\frac{6}{\beta}\mu^2\gamma-\tfrac{1}{2}\mu^2\varphi^2+\frac{1-\beta}{2}e^{-\nu}\dot{\varphi}^2-\frac{1-\beta}{2}e^{-\lambda}\varphi'^2\right], \tag{12}$$

where an overdot denotes a derivative with respect to the coordinate t and a prime denotes a derivative with respect to r. When the massless scalar field φ (i.e., $\mu=0$) is a function of r only, that is,

$$\dot{\varphi}=0\ , \tag{13}$$

then from Eq. (11) we have

$$\dot{\lambda}\left[\gamma-\frac{\beta}{12}\varphi^2+\frac{\beta r}{24}(\varphi^2)'\right]=0 \tag{14}$$

which implies that either

$$\dot{\lambda}=0 \tag{15}$$

or

$$\gamma-\frac{\beta}{12}\varphi^2+\frac{\beta r(\varphi^2)'}{24}=0,\ \text{ i.e., } \varphi^2=\varphi_0 r^2+\frac{12\gamma}{\beta},\quad \varphi_0=\text{const}\ . \tag{16}$$

Now, from Eqs. (8), (10), (13), and (16) it follows that

$$e^{\lambda}=\frac{[r(\lambda'-\nu')-2]\beta\varphi_0}{6(q^2+m^2)-\beta\varphi_0}\ . \tag{17}$$

Using (13) and (16) in Eq. (12) we get

$$\lambda'-\nu'=\frac{r\beta(\beta-1)}{12\gamma\left[\varphi_0 r^2+\frac{12\gamma}{\beta}\right]^{1/2}+\beta(\beta-1)\left[\varphi_0 r^2+\frac{12\gamma}{\beta}\right]^{3/2}}+\frac{6}{r}\ . \tag{18}$$

Substituting this value in Eq. (17) we see that λ is a function of r only. Therefore

$$\dot{\lambda}=0\ .$$

Now differentiation of (18) with respect to t along with the use of Eq. (15) gives

$$\dot{\nu}'=0\ . \tag{19}$$

From this, one has

$$\nu=f(r)+g(t)\ , \tag{20}$$

where f and g are arbitrary functions of r and t, respectively. Then, a simple redefinition of the time coordinate leads ν into a function of r only. This together with (15) reduces the matric (4) to the static case.

Thus we have shown that in the presence of electromagnetic fields in a new scalar-tensor theory proposed by Schmidt *et al.*[2] when the massless scalar field is independent of time, the spherically symmetric gravitational and electromagnetic fields become static. This leads to the conclusion that, possibly, the interaction of the time-dependent scalar field with the electromagnetic field may produce electromagnetic monopole radiation.

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