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Strengths of singularities in Vaidya spacetimes

Krishna Rajagopal and Kayll Lake
Department of Physics, Queen's University at Kingston, Kingston, Ontario, Canada K7L 3N6
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The strengths of shell-focusing singularities in the ingoing Vaidya spacetimes are examined in terms of limiting focusing conditions. The singularities are found to be strong-curvature singularities only for mass functions which are initially linear functions of the advanced time.

In a recent paper Waugh and Lake[1] have given a regular covering of the Vaidya metric for a linear mass function including the cases with a naked shell-focusing singularity. The purpose of this report is to examine the strengths of shell-focusing singularities in the Vaidya metric. We find that strong-curvature singularities, in the sense of Tipler,[2] arise only for mass functions which are initially linear functions of the advanced time.

The ingoing Vaidya spacetime (in standard geometrical units) is given by

$$ds^2=2\,dr\,dv-[1-2m(v)/r]dv^2+r^2d\Omega^2\ , \tag{1}$$

where $d\Omega^2=d\theta^2+\sin^2\theta\, d\phi^2$ and m is a non-negative monotone increasing function of the advanced time v. The metric (1) represents the solution to the Einstein equations for a radial flux of unpolarized radiation with the eikonal energy tensor

$$8\pi T_{\alpha\beta}=\frac{2}{r^2}\frac{dm}{dv}\delta^v_\alpha\delta^v_\beta\ . \tag{2}$$

The affine radial null geodesic equations (with parameter λ) of the "backscattered" test field ($v\neq$const) reduce to[3]

$$\frac{dr}{d\lambda}=\frac{r-2m(v)}{\lambda}\ , \tag{3}$$

with

$$\frac{dv}{d\lambda}=\frac{2r}{\lambda}\ . \tag{4}$$

In this report we consider these geodesics with the (assumed) conditions $d\lambda>0$, $\lambda\geq 0$, and $v=r=m=0$ at $\lambda=0$ (the shell-focusing singularity). Though the coordinates used in the metric (1) do not unfold the shell-focusing singularities,[1] they are adequate and convenient for the examination of their strengths.

Along a null geodesic, affinely parametrized by λ, with 4-tangent l^α let

$$\psi(\lambda)\equiv R_{\alpha\beta}l^\alpha l^\beta\ , \tag{5}$$

where $R_{\alpha\beta}$ is the Ricci tensor and the geodesic terminates at $\lambda=0$. For the purpose of this report we consider the following focusing conditions:[4] the limiting focusing condition (LFC)

$$\lim_{\lambda\to 0}\lambda\psi>0\ , \tag{6}$$

and the strong LFC

$$\lim_{\lambda\to 0}\lambda^2\psi>0\ , \tag{7}$$

which is equivalent to the termination of the geodesic in a strong-curvature singularity in the sense of Tipler.[4,2]

From Eqs. (2) and (5) it follows that $\psi=0$ along all geodesics $v=$const. Along the backscattered test field, however, it follows from Eqs. (2), (4), and (5) that

$$\psi=\frac{8}{\lambda^2}\frac{dm}{dv}\ . \tag{8}$$

As a result, from Eq. (7), it follows that the strong LFC holds as long as $dm/dv\,|_{v=0}(\equiv\mu)>0$, that is

$$m\sim\mu v \tag{9}$$

as $v\to 0$ [$\lim_{v\to 0}(m/v)=\mu$]. It is worth noting that the strong LFC holds for all the geodesics which terminate at the shell-focusing singularity $v=r=0$. The character of this singularity (other than its "strong" nature) depends on the exact form of $m(v)$. (For example, with $m=\mu v$ it is known that the singularity is globally naked for $\mu<\frac{1}{16}$.)[5,6]

Condition (9) is a very strong restriction on $m(v)$. It is of interest, therefore, to examine the (albeit not strong) focusing character of a more general function. Here we examine

$$m\sim\epsilon v^n\ ,\quad n>1 \tag{10}$$

as $v\to 0$ (where $\epsilon=$const$\,>0$) so that from Eq. (8)

$$\psi\sim\frac{8n\epsilon v^{n-1}}{\lambda^2} \tag{11}$$

as $\lambda\to 0$. Equation (11) shows that details of the geodesic history are required [in particular, $v(\lambda)$ as $\lambda\to 0$]. Following standard techniques,[7] one can show that the system of Eqs. (3) and (4) has the regular critical direction

$$v=2r\ . \tag{12}$$

That is, a single null geodesic (the Cauchy horizon) leaves the origin tangent to this direction.[8]

As a result, from Eqs. (4), (11), and (12) we find that

$$\psi\sim\frac{8n\epsilon}{\lambda^{3-n}} \tag{13}$$

as $\lambda\to 0$ along the Cauchy horizon. With the limiting form (10) then the past most point of a shell-focusing singularity satisfies the LFC only for $1<n\leq 2$.

With the above limitation it is of interest to examine the remainder of a shell-focusing singularity. Again, with the aid of standard techniques, it follows from Eqs. (3) and (4) that, with the form (10),

$$\frac{dr}{dv} \sim 0 \tag{14}$$

along the remaining geodesics which reach $v = r = 0$. From Eqs. (3), (4), (11), and (14) then

$$\psi \sim \frac{2}{\epsilon(n-1)\lambda^2(-\ln\lambda)} \tag{15}$$

as $\lambda \to 0$ and so the LFC is satisfied along the remainder of a shell-focusing singularity.

Of course the limiting form (10) need not hold. An example has been given by Lake.[9] For this example, however, we find that ψ grows simply as in (15) but with $n = 1/\epsilon = 2$.

As a final remark, we note that with the limiting form (15) ψ grows faster than $1/\lambda^\beta$ for all $\beta < 2$, and so the remainder of the shell-focusing singularity just fails to be strong in the sense of Tipler.

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[1]B. Waugh and K. Lake, Phys. Rev. D **34**, 2978 (1986).

[2]See, for example, F. J. Tipler, C. J. S. Clarke, and G. F. R. Ellis, in *General Relativity and Gravitation,* edited by A. Held (Plenum, New York, 1980).

[3]B. Waugh and K. Lake, Phys. Lett. **116A**, 154 (1986).

[4]For a detailed discussion of these conditions, see C. J. S. Clarke and A. Krolak, J. Geom. Phys. **2**, 127 (1986). These conditions have been used recently in a study of the Tolman (spherical dust) solutions by R. P. A. C. Newman, Class. Quantum Gravit. **3**, 527 (1986).

[5]See, for example, W. Israel, Can. J. Phys. **64**, 120 (1986).

[6]It has been shown by G. Hollier, Class. Quantum Gravit. **3**, L111 (1986), that with $m = \mu v$, the Cauchy horizon of the shell-focusing singularity satisfies the strong LFC. This result is a special case of the result given here. The present result has been anticipated by Israel (Ref. 5).

[7]See, for example, V. V. Nemytskii and V. V. Stepanov, *Qualitative Theory of Differential Equations* (Princeton University Press, Princeton, NJ, 1960).

[8]From Eq. (12) it follows that the condition $m(v) < v/4$ for all v is a necessary condition for the visibility of the origin from spatial infinity.

[9]K. Lake, Phys. Lett. **116A**, 17 (1986).