Strong-coupling analysis of the critical dimensionality of space-time in SU(2) lattice gauge theory

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Using high-order strong-coupling developments we have analyzed the phase structure of the SU(2) lattice gauge theory at different space-time dimensionalities, considering the action defined through different representations of the SU(2) group. For the Wilson action, the strong-coupling analysis is able to determine unambiguously the five-dimensional first-order phase transition. We have observed also the persistence of this first-order phase transition at higher dimensions. Strong-coupling methods have been shown to be a valuable instrument to study the behavior at high dimensionalities. In respect to the other group representations, phase transitions of first order are clearly detected for all dimensionalities higher than four. In these cases, the most difficult point is the determination of the critical dimensionality above which phase transitions are present. Only for the spin-2 representation action does a first-order phase transition at three dimensions seem to be present.

I. INTRODUCTION

Space-time dimensionality plays an important role in confinement dynamics.[1] Indeed, the phase structure of a lattice gauge theory changes when the space-time dimension varies. In general, one expects, from mean-field arguments, that for a sufficiently large space-time dimensionality all lattice gauge theories will exhibit a first-order phase transition.[2] Monte Carlo simulations are the usual way to study the phase structure of a lattice gauge theory.[3–5] Nevertheless, serious practical problems arise when one tries to simulate the behavior at high dimensions due to the size of the lattice. On the other hand, since the phase transitions at high dimensions are usually of first order, strong-coupling methods,[2] that are known to work better dealing with second-order transitions, seem, in principle, to be not very useful.

In this paper we apply strong-coupling methods to determine the phase structure of the SU(2) lattice gauge theory at different space-time dimensions. We have found that strong-coupling methods are valuable to determine the presence of phase transitions at high dimensions. Padé approximants[6] for both the free energy and the internal energy give very consistent results. The most difficult part of this analysis is the determination of the critical dimensionality of the space-time, i.e., the dimension above which phase transitions are present. The reason is that in the immediate below dimension, a shadow of the phase transition is usually present, giving a singular behavior in the Padé analysis. In such cases, a careful analysis must be performed to determine whether a continuous phase transition is present. In some cases, a Monte Carlo analysis becomes necessary.

The plan of the paper is as follows. In Sec. II the different lattice actions considered are presented and the strong-coupling series generated. Section III contains the results of the analysis of these series, leaving for Sec. IV the study of the Padé extrapolations. Finally Sec. V contains the conclusions of the work.

II. STRONG-COUPLING DEVELOPMENTS

The action that we have considered for the strong-coupling analysis is just

$$S_l(U_p)=\frac{\beta_l}{2l+1}\chi_l(U_p)\ , \tag{1}$$

where $\chi_l(U_p)=\mathrm{tr}_l(U_p)$ represents the trace on the l representation of the SU(2) group, for the product of the matrices belonging to an elementary square of the lattice. In the present paper we have considered the cases

$$l=\tfrac{1}{2},1,\tfrac{3}{2},2\ .$$

The first case corresponds to the pure Wilson action and the second to the SO(3) theory. A summary of the results obtained for the SO(3) case can be found in Ref. 7. The Wilson action[8] has been already studied in four dimensions using different approaches.[9] Nevertheless, its behavior at high dimensionalities by strong-coupling methods has not been determined as yet.

The strong-coupling analysis has been performed in a rather general way. To this purpose we have considered the SU(2) lattice action defined through the sum over all SU(2) group representations

$$S(U_p)=\sum_{l=1/2,1,\dots} S_l(U_p) \tag{2}$$

and we applied the usual character expansion[2] that gives

$$e^{S_l(U_p)}=\sum_{r=0,1/2,1,\dots}\bar{\beta}_{lr}\chi_r(U_p) \tag{3}$$

or, inversely,

$$\bar{\beta}_{lr}=\int DU\,\chi_l(U)e^{S_l(U)}\ , \tag{4}$$

where DU is the Haar measure of the SU(2) group.

The first step in such a computation is the evaluation of the $\bar{\beta}_{lr}$ functions. This has been performed analytically, obtaining the following expressions $[x_r=\beta_r/(2r+1)]$:

TABLE I. Coefficients of the four different series for the free energies. See formula (6). D is the space-time dimensionality.

n	$S_{1/2}$	S_1
2	$\frac{1}{2}$	$\frac{1}{2}$
3	0	$\frac{1}{6}$
4	$-\frac{1}{24}$	0
5	0	$-\frac{1}{30}$
6	$\frac{1}{48}D-\frac{5}{144}$	$\frac{1}{243}D-\frac{43}{1944}$
7	0	$\frac{1}{81}D-\frac{197}{9072}$
8	$-\frac{1}{48}D+\frac{29}{720}$	$\frac{5}{324}D-\frac{133}{5184}$
9	0	$\frac{1}{162}D-\frac{71}{6480}$
10	$\frac{1}{128}D-\frac{49}{2304}D+\frac{1001}{86\,400}$	$\frac{2}{6561}D^2-\frac{115}{11\,664}D+\frac{91\,997}{5\,248\,800}$
11	0	$\frac{4}{2187}D^2-\frac{29}{1296}D+\frac{140\,779}{3\,849\,120}$
12 11	$-\frac{7}{512}D^2+\frac{32\,131}{622\,080}D-\frac{211\,991}{4\,354\,560}$	$\frac{1775}{354\,294}D^2-\frac{101\,981\,413}{3\,542\,940\,000}D+\frac{44\,229\,817}{1\,180\,980\,000}$
13	0	$\frac{901}{118\,098}D^2-\frac{17\,223\,067}{590\,490\,000}D+\frac{670\,147\,489}{23\,882\,040\,000}$
14	$\frac{5}{1024}D^3-\frac{43}{2048}D^2+\frac{5341}{184\,320}D-\frac{264\,497}{20\,321\,280}$	$\frac{20}{531\,441}D^3+\frac{10\,807}{2\,125\,764}D^2-\frac{32\,568\,827}{2\,125\,764\,000}D+\frac{16\,900\,287\,011}{1\,666\,598\,976\,000}$
15	0	$\frac{544}{1\,594\,323}D^3-\frac{8615}{1\,594\,323}D^2+\frac{222\,862\,051}{10\,628\,820\,000}D-\frac{1\,356\,188\,041}{58\,786\,560\,000}$
16	$-\frac{47}{4096}D^3+\frac{7\,030\,933}{106\,168\,320}D^2-\frac{97\,100\,911}{743\,178\,240}D+\frac{1\,474\,972\,157}{16\,721\,510\,400}$	$\frac{2323}{1\,594\,323}D^3-\frac{9\,232\,746\,743}{430\,467\,210\,000}D^2+\frac{95\,695\,307\,951}{1\,377\,495\,072\,000}D-\frac{66\,708\,447\,081\,853}{1\,028\,529\,653\,760\,000}$

$$\bar{\beta}_{1/2r}=(2r+1)I_{2r+1}(2x_{1/2})/x_{1/2}\ , \tag{5a}$$

$$\bar{\beta}_{1r}=\frac{1+(-1)^{2r}}{2}e^{x_1}[I_r(2x_1)-I_{r+1}(2x_1)]\ , \tag{5b}$$

$$\bar{\beta}_{3/2r}=\frac{2r+1}{x_{3/2}}I_{2r+1}(2x_{3/2})I_0(2x_{3/2})+\sum_{m=1}^{\infty}I_m(2x_{3/2})\left[\sum_{k=|r-3m/2|}^{r+3m/2}-\sum_{k=|r-3m/2+1|}^{r+3m/2-1}\right]\frac{2k+1}{x_{3/2}}I_{2k+1}(2x_{3/2})\ , \tag{5c}$$

$$\bar{\beta}_{2r}=\frac{1+(-1)^{2r}}{2}e^{x_2}\sum_{k=0}^{\infty}\left\{I_k(2x_2)\sum_{m=|r-2k|}^{|r+2k|}[I_m(2x_2)-I_{m+1}(2x_2)]-I_{k+1}(2x_2)\sum_{m=|r-2k-1|}^{r+2k+1}[I_m(2x_2)-I_{m+1}(2x_2)]\right\}\ , \tag{5d}$$

where $I_n(z)$ are the modified Bessel functions.[10]

Introducing these expressions in the rather general formulas for the free energy of Refs. 11 and 12, one obtains the strong-coupling series that, up to order 16, are

$$F_l=\sum C_n^l(D)(x_l)^n\ , \tag{6}$$

where the coefficients for the different actions considered, depending on the space-time dimensionality, are shown in Table I.

From the series for the free energy is possible to determine the corresponding series for the internal energy. We have used the following normalization:

$$E_l=1-\frac{1}{2l+1}\frac{d}{dx_l}F_l\ , \tag{7}$$

where $(2l+1)$ is the dimensionality of the corresponding group representation.

III. ANALYSIS OF THE SERIES

We have represented in Figs. 1–4 the behavior of the series for the free energy for the different lattice actions considered, from $d=2$ to $d=7$. In each figure, there are represented the series of order 10–16. The two-dimensional case has been included as a reference to compare the different behaviors since we know that it is trivial; i.e., there are no phase transitions present.

In all the different cases, the series at high dimensions show a very clear singular behavior, all orders diverging at, very approximately, the same point. This behavior contrast with that of the two-dimensional case, where a very smooth behavior is shown. In addition, it can be also observed the fact that this singular point approaches the origin when the space-time dimensionality grows. This fact is expected from general mean-field arguments. The most difficult point is the analysis of the three- and four-dimensional cases, where a quasisingular behavior ap-

TABLE I. (*Continued*).

n	$S_{3/2}$	S_2
2	$\frac{1}{2}$	$\frac{1}{2}$
3	0	$\frac{1}{6}$
4	$\frac{1}{24}$	$\frac{1}{12}$
5	0	$\frac{1}{20}$
6	$\frac{1}{768}D+\frac{17}{5760}$	$\frac{1}{1875}D+\frac{577}{45\,000}$
7	0	$\frac{1}{625}D-\frac{7141}{630\,000}$
8	$\frac{1}{768}D-\frac{29}{5760}$	$\frac{23}{7500}D-\frac{3229}{180\,000}$
9	0	$\frac{3}{625}D-\frac{9059}{540\,000}$
10	$\frac{37}{46\,080}D-\frac{1373}{806\,400}$	$\frac{569}{90\,000}D-\frac{28\,073}{1\,890\,000}$
11	0	$\frac{3163}{450\,000}D-\frac{11\,177\,323}{831\,600\,000}$
12	$\frac{40\,171}{155\,520\,000}D-\frac{122\,611}{272\,160\,000}$	$\frac{161\,701}{24\,300\,000}D-\frac{32\,256\,149}{2\,721\,600\,000}$
13	0	$\frac{5497}{1\,012\,500}D-\frac{343\,659\,461}{35\,380\,800\,000}$
14	$\frac{107\,117}{1\,161\,216\,000}D-\frac{7\,658\,449}{44\,706\,816\,000}$	$\frac{443\,239}{113\,400\,000}D-\frac{7\,750\,753}{1\,058\,400\,000}$
15	0	$\frac{63\,151}{24\,300\,000}D-\frac{169\,779\,613}{32\,659\,200\,000}$
16	$\frac{275\,633}{11\,612\,160\,000}D-\frac{383\,221}{7\,464\,960\,000}$	$\frac{3\,682\,171}{2\,268\,000\,000}D-\frac{24\,797\,723\,581}{7\,185\,024\,000\,000}$

pears. To elucidate these cases, the Padé analysis turns out to be more efficient than a simple direct inspection of the series.

One has to remark that dealing with first-order phase transitions, the point where a strong-coupling series diverges, even the Padé extrapolations, can correspond, actually, to the point where the metastable phase ends, instead of to the true critical point. This is also the origin of the fact that free energy and internal energy analysis show slightly different singular points.

IV. PADÉ ANALYSIS OF THE SERIES

Wishing to extract the maximum possible information from the strong-coupling series that we have generated, the Padé approximants technique has become very useful. Indeed, Padé approximants provide information about the analytic structure of functions that are known only through its series developments. In our case, singularities in the thermodynamical quantities, as the free energy, and the internal energy can point out the presence of phase transitions. It is expected that second-order transitions give a singularity in the internal-energy Padé approximants. By contrast, first-order transitions can be detected as a singular behavior in the free energy, although a reflection of this transition can be manifested also in the other thermodynamical quantities as internal energy or specific heat.

We denote by $[I,J]$ the Padé approximant[6]

$$[I,J]=P^{(I)}(x)/Q^{(J)}(x)\ , \tag{8}$$

where $P^{(I)}(x)$ and $Q^{(J)}(x)$ are polynomials of order I and J in x, in such a way that $I+J\leq N$ where N is the order of the series. One expects that true singularities can be detected as congruences in the apparition of poles of approximants of different orders. In the case of the adjoint and spin-2 representations, we are interested in the diagonal of the Padé table: $[I,I]$ and its nearest ones $[I,I+1],[I+1,I]$. For the fundamental and spin-$\frac{3}{2}$ representations, however, due to the fact that

$$F_{n+1/2}(-x_{n+1/2})=F_{n+1/2}(x_{n+1/2})\ , \tag{9}$$

the independent Padé approximants are only $[2n,2n]$, $[2n+2,2n]$, and $[2n,2n-2]$. In this case, the Padé table will contain less independent approximants.

Dealing with series of order 16 for the free energy and order 15 for the internal energy and considering all the significant Padé approximants, the number of poles detected is high enough to allow for a statistical treatment in order to detect congruences. To this purpose we have represented in Figs. 5–8 the corresponding histograms, one for each dimension, indicating the number of poles found, for each value of the coupling. A first look to these figures leads to the satisfactory fact that the Padé statistical method gives a unique value of the poles. Its average location, with an estimation of the statistical variance, can be found in Tables II and III.

The information that can be deduced from the analysis

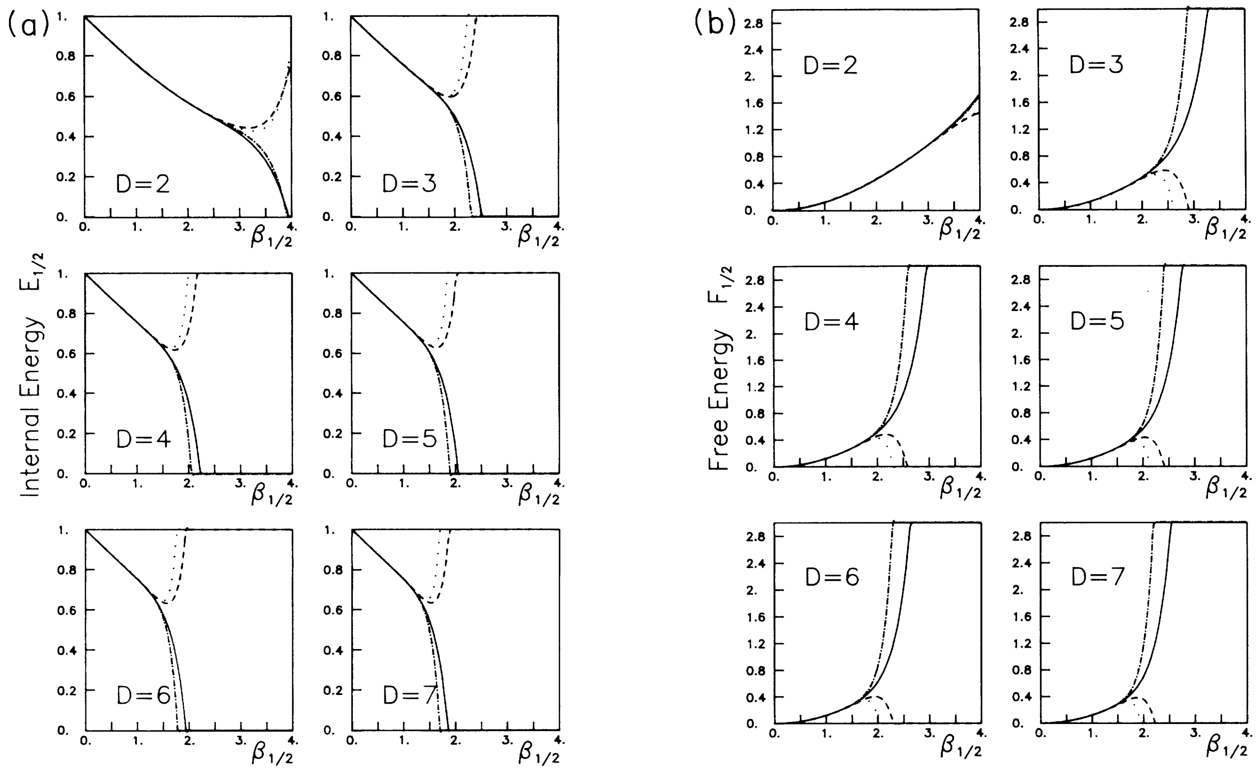

FIG. 1. Strong-coupling results for the (a) internal and (b) free energies at orders (a) 9,11,13,15 and (b) 10,12,14,16, for the fundamental case. The order of the series corresponds to the full, dashed, dashed-dotted, and dotted lines.

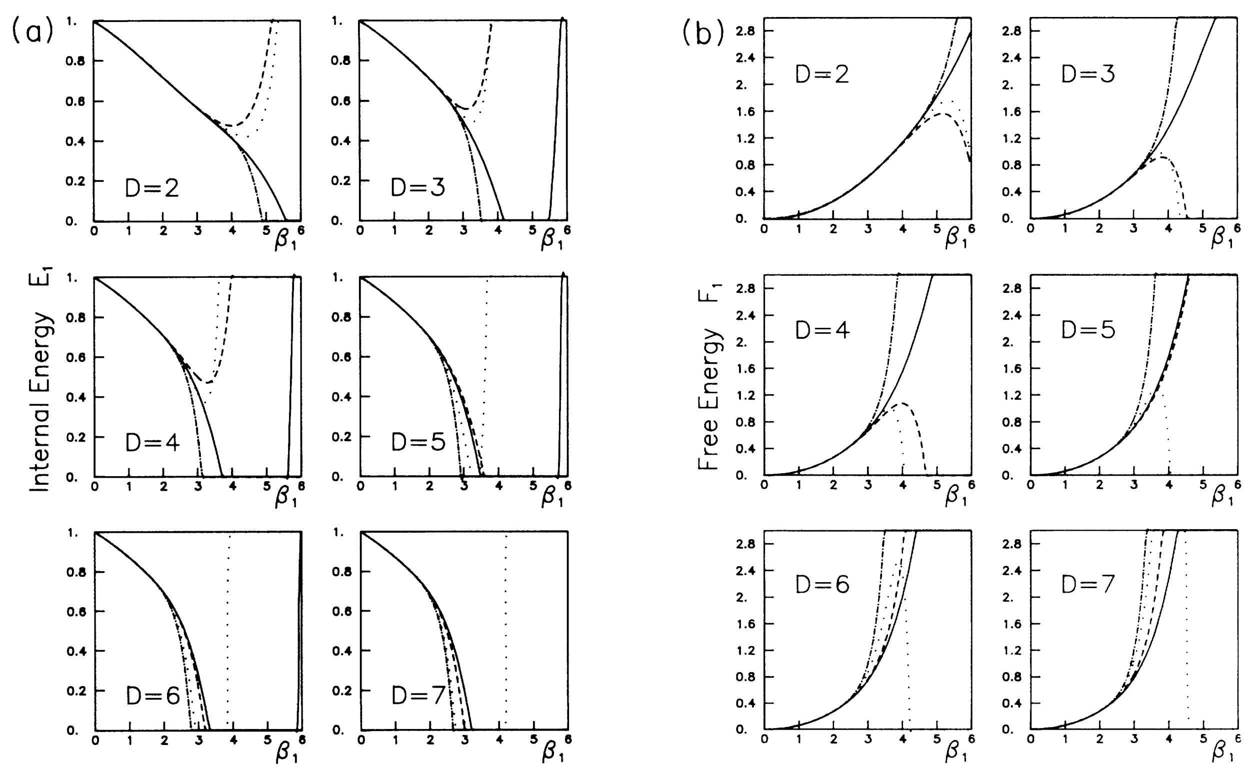

FIG. 2. Strong-coupling results for the adjoint case. The notation is the same as in Fig. 1.

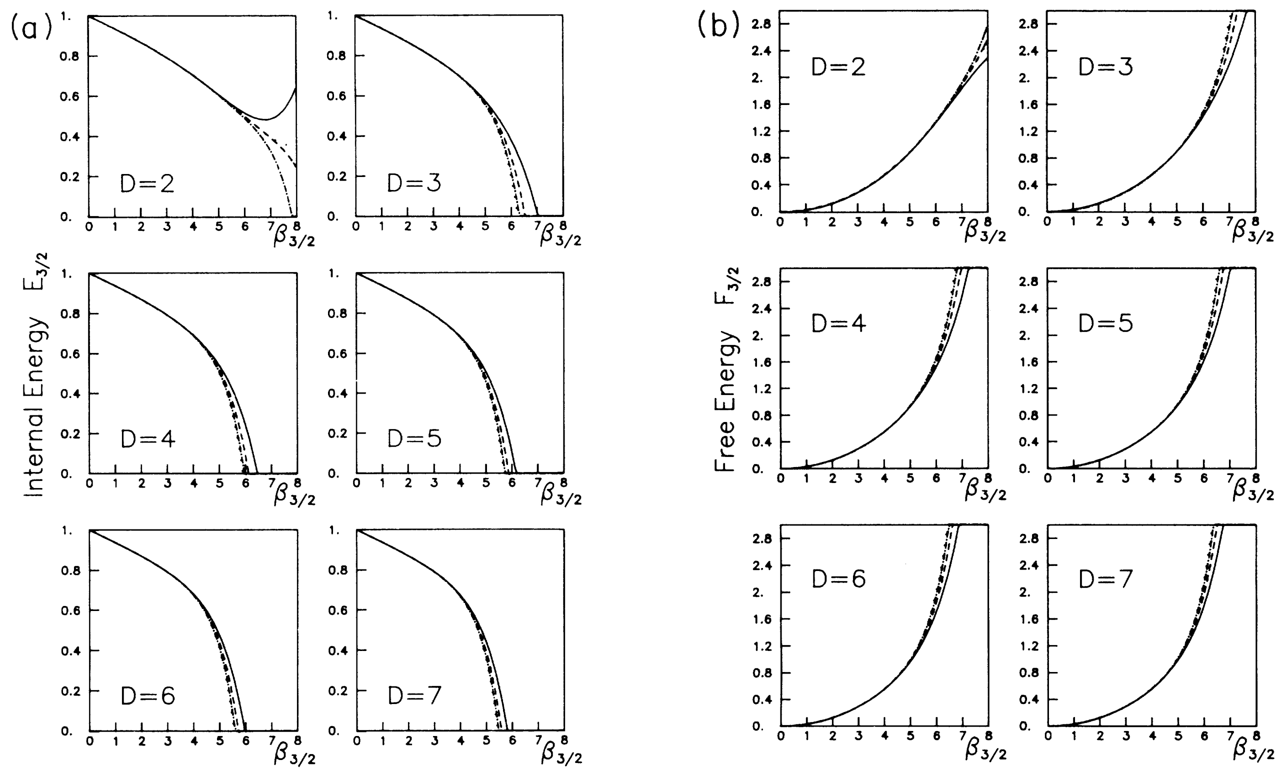

FIG. 3. Strong-coupling results for the spin-$\frac{3}{2}$ case. The notation is the same as in Fig. 1.

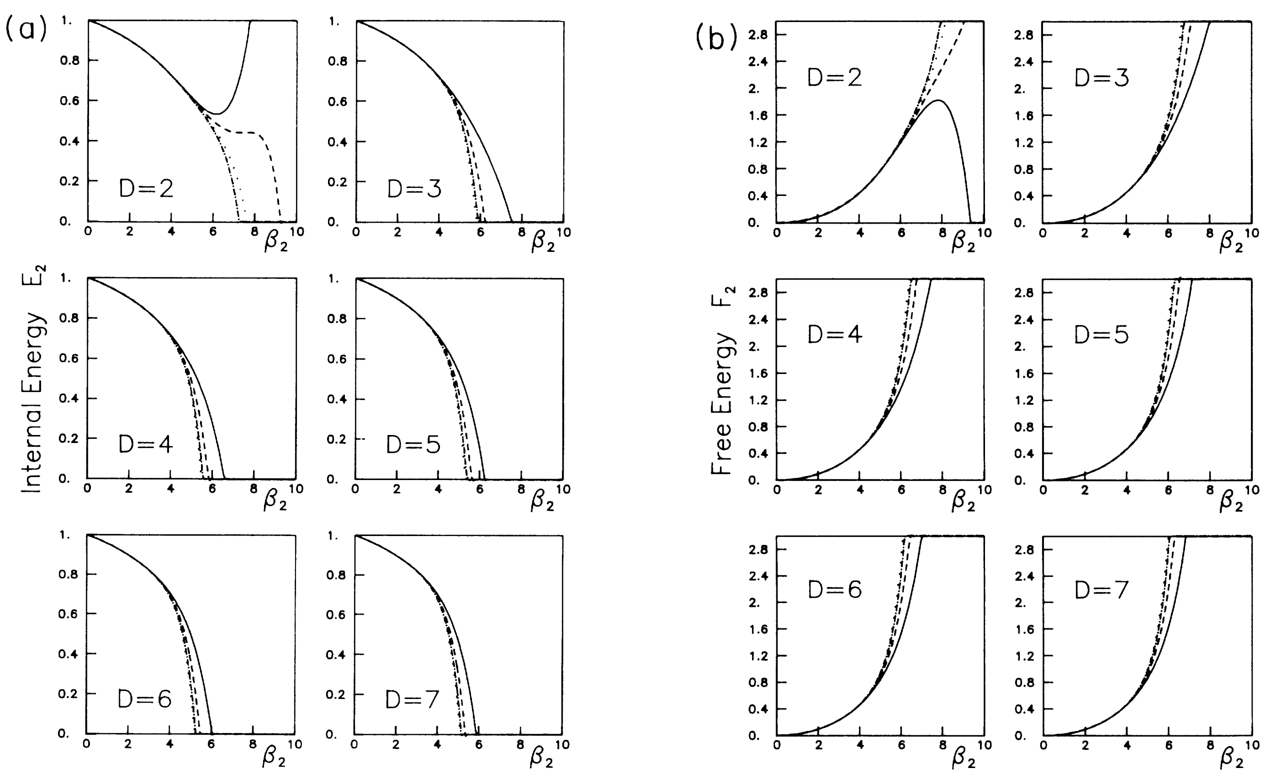

FIG. 4. Strong-coupling results for the spin-2 case. The notation is the same as in Fig. 1.

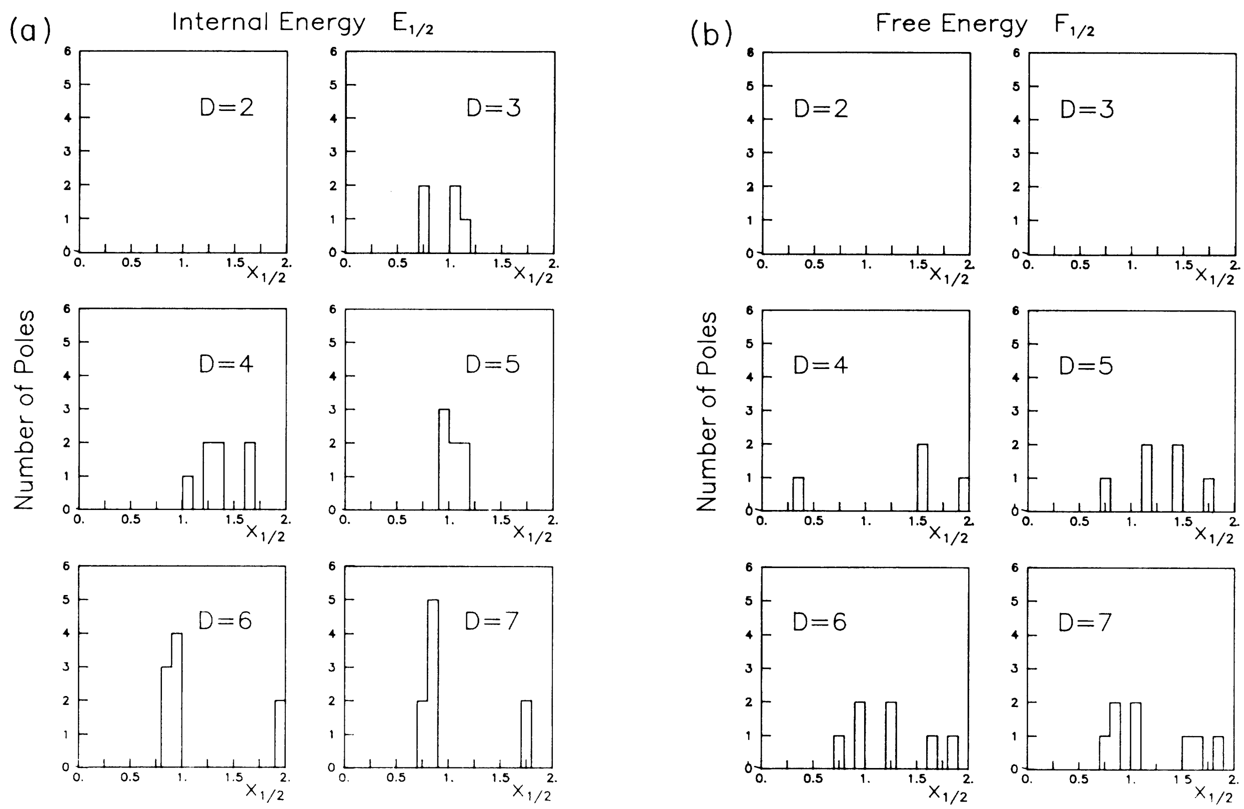

FIG. 5. Number of poles of the Padé approximants for different dimensionalities for the fundamental case.

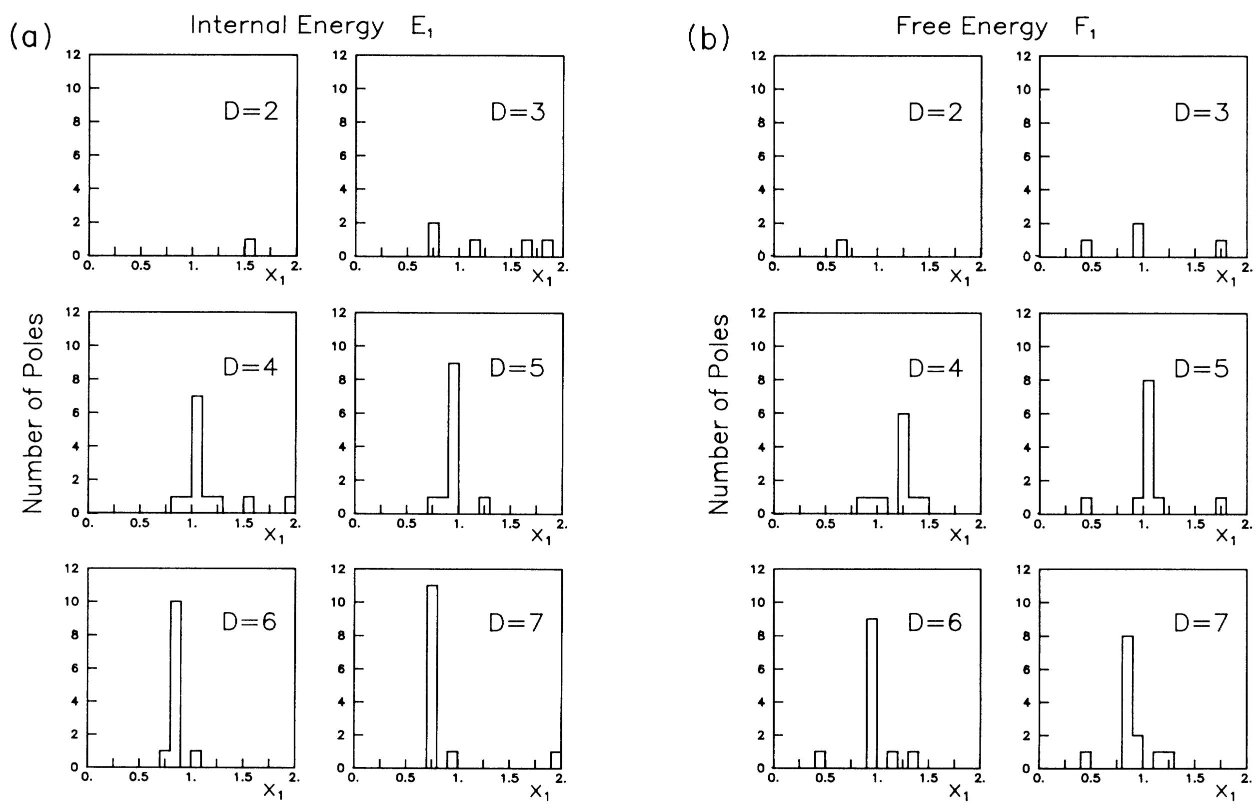

FIG. 6. Number of poles of the Padé approximants for the adjoint case.

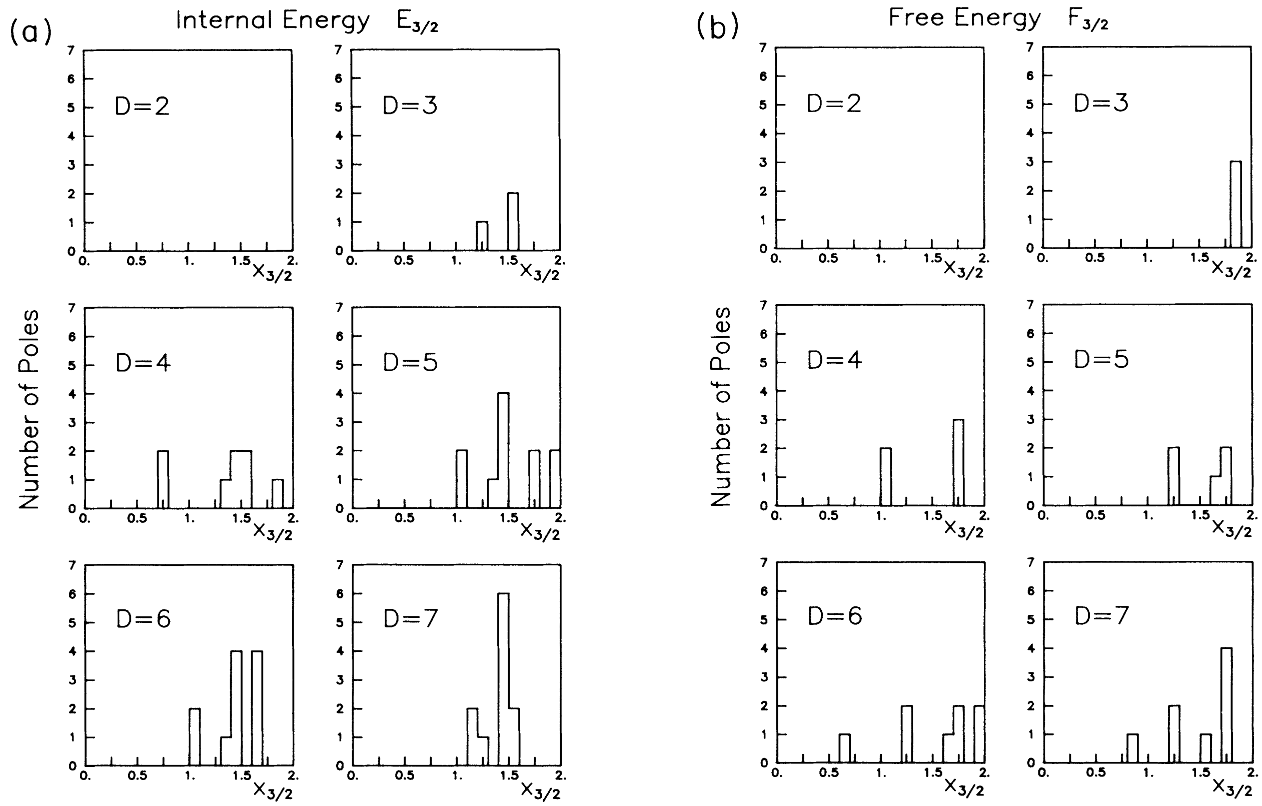

FIG. 7. Number of poles of the Padé approximants for the spin-$\frac{3}{2}$ case.

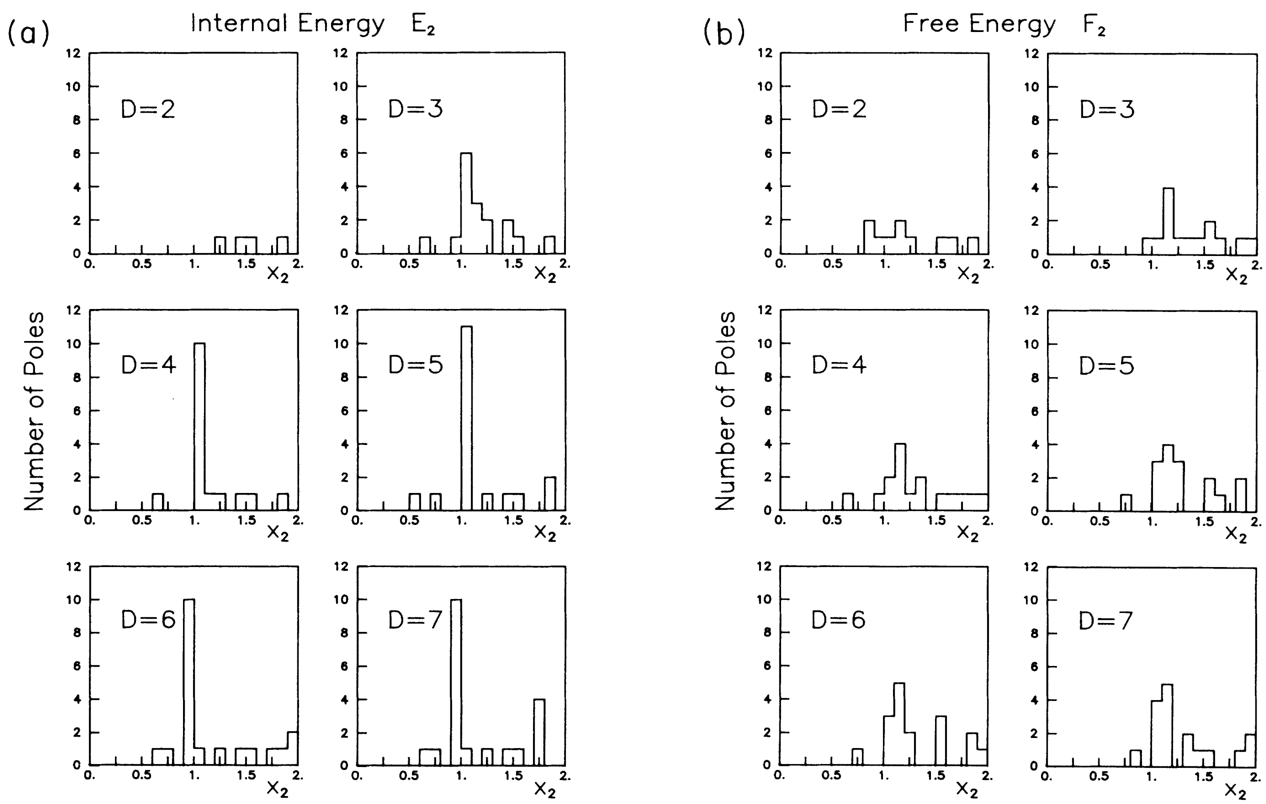

FIG. 8. Number of poles of the Padé approximants for the spin-2 case.

TABLE II. Pondered average of the location of the poles of the Padé approximants for the internal energy.

D	$x_{1/2}$	x_1	$x_{3/2}$	x_2
2				1.23 ± 0.33
3				1.25 ± 0.19
4	1.35 ± 0.60	1.19 ± 0.17	1.47 ± 0.34	1.20 ± 0.16
5	1.19 ± 0.26	1.05 ± 0.04	1.53 ± 0.23	1.18 ± 0.20
6	1.13 ± 0.29	1.01 ± 0.12	1.65 ± 0.27	1.20 ± 0.22
7	1.11 ± 0.33	0.93 ± 0.13	1.49 ± 0.32	1.18 ± 0.18

of the Padé extrapolation results, for all the different cases, can be summarized as follows.

A. Spin-$\frac{1}{2}$ representation. Fundamental action

The analysis of this case is complicated by the fact that there exist only a rather small number of independent approximants. Nevertheless, a clear peak in the Padé histograms for both the free and the internal energies is seen for the space-time dimensionalities from $d=5$ to $d=7$, suggesting, then, a first-order phase transition. The five-dimensional first-order transition was already detected by Monte Carlo simulations.[1]

The two- and the three-dimensional cases look very similar, with no concentration of poles, denoting the absence of phase transitions. The four-dimensional case, finally, is the most difficult to interpret, as it happened in previous strong-coupling analysis of this case.[9] The slightly singular behavior shown by the Padé extrapolated series corresponds to the shadow of the singularity of the five-dimensional case.

As a general characteristic, shown also by the other group representations, the approach to the origin of the singularity when the space-time dimensionality grows is clearly reflected in the values of the averages of Tables II and III. These results show the better convergence of the Padé method to determine singularities at high dimensionalities, since the statistical errors of the pole locations are very small. For instance, the four-dimensional case has a greater statistical error, denoting, then, that it is not a true singularity.

B. Spin-1 representation. SO(3) lattice gauge theory

The statistical analysis of this case is very efficient due to the great number of independent approximants. Figure 6 clearly shows the existence of stable singularities for the space-time dimensions from 4 to 7. The singularity present in the four- and five-dimensional cases has been already determined also by a direct Monte Carlo analysis.[13,14]

The three-dimensional case shows a quasisingular behavior, which may correspond to a singularity. In a previous work[7] we associated this singular behavior with a possible continuous phase transition. However, the comparison with the behavior of the fundamental case suggests that this may correspond also to the shadow of the phase transition present in the next higher dimension. The definitive conclusion is a matter of the Monte Carlo analysis.[15]

As expected, the two-dimensional case shows a total absence of stable poles.

C. Spin-$\frac{3}{2}$ representation

The analysis of this case has the same difficulties as the fundamental action due to the small number of independent poles. Nevertheless, for high dimensionalities, $d=5,6,7$, the presence of a phase transition is clearly shown. The four-dimensional case shows also a singular behavior. This case was already studied in Ref. 16 where the full four-dimensional mixed $(\frac{1}{2},\frac{3}{2})$ action was considered, showing a first-order phase transition for the spin-$\frac{3}{2}$ axis.

The two- and the three-dimensional cases show no phase transitions, although some poles appear at $d=3$, possibly corresponding again to the shadow of the four-dimensional phase transition.

D. Spin-2 representation

The results of the statistical analysis for this case show the most spectacular phase transition determinations.

TABLE III. Pondered average of the location of the poles of the Padé approximants for the free energy.

D	$x_{1/2}$	x_1	$x_{3/2}$	x_2
2				
3	0.95 ± 0.17	0.88 ± 0.19	1.45 ± 0.14	1.19 ± 0.26
4	1.36 ± 0.20	1.09 ± 0.17	1.34 ± 0.36	1.14 ± 0.16
5	1.04 ± 0.08	0.95 ± 0.11	1.51 ± 0.30	1.10 ± 0.18
6	0.91 ± 0.04	0.86 ± 0.06	1.44 ± 0.21	1.04 ± 0.21
7	0.82 ± 0.05	0.77 ± 0.06	1.40 ± 0.14	1.04 ± 0.21

From all dimensionalities from $d=3$ to $d=7$, a clear accumulation of poles is visible for both the free and the internal energies. This result suggests, then, the presence of first-order phase transitions for all these dimensionalities, including the three-dimensional case. To confirm the existence of this phase transition, a Monte Carlo analysis is, therefore, necessary.

V. CONCLUSIONS

In this paper we have performed a high-order strong-coupling analysis of the SU(2) lattice gauge theory with different action representations at space-time dimensionalities, from $d=2$ to $d=7$. Rather general analytical formulas for the strong-coupling developments have been deduced. In addition to the direct series analysis, the Padé approximants have been also computed. The order of the series has allowed for a statistical treatment for the determination of the Padé singularities. From this analysis we have shown the existence of first-order phase transitions from $d=5$ to $d=7$ for all lattice actions considered, fundamental, adjoint, spin-$\frac{3}{2}$ and spin-2. The location of these singularities approached the origin when the space-time dimensionality increased, a type of behavior already expected from general mean-field arguments. One has to remark that the locations detected by the strong-coupling methods may not correspond exactly to the true singularity, since, dealing with first-order transitions, the strong-coupling series show its singular behavior near the end of the metastable phase.

With respect to the determination of the critical dimensionality, i.e., the dimension above which phase transitions are present, we have pointed out the problem of identifying the shadow of the phase transitions of high dimensions at the critical dimensionality. Nevertheless, our analysis has been able to determine the absence of phase transitions for $d=2$, 3, and 4 in the fundamental case, $d=4$ being the critical dimension. For the adjoint case, a first-order transition is present at $d=4$, with $d=3$ being the critical dimensionality. This is also the situation for the spin-$\frac{3}{2}$ action, contrary to the spin-2 action where a phase transition seems to be present also in three dimensions.

The general conclusion that can be deduced from the above analysis is that strong-coupling methods are a valuable instrument to determine the presence of phase transitions, even of first order, in lattice gauge theories. The Padé analysis becomes very efficient in the determination of singularities, especially at high dimensions. With respect to the critical dimensionality determination, strong-coupling methods have the problem of identifying, in the Padé approximants, the shadow of the phase transition present at higher dimensions. Nevertheless, the phase transition picture obtained in the cases analyzed in the present work looks completely consistent.

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[1] M. Creutz, Phys. Rev. Lett. **43**, 553 (1979).
[2] J. M. Drouffe and J. B. Zuber, Phys. Rep. **102**, 1 (1983).
[3] M. Creutz, Phys. Rev. D **21**, 2308 (1980).
[4] C. Rebbi, Phys. Rev. D **21**, 3350 (1980).
[5] M. Creutz, L. Jacobs, and C. Rebbi, Phys. Rep. **95**, 201 (1983).
[6] G. A. Baker, *Essentials of Padé Approximants* (Academic, New York, 1975).
[7] M. Baig, Phys. Lett. **168B**, 267 (1986).
[8] K. Wilson, Phys. Rev. D **10**, 2445 (1974).
[9] M. Falcioni, E. Marinari, M. L. Paciello, G. Parisi, and B. Taglienti, Nucl. Phys. **B170** [FS3], 782 (1981).
[10] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1965).
[11] J. M. Drouffe, Nucl. Phys. **B170**, 91 (1980).
[12] R. Dashen, U. M. Heller, and H. Neuberger, Nucl. Phys. **B215** [FS7], 360 (1980).
[13] J. Greensite and B. Lautrup, Phys. Rev. Lett. **47**, 9 (1981).
[14] M. Baig, Phys. Rev. Lett. **54**, 167 (1985).
[15] M. Baig and A. Cuervo, Report No. Pre-UAB-FT-157, 1986 (unpublished).
[16] C. Ayala and M. Baig, Phys. Rev. D **34**, 1148 (1986).