Quantum inverse problem for an extended derivative nonlinear Schrödinger system

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The technique of the quantum inverse spectral method is applied to an extended version of the derivative nonlinear Schrödinger equation (DNLSE). The simple DNLSE does not permit a quantization in the R-matrix framework of Faddeev *et al.* due to its noncanonical and nonultralocal nature. The extended DNLSE is canonical and we have explicitly obtained the R matrix, the commutation rule for the scattering data, the excitation spectrum, and the integral equation for the eigenvalues. Some comments are also added about the equation satisfied by the scattering matrix.

I. INTRODUCTION

With the advent of the inverse spectral transform there have been various attempts to formulate a quantum-mechanical version of it. Recently, two important and parallel methodologies have been advocated by Thacker, Wilkinson, and co-workers[1] at Fermilab and by Faddeev and his collaborators[2] in the U.S.S.R. That both approaches lead to the same result, at least in the cases of the nonlinear Schrödinger equation (NLSE) and the massive Thirring model[3] (MTM), has been demonstrated. But some restrictions do exist for the applicability of the latter approach which is often referred to as the quantum R matrix method. The first such restriction is that the theory should be canonical and ultralocal. That is, the basic commutation relation must not contain any derivative of δ functions. Also a second and very severe restriction is that the space part of the Lax equation, that is, the L operator ($\psi_x = L\psi$), must contain canonically conjugate variables. The equation in two space-time dimensions which is very similar to the NLSE but differs in that the nonlinear term is the derivative nonlinear Schrödinger equation (DNLSE). Unfortunately the Kaup-Newell spectral problem[4] which is the space part of the Lax pair for the DNLSE leads to a noncanonical symplectic structure for the corresponding Hamiltonian flow. And so the R-matrix approach was not applicable. But here we demonstrate that an extended version[5] of DNLSE can be interpreted as a usual canonical Hamiltonian flow and hence can be quantized following the R-matrix formalism in an elegant fashion. In Sec. II we discuss the equation and its Hamiltonian structure very tersely. In Sec. III the R matrix is obtained. In Sec. IV the commutation of the scattering data is deduced and the diagonalization of the Hamiltonian is performed. Lastly we deduce the integral equations satisfied by the S matrix and eigenvalues of the excitation spectrum.

II. FORMULATION AND BASIC HAMILTONIAN STRUCTURE

The simplest DNLSE is written as

$$iq_t + q_{xx} + \epsilon(|q|^2 q)_x = 0 \ . \tag{1}$$

The isospectral problem associated with this equation is given as

$$\psi_x = \begin{bmatrix} i\Lambda^2 & \Lambda q \\ \Lambda q^* & -i\Lambda^2 \end{bmatrix} \psi \tag{2}$$

which is known as the Kaup-Newell spectral problem in the literature. Then any member of the infinite number of conservation laws can be used as the Hamiltonian. But it is seen that the symplectic structure of (1) is

$$q_t = \frac{\partial}{\partial x} \frac{\delta H}{\delta q^*} \tag{3}$$

which leads to a noncanonical nonultralocal commutation rule and the attempt to quantize such a theory within the methodology of R-matrix formalism becomes a total failure. But recently by starting from a 2×2 quadratic bundle it has been shown by Gerdjikov *et al.*[5] that it is possible to generate a coupled set of generalized DNLSE (which we will refer to as a generalized derivative nonlinear Schrödinger equation).

The quadratic bundle is

$$\bar{L}\psi = \left[i\sigma_3 \frac{d}{dx} + Q_0 + \Lambda Q_1 + r_0 - \Lambda^2 \right]$$

with

$$Q_i = \begin{bmatrix} 0 & q_i \\ p_i & 0 \end{bmatrix}, \quad i = 0,1, \quad r_0 = -\tfrac{1}{2} q_1 p_1 \ . \tag{4}$$

(p_1, q_1), (p_0, q_0) are two sets of nonlinear fields. Then assuming

$$\psi_t = V\psi$$

and V taken to be a fourth-degree polynomial in the spectral parameter Λ it is shown in Ref. 5 that the equations generated are

$$iq_{1t} + q_{1xx} + i\epsilon_1 q_1{}^2 q_{1x}^* + V_1 q_1 - 2V_0 q_0 = 0 \ , \tag{5}$$

$$iq_{0t} + q_{0xx} - i\epsilon_0 q_1{}^2 q_{0x}^* - 2iV_0 q_{1x} + \epsilon |q_1|^2 V_0 q_1 + V_1 q_0 = 0 \ , \tag{6}$$

where

$$p_0=\epsilon_0 q_0^*,\quad p_1=\epsilon q_1^*,\quad \epsilon_0^2=\epsilon_1^2,\quad \epsilon_1^*=\epsilon_1\ ,$$

and

$$\begin{aligned} V_0&=\epsilon_1 q_0 q_1^* +\epsilon_0 q_1 q_0^*\ ,\\ V_1&=\frac{\epsilon_1^2}{2}|q_1|^4-2\epsilon_0|q_0|^2\ . \end{aligned} \tag{7}$$

For $q_1=0$ this system is reduced to the case of usual NLSE and for $q_0=0$ we get

$$iq_{1\epsilon}+q_{1xx}+i\epsilon_1 q_1{}^2 q_x^* +\frac{\epsilon_1^2}{2}|q_1|^4 q_1=0\ . \tag{8}$$

If we now make the change of variables

$$q_i'=q_i e^{-2i\phi},\quad p_i'=p_i e^{2i\phi}$$

with ϕ given as

$$\phi=\frac{\epsilon_1}{2}\int_x^\infty dy\,|q_1|^2\ , \tag{9a}$$

then we get

$$iq_{1t}'+q_{1xx}'+i\epsilon_1(|q_1'|^2 q_1)_x=0\ , \tag{9b}$$

the derivative NLSE. It can then be deduced that the set of Eqs. (5) and (6) can be put into a Hamiltonian form via the Poisson-brackets relations:

$$\begin{aligned} \{\epsilon_0 q_0^*(x),q_1(x')\}&=\delta(x-x')\ ,\\ \{\epsilon_1 q_0(x),q_1^*(x')\}&=-\delta(x-x')\ . \end{aligned} \tag{10}$$

But these structures make a transition to the form

$$\{q_i'^*(x),q_i'(x')\}=\frac{\partial}{\partial x}\delta(x-x') \tag{11}$$

under the reductions noted in (9) and the theory becomes nonultralocal and noncanonical. So we formulate the quantum inverse scattering method (QISM) for the coupled set (5) and (6) rather than for (9b).

III. THE QUANTUM INVERSE PROBLEM

The quantum inverse problem is always formulated by discretizing the space coordinate. To proceed with the calculation we rewrite Eq. (4) as

$$\psi_x=i\begin{bmatrix} r_0-\Lambda^2 & \Lambda q_1+q_0\\ -\Lambda p_1-p_0 & \Lambda^2-r_0 \end{bmatrix}\psi=L\psi\ . \tag{12}$$

If we solve Eq. (12) formally we can write[6]

$$\psi=e^{\int L(x,\Lambda)dx}\psi_0\ , \tag{13}$$

under the assumption of asymptotically nonvanishing nonlinear fields. Because in the formalism that we will be following the system is assumed to be periodic over the strip $-L<x<L$. If this length is divided into n equal subdivisions of length Δ then $n\Delta=2L$ and we consider then an infinitesimal version of (13), written as

$$\psi=\left[1+\int_{x_{n-1}}^{x_n} L(x,\Lambda)\right]\psi_0\ . \tag{14}$$

It is then customary to define the operator at the nth point as

$$L_n(\Lambda)=\left[1+\int_{x_{n-1}}^{x_n} L(x,\Lambda)dx\right]\ . \tag{15}$$

If we define the average over the fields and their products as

$$\begin{aligned} &\int_{x_{n-1}}^{x_n} q_1 dx=\Delta q_{1n},\quad \int_{x_{n-1}}^{x_n} q_0 dx=\Delta q_{0n}\ ,\\ &\int_{x_{n-1}}^{x_n} q_1 q_1^* dx=\Delta q_{1n} q_{1n}^*\ , \end{aligned} \tag{16}$$

then the commutation rules [obtained from the Poisson brackets (10)] become

$$\begin{aligned} [\epsilon_0 q_{0n}^*,q_{1m}]&=\frac{1}{\Delta}\delta_{nm}\ ,\\ [\epsilon_1 q_{0n},q_{1m}^*]&=\frac{-1}{\Delta}\delta_{nm}\ , \end{aligned} \tag{17}$$

and Eq. (15) can be explicitly written as

$$L_n(\Lambda)=i\begin{bmatrix} 1-\frac{1}{2}\epsilon_1\Delta q_{1n}q_{1n}^*-\Lambda^2\Delta & \Lambda\Delta q_{1n}+\Delta q_{0n}\\ -\epsilon\Lambda\Delta q_{1n}^*-\epsilon_0\Delta q_{0n}^* & 1+\frac{1}{2}\epsilon_1\Delta q_{1n}q_{1n}^*+\Lambda^2\Delta \end{bmatrix}\ . \tag{18}$$

up to terms first order in Δ.

We then construct the direct product of the matrices [7]

$$\begin{aligned} L_n'(\Lambda)&=L_n(\Lambda)\otimes 1\ ,\\ L_n''(\Lambda)&=1\otimes L_n(\Lambda)\ , \end{aligned} \tag{19}$$

defined in accordance with the prescription given in (3). Then the most important object to consider is

$$L_n'(\Lambda)\times L_n''(\mu) \tag{20}$$

After an elaborate calculation we arrive at

$$L_n(\Lambda)\otimes L_n(\mu)=\begin{bmatrix} L_{11n}(\Lambda,\mu) & L_{12n}(\Lambda,\mu)\\ L_{21n}(\Lambda,\mu) & L_{22n}(\Lambda,\mu) \end{bmatrix}\ , \tag{21}$$

where each $L_{ijn}(\Lambda,\mu)$ are 2×2 matrices written as

$$L_{11n}(\Lambda,\mu)=\begin{bmatrix}1-\epsilon_1 q_{1n}q_{1n}^*\Delta-(\mu^2+\Lambda^2)\Delta & \mu q_{1n}\Delta+q_{0n}\Delta\\ -\epsilon_1(\mu+\frac{1}{2})q_{1n}^*\Delta-\epsilon_0 q_{0n}^*\Delta & 1+(\mu^2-\Lambda^2)\Delta\end{bmatrix},$$

$$L_{12n}(\Lambda,\mu)=\begin{bmatrix}(\Lambda+\frac{1}{2})q_{1n}\Delta+q_0\Delta & 0\\ (\mu+\Lambda)\Delta & (\Lambda-\frac{1}{2})q_{1n}\Delta+q_{0n}\Delta\end{bmatrix},$$

$$L_{21n}(\Lambda,\mu)=\begin{bmatrix}-\epsilon_1\Lambda q_{1n}^*\Delta-\epsilon_0 q_{0n}^*\Delta & 0\\ 0 & -\epsilon_1\Lambda q_{1n}^*\Delta-\epsilon_0 q_{0n}^*\Delta\end{bmatrix},\tag{22}$$

$$L_{22n}(\Lambda,\mu)=\begin{bmatrix}1-(\mu^2-\Lambda^2)\Delta & \mu q_{1n}\Delta+q_{0n}\Delta\\ -\epsilon_1(\mu-\frac{1}{2})q_{1n}^*\Delta-\epsilon_0 q_{0n}^*\Delta & 1+\epsilon_1 q_{1n}q_{1n}^*\Delta+\Delta(\mu^2+\Lambda^2)\end{bmatrix}.$$

Similar calculation can also be done for $L_n(\mu)\otimes L_n(\Lambda)$. Next an important object of our study is the quantum R matrix which satisfies

$$RL_n(\Lambda)\otimes L_n(\mu)=L_n(\mu)\otimes L_n(\Lambda)R\ .\tag{23}$$

We consider R of the form

$$R=\begin{bmatrix}a&0&0&0\\0&b&c&0\\0&d&e&0\\0&0&0&f\end{bmatrix}.\tag{24}$$

Then Eq. (23) yields the solution

$$R=\begin{bmatrix}1&0&0&0\\0&\alpha&\beta&0\\0&\beta&\alpha&0\\0&0&0&1\end{bmatrix},\tag{25}$$

$$\alpha(\mu,\Lambda)=\frac{2(\mu-\Lambda)}{1+2(\mu-\Lambda)},\qquad \beta(\mu,\Lambda)=\frac{1}{1+2(\mu-\Lambda)}.\tag{26}$$

We now obtain the commutation rules for the scattering data via this R matrix through

$$RT(\Lambda)\otimes T(\mu)=T(\mu)\otimes T(\Lambda)R\ ,\tag{27}$$

where the scattering matrix $T(\Lambda)$ is written as

$$T(\Lambda)=\begin{bmatrix}a(\Lambda) & -\bar{b}(\Lambda)\\ b(\Lambda) & +\bar{a}(\Lambda)\end{bmatrix}.\tag{28}$$

Evaluating (27) by (28) and (25) we arrive at

$$\begin{aligned}
&a(\Lambda)a(\mu)=a(\mu)a(\Lambda)\ ,\\
&\bar{b}(\Lambda)\bar{b}(\mu)=\bar{b}(\mu)\bar{b}(\Lambda)\ ,\\
&\bar{a}(\Lambda)\bar{a}(\mu)=\bar{a}(\mu)\bar{a}(\Lambda)\ ,\\
&a(\Lambda)\bar{b}(\mu)=\alpha(\mu,\Lambda)\bar{b}(\mu)a(\Lambda)+\beta(\mu,\Lambda)a(\mu)\bar{b}(\Lambda)\ ,\\
&b(\mu)a(\Lambda)=\alpha(\mu,\Lambda)a(\Lambda)b(\mu)+\beta(\mu,\Lambda)b(\Lambda)a(\mu)\ .\\
&\bar{a}(\mu)\bar{b}(\Lambda)=\alpha(\mu,\Lambda)\bar{b}(\Lambda)\bar{a}(\mu)+\beta(\mu,\Lambda)\bar{a}(\Lambda)\bar{b}(\mu)\ ,\\
&b(\Lambda)\bar{a}(\mu)=\alpha(\mu,\Lambda)\bar{a}(\mu)b(\Lambda)+\beta(\mu,\Lambda)\bar{a}(\mu)b(\Lambda)\ .
\end{aligned}\tag{29}$$

IV. CONSTRUCTION OF THE EIGENSTATES

The eigenstates of the quantized system can be constructed by starting with a postulated vacuum. Let us say that we designate the vacuum by

$$q_{1n}^*\,|\,0\rangle_x=q_{0n}^*\,|\,0\rangle_x=0\ .\tag{30}$$

Then let us now observe how the $L_n(\Lambda)$ operator operates on the vacuum so defined. At the nth lattice sites we have

$$L_n(\Lambda)\,|\,0\rangle_n=i\begin{bmatrix}1-\Lambda^2\Delta & \Lambda\Delta q_{1n}+\Delta q_{0n}\\ 0 & 1+\Lambda^2\Delta\end{bmatrix}|\,0\rangle\ .\tag{31}$$

Let us now consider the effect of two consecutive L operators on $|\,0\rangle_n$. It is

$$L_{n+1}(\Lambda)L_n(\Lambda)\,|\,0\rangle_n'=(i)^2\begin{bmatrix}1-2\Lambda^2\Delta & A(q_{1n},q_{1n+1})\\ 0 & 1+2\Lambda^2\Delta\end{bmatrix}|\,0\rangle_n'\ ,\tag{32}$$

where A is some polynomial $q_{1n}q_{1n+1}$. This property of L_n when operating on the local vacuums is of utmost importance. Because as we proceed from one end of the lattice to the other, which is really a circle due to periodicity condition assumed, the product $L_n(\Lambda)L_{n+1}(\Lambda)\cdots L_N(\Lambda)$ is nothing but the scattering matrix $T(\Lambda)$. Furthermore due to Eq. (32) the product remains triangular. Hence, since

$$T(\Lambda)=\prod_{n=1}L_n(\Lambda)=\begin{bmatrix}a(\Lambda) & -\bar{b}(\Lambda)\\ b(\Lambda) & +\bar{a}(\Lambda)\end{bmatrix},\tag{33}$$

we can infer the effect of $a,b,\bar{a},\bar{b}$ on $|\,0\rangle$. Then in the limit of large N such that $N\Delta=L'$ is fixed where L' is the length of the lattice, the diagonal elements of (32) lead to the eigenvalues $e^{-\Lambda^2 L'}$ and $e^{\Lambda^2 L'}$ via the limit

$$\lim_{N\to\infty}\left[1-\frac{\Lambda^2 L'}{N}\right]^N = e^{-\Lambda^2 L'} .$$

So finally we write

$$a(\Lambda)\,|\,0\rangle = e^{-\Lambda^2 L'}\,|\,0\rangle ,$$
$$\bar{a}(\Lambda)\,|\,0\rangle = +e^{+\Lambda^2 L'}\,|\,0\rangle , \tag{34}$$
$$b(\Lambda)\,|\,0\rangle = 0 ,$$

and $\bar{b}(\Lambda)$ will create states from a vacuum. Let us now consider a series of physical states of the form

$$\Omega_1(\Lambda_1)=\bar{b}(\Lambda_1)\,|\,0\rangle ,$$
$$\Omega_2(\Lambda_1,\Lambda_2)=\bar{b}(\Lambda_1)\bar{b}(\Lambda_2)\,|\,0\rangle , \tag{35}$$
$$\Omega_3(\Lambda_1,\Lambda_2,\Lambda_3)=\bar{b}(\Lambda_1)\bar{b}(\Lambda_2)\bar{b}(\Lambda_3)\,|\,0\rangle ,$$

and so on. The important job is to ascertain the eigenmomenta and the eigenenergies of Ω_1, Ω_2, Ω_3. For that we operate with $a(\mu)$ and $\bar{a}(\mu)$ on Ω_1, Ω_2, Ω_3 and utilize the commutation rules to shift $a(\mu)$ so as to operate on $|\,0\rangle$ to get (we illustrate here the case of two-particle states)

$$a(\mu)\Omega_2(\Lambda_1,\Lambda_2)=\frac{e^{-\mu^2 L'}}{\alpha(\mu,\Lambda_1)\alpha(\mu,\Lambda_2)}\Omega_2(\Lambda_1,\Lambda_2)-\frac{\beta(\mu,\Lambda_2)}{\alpha(\mu,\Lambda_1)\alpha(\mu,\Lambda_2)}e^{-\Lambda_2{}^2 L'}\Omega_2(\Lambda_1,\mu)$$
$$-\frac{\beta(\mu,\Lambda_1)}{\alpha(\mu,\Lambda_1)\alpha(\Lambda_1,\Lambda_2)}e^{-\Lambda_1{}^2 L'}\Omega_2(\mu,\Lambda_2)+\frac{\beta(\mu,\Lambda_1)\beta(\Lambda_1,\Lambda_2)}{\alpha(\mu,\Lambda_1)\alpha(\Lambda_1,\Lambda_2)}e^{-\Lambda_2{}^2 L'}\Omega_2(\mu,\Lambda_1) . \tag{36}$$

Similarly,

$$\bar{a}(\mu)\Omega_2(\Lambda_1,\Lambda_2)=\frac{+e^{u^2 L'}}{\alpha(\Lambda_1,\mu)\alpha(\Lambda_2,\mu)}\Omega_2(\Lambda_1,\Lambda_2)-\frac{\beta(\Lambda_2,\mu)}{\alpha(\Lambda_1,\mu)\alpha(\Lambda_2,\mu)}e^{\Lambda_2{}^2 L'}\Omega_2(\Lambda,\mu)$$
$$-\frac{\beta(\Lambda_1,\mu)}{\alpha(\Lambda_1,\mu)\alpha(\Lambda_2,\Lambda_1)}e^{\Lambda_1{}^2 L'}\Omega_2(\mu\Lambda_2)+\frac{\beta(\Lambda_1\mu)\beta(\Lambda_2\Lambda_1)}{\alpha(\Lambda_1\mu)\alpha(\Lambda_2\Lambda_1)}e^{\Lambda_2{}^2 L'}\Omega_2(\mu\Lambda_1) . \tag{37}$$

Now $\mathrm{Tr}T(\Lambda)$ is nothing but the Hamiltonian so we demand $\Omega_2(\Lambda_1\Lambda_2)$ to be an eigenstate of $a(\mu)+\bar{a}(\mu)$ which leads to the following equations, determining the eigenmomenta and energy eigenvalue of the two-particle state:

$$[a(\mu)+\bar{a}(\mu)]\Omega_2(\Lambda_1\Lambda_2)=\left[\frac{e^{-\mu^2 L'}}{\alpha(\mu,\Lambda_1)\alpha(\mu,\Lambda_2)}+\frac{e^{\mu^2 L'}}{\alpha(\Lambda_1\mu)\alpha(\Lambda_2\mu)}\right]\Omega_2(\Lambda_1\Lambda_2) \tag{38}$$

along with the condition

$$e^{2\Lambda_1{}^2 L'}=\frac{1+2(\Lambda_1-\Lambda_2)}{-1+2(\Lambda_1-\Lambda_2)} . \tag{39}$$

In general for the n particle configuration we have the condition

$$e^{2\Lambda_i{}^2 L'}=\prod_{j\neq i}\left[\frac{1+2(\Lambda_i-\Lambda_j)}{-1+2(\Lambda_i-\Lambda_j)}\right] \tag{40}$$

which is nothing but the algebraic Bethe ansatz.[8] In practice it is really very difficult to analyze an equation of the form (40) so we now make a transition to the usual form of a Fredholm-type integral equation from (40). Let us rewrite (40) as

$$e^{2\Lambda_i{}^2 L'}=\prod_{j\neq i} e^{\Delta(\Lambda_i-\Lambda_j)}$$

with

$$\Delta(\Lambda_i-\Lambda_j)=\ln\left[\frac{1+2(\Lambda_i-\Lambda_j)}{-1+2(\Lambda_i-\Lambda_j)}\right] .$$

Taking the logarithm of both sides we get

$$2\Lambda_i{}^2 L'=\sum_{j\neq i}\Delta(\Lambda_i-\Lambda_j)+2\pi n_i , \tag{41}$$

where n_i are integers. Let us consider Eq. (41) with i replaced by $i+1$:

$$2\Lambda_{i+1}{}^2 L'=\sum_{j\neq i}\Delta(\Lambda_{i+1}-\Lambda_j)+2\pi n_{i+1} . \tag{42}$$

Subtracting (41) from (42) we obtain

$$\Lambda_{i+1}{}^2-\Lambda_i{}^2=\frac{1}{2L'}\left[\sum_{j\neq 1}\Delta(\Lambda_{i+1}-\Lambda_j)-\Delta(\Lambda_i-\Lambda_j)\right]$$
$$+\frac{2\pi}{2L'}(n_{i+1}-n_i) . \tag{43}$$

As $L'\to\infty$, the Λ_i's become infinitesimally spaced so we define

$$\rho(\Lambda_i)=\frac{1}{L'(\Lambda_{i+1}-\Lambda_i)}$$

which is supposed to reach a finite limit in the continuous limit. So we deduce from (43)

$$\pi\rho(\Lambda)+\tfrac{1}{2}\int_{-\Lambda_F}^{\Lambda_F}K(\Lambda-\Lambda')\rho(\Lambda')d\Lambda'=2\Lambda , \tag{44}$$

where

$$K(\Lambda)=\frac{d\Delta}{d\Lambda}=\frac{4}{1-4\Lambda^2}\ , \tag{45}$$

where Λ_F is determined by the condition of particle density written as

$$\int_{-\Lambda_F}^{\Lambda_F}\rho(K)dK=\frac{N}{L}\ . \tag{46}$$

Equation (44) is of utmost importance for discussing the thermodynamic spectrum of the model which is outside the scope of the present communication. Lastly we may add without derivation some comments about the scattering matrix between two pseudoparticles. Suppose in the state of the system represented by Eq. (41) we introduce an extra particle with momentum Λ' which alters the momenta of the ith excitation from Λ_i to $\bar{\Lambda}_i$, then we may write

$$2\bar{\Lambda}_i{}^2L'=\sum_{i\neq j}\Delta(\bar{\Lambda}_i-\bar{\Lambda}_j)+2\pi\bar{\eta}_i+\Delta(\bar{\Lambda}_i-\Lambda')\ . \tag{47}$$

If we now define as before

$$w(\Lambda_i)=(\bar{\Lambda}_i'-\Lambda_i)L,\quad F(\Lambda_i)=\rho(\Lambda_i)w(\Lambda_i)\ ,$$

then proceeding as before we can deduce

$$2\pi F(\Lambda)=-\int d\Lambda' F(\Lambda')\frac{\partial}{\partial\Lambda}\Delta(\Lambda-\Lambda')+\Delta(\bar{\Lambda}'-\Lambda)\ . \tag{48}$$

V. QUANTUM INTEGRALS OF MOTION AND TRACE IDENTITIES

In the previous section we have used the trace of $T(\Lambda)$ as the Hamiltonian for the nonlinear system under consideration and have constructed the eigenstates. The justification of taking the trace $T(\Lambda)$ as the Hamiltonian lies in the derivation of the quantum trace identities following Ref. 9 which will yield the quantum integrals of motion and hence also the Hamiltonian proper.

To proceed with the derivation we start from the discrete transfer operator $L_n(\Lambda)$ written as

$$L_n(\Lambda)=i\begin{pmatrix}1-\frac{1}{2}\epsilon_1\Delta q_{1n}q_{1n}^*-\Lambda^2\Delta & \Lambda\Delta q_{1n}+\Delta q_{0n}\\ -\epsilon_1\Lambda\Delta q_{1n}^*-\epsilon_0\Delta q_{0n}^* & 1+\frac{1}{2}\epsilon_1\Delta q_{1n}q_{1n}^*+\Lambda^2\Delta\end{pmatrix}\ . \tag{49}$$

The monodromy matrix $T_L(\Lambda)$ for the interval $[-L,L]$ is defined as

$$T_L(\Lambda)=L_N(\Lambda)\cdots L_1(\Lambda)=\begin{pmatrix}A(\Lambda) & B(\Lambda)\\ C(\Lambda) & D(\Lambda)\end{pmatrix}\ . \tag{50}$$

By the standard procedure in the quantum inverse scattering method one can show that the generating functional $\tau=(A+D)$ (the transfer matrix) of the integrals of the motion have the property

$$[\tau(\Lambda),\tau(\mu)]=0\ . \tag{51}$$

The spectrum of the operator $\tau(\Lambda)$ is calculated in the limit $\Delta\to 0$, $N\to\infty$, $\Delta N=L=$const. We shall show, in this continuous limit, that the local integrals of the motion can be obtained from the expansion of $\ln\tau(\Lambda)$ in reciprocal powers of Λ as $\Lambda'\to\infty$:

$$\ln[\tau(\Lambda)\exp(\Lambda^2L)]=\sum_{\substack{k=1\\ \Lambda\to\infty}}^{\infty}C_K\Lambda^{-K}\ . \tag{52}$$

[The factor $\exp(\Lambda^2L)$ corresponds to ordinary canceling of the plane wave.] To express the coefficients $C_1,C_2,C_3,\ldots$ in the limit $\Delta\to 0$, $L\to\infty$ in terms of the quantum particle number, the quantum momentum, and the quantum Hamiltonian using the fields.

We transform the infinitesimal monodromy matrix L_n by means of a gauge transformation U_n, requiring the transformed monodromy matrix $\tilde{L}_n$ to be diagonal:

$$\tilde{L}_n=U_n{}^{-1}L_nU_{n-1}\ , \tag{53a}$$

where the 2×2 matrix U_n is in the form of a formal asymptotic series,

$$U_n=I+\sum_{K=1}^{\infty}\Lambda^{-K}Q_n^{(K)}\ , \tag{53b}$$

where

$$Q_n^{(K)}=\begin{pmatrix}0 & \beta_n^{(K)}\\ \gamma_n^{(k)} & 0\end{pmatrix}\ .$$

Although the matrix elements U_n are operators, the inverse matrix $U_n{}^{-1}$ can be obtained by formal inversion of the series, since it begins with the unit matrix I.

The condition of diagonality of the matrix $\tilde{L}_n$ leads to the following equations for β and γ:

$$\beta_n^K+\beta_{n-1}^K=B_n^K,\quad \gamma_n^K+\gamma_{n-1}^K=G_n^K\ . \tag{54}$$

The quantities B_n^K and G_n^K can be expressed in terms of $\beta_m^{(l)}$ and $\gamma_m^{(l)}$ with $l<K$, respectively, and the fields. We give here only the expressions for $G^{(1,2,3,4,5)}$, for which we require

$$\begin{aligned}
&G_n^{(1)}=\epsilon_1 q_{1n}^*,\quad G_n^{(2)}=\epsilon_0 q_{0n}^*\ ,\\
&G_n^{(3)}=-\tfrac{1}{2}\epsilon_1{}^2 q_{1n}^* q_{1n} q_{1n}^* +\gamma_n^1 q_{1n}\gamma_{n-1}^1+\frac{1}{\Delta}(\gamma_n^1-\gamma_{n-1}^1)\ ,\\
&G_n^{(4)}=-\tfrac{1}{2}\epsilon_0\epsilon_1 q_{0n}^* q_{1n} q_{1n}^* +\gamma_n^1 q_{0n}\gamma_{n-1}^1+\frac{1}{\Delta}(\gamma_n^2-\gamma_{n-1}^2)\ ,\\
&G_n^5=\tfrac{1}{2}\epsilon_1{}^3 q_{1n}^* q_{1n} q_{1n}^* q_{1n} q_{1n}^* -\frac{\epsilon_1}{2}\gamma_n^1 q_{1n}\gamma_{n-1}^1 q_{1n} q_{1n}^* +\gamma_n^2 q_{1n}\gamma_{n-1}^2-\frac{1}{\Delta}\left[\frac{\epsilon_1}{2}(\gamma_n'-\gamma_{n-1}')q_{1n}q_{1n}^*-(\gamma_n^3-\gamma_{n-1}^3)\right],\\
&G_n^{(6)}=\tfrac{1}{2}\epsilon_0\epsilon_1{}^2 q_{0n}^* q_{1n} q_{1b}^* q_{1n} q_{1n}^* -\frac{\epsilon_1}{2}\gamma_n^1 q_{0n}\gamma_{n-1}^1 q_{1n} q_{1n}^* +\gamma_n^2 q_{0n}\gamma_{n-1}^2-\frac{1}{\Delta}\left[\frac{\epsilon_1}{2}(\gamma_n^2-\gamma_{n-1}^2)q_{1n}q_{1n}^*-(\gamma_n^4-\gamma_{n-1}^4)\right].
\end{aligned}\tag{55}$$

On β and γ we impose the natural boundary conditions

$$\gamma_N^K=0,\quad \beta_0^K=0\quad \text{for all } K\ . \tag{56}$$

This choice leads to a normally ordered form of β and γ when Eqs. (54) are solved and the solution for γ is

$$\gamma_n^{(K)}=z(-1)^{n-1}\sum_{l=n+1}^{N}(-1)^l G_l^{(K)}\ . \tag{57}$$

Using (55) we obtain, from this,

$$\begin{aligned}
&\gamma_n^1=(-1)^{n+1}\epsilon_1\sum_{l=n+1}^{N}(-1)^l q_{1l}^*,\quad \gamma_n^2=(-1)^{n+1}\epsilon_0\sum_{l=n+1}^{N}(-1)^l q_{0l}^*\ ,\\
&\gamma_n^3=(-1)^{n+1}\sum_{l=n+1}^{N}(-1)^l\left[-\tfrac{1}{2}\epsilon_1{}^2 q_{1l}^* q_{1l} q_{1l}^*+\gamma_l^1 q_{1l}\gamma_{l-1}^1+\frac{1}{\Delta}(\tau_l^1-\gamma_{l-1}^1)\right],\\
&\gamma_n^4=(-1)^{n+1}\sum_{l=n+1}^{N}(-1)^l\left[-\tfrac{1}{2}\epsilon_0\epsilon_1 q_{0l}^* q_{1l} q_{1l}^*+\gamma_l^1 q_{0l}\gamma_{l-1}^1+\frac{1}{\Delta}(\gamma_l^2-\gamma_{l-1}^2)\right],\\
&\gamma_n^5=(-1)^{n+1}\sum_{l=n+1}^{N}(-1)^l\left[-\tfrac{1}{2}\epsilon_1{}^3 q_{1l}^* q_{1l} q_{1l}^* q_{1l} q_{1l}^*-\frac{\epsilon_1}{2}\gamma_l^1 q_{1l}\gamma_{l-1}^1 q_{1l} q_{1l}^*\right.\\
&\qquad\left.+\gamma_l^2 q_{1l}\gamma_{l-1}^2-\frac{\epsilon_1}{2\Delta}(\gamma_l^1-\gamma_{l-1}^1)q_{1l}q_{1l}^*+\frac{1}{\Delta}(\gamma_l^3-\gamma_{l-1}^3)\right],\\
&\gamma_n^6=(-1)^{n+1}\sum_{l=n+1}^{N}(-1)^l[G_l{}^6]\ .
\end{aligned}\tag{58}$$

It is easy to express the elements of the matrices $\widetilde{L}_n$ in terms of β and γ, i.e.,

$$\begin{aligned}
(\widetilde{L}_n)_{11}&=i\left[1-\tfrac{1}{2}\epsilon_1\Delta q_{1n}q_{1n}^*-\Lambda^2\Delta+\Delta q_{1n}\sum_{K=1}^{\infty}\Lambda^{-K+1}\gamma_{n-1}^K+\Delta q_{0n}\sum_{K=1}^{\infty}\Lambda^{-K}\gamma_{n-1}^K\right]\\
&\simeq e^{-\Lambda^2\Delta}\left[1-\tfrac{1}{2}\epsilon_1\Delta q_{1n}q_{1n}^*+\Delta q_{1n}\sum_{K=1}^{\infty}\Lambda^{-K+1}\gamma_{n-1}^K+\Delta q_{0n}\sum_{K=1}^{\infty}\Lambda^{-K}\gamma_{n-1}^K\right].
\end{aligned}\tag{59}$$

We have written down the last equation with allowance for $\Delta^2\to 0$. Similarly,

$$(\widetilde{L}_n)_{22}\simeq e^{\Lambda^2\Delta}\left[1+\tfrac{1}{2}\epsilon_1 q_{1n}q_{1n}^*-\Delta\epsilon_1 q_{1n}^*\sum_{K=1}^{\infty}\Lambda^{-k+1}\beta_{n-1}^K-\Delta\epsilon_0 q_{0n}^*\sum_{K=1}^{\infty}\Lambda^{-K}\beta_{n-1}^K\right]. \tag{60}$$

We next go back to the derivation of (52). It follows from (53a) that

$$\begin{aligned}
&T_L(\Lambda)=U_N(\Lambda)\widetilde{T}_L(\Lambda)U_0{}^{-1}(\Lambda)\ ,\\
&\widetilde{T}_L(\Lambda)=\widetilde{L}_N(\Lambda)\cdots\widetilde{L}_1(\Lambda)=\begin{bmatrix}\widetilde{A}(\Lambda) & 0\\ 0 & \widetilde{D}(\Lambda)\end{bmatrix}.
\end{aligned}\tag{61}$$

Using (59) and (60), we obtain, accurate up to order of Δ^2,

$$\widetilde{A}(\Lambda)=e^{-\Lambda^2 L}\Bigg[1-\tfrac{1}{2}\epsilon_1\Delta\sum_{n=1}^{N} q_{1n}q_{1n}^{*}+\Delta\sum_{n=1}^{N} q_{1n}\sum_{K=1}^{\infty}\Lambda^{-K+1}\gamma_{n-1}^{K}+\Delta\sum_{n=1}^{N} q_{0n}\sum_{K=1}^{\infty}\Lambda^{-K}\gamma_{n-1}^{K}$$
$$+\tfrac{1}{4}\epsilon_1{}^2\Delta^2\sum_{\substack{n_i n_j\\ i<j}}\sum q_{1n_i}q_{1n_i}^{*}q_{1n_j}q_{1n_j}^{*}-\tfrac{1}{2}\epsilon_1\Delta^2\sum_{\substack{n_i n_j\\ i<j}}\sum q_{1n_i}q_{1n_i}^{*}q_{1n_j}\sum_{K=1}^{\infty}\Lambda^{-K+1}\gamma_{n_j-1}^{K}$$
$$-\tfrac{1}{2}\epsilon_1\Delta^2\sum_{\substack{n_i n_j\\ i<j}}\sum q_{1n}q_{1n_i}^{*}q_{0n_i}^{*}q_{0n_j}\sum_{K=1}^{\infty}\Lambda^{-K}\gamma_{n_j-1}^{K}-\tfrac{1}{2}\epsilon_1\Delta^2\sum_{\substack{n_i n_j\\ i<j}}\sum q_{1n_i}\sum_{K=1}^{\infty}\Lambda^{-K+1}\gamma_{n_i-1}^{K}q_{1n_j}q_{1n_j}^{*}$$
$$-\tfrac{1}{2}\epsilon_1\Delta^2\sum_{\substack{n_i n_j\\ i<j}}\sum q_{0n_i}\sum_{K=1}^{\infty}\Lambda^{-K}\gamma_{n_i-1}^{K}q_{1n_j}q_{1n_j}^{*}+\Delta^2\sum_{\substack{n_i n_j\\ i<j}}\sum q_{1n_i}\sum_{K=1}^{\infty}\Lambda^{-2K+2}\gamma_{n_i-1}^{K}q_{1n_j}\gamma_{n_j-1}^{K}$$
$$+\Delta^2\sum_{\substack{n_i n_j\\ i<j}}\sum q_{1n_i}\sum_{K=1}^{\infty}\Lambda^{-2K+1}\gamma_{n_i-1}^{K}q_{0n_j}\gamma_{n_j-1}^{K}+\Delta^2\sum_{\substack{n_i n_j\\ i<j}}\sum q_{0n_i}\sum_{K=1}^{\infty}\Lambda^{-2K+1}\gamma_{n_i-1}^{K}q_{1n_j}\gamma_{n_j-1}^{K}$$
$$+\Delta^2\sum_{\substack{n_i n_j\\ i<j}}\sum q_{0n_i}\sum_{K=1}^{\infty}\Lambda^{-2K}\gamma_{n_i-1}^{K}q_{0n_j}\gamma_{n_j-1}^{K}\Bigg]\,. \tag{62}$$

We obtain a similar expression for $\widetilde{D}(\Lambda)$ but, in the limit $\Lambda\rightarrow\infty$, $\widetilde{D}(\Lambda)$ is exponentially small. In expression (61) for $T_L(\Lambda)$ in terms of β and γ we ignore the boundary terms, which correspond to the transition $L\rightarrow\infty$, and we note that conditions (56) lead to the equation $\widetilde{A}(\Lambda)=A(\Lambda)$. Then, as $L\rightarrow\infty$, we obtain

$$\ln[\exp(\Lambda^2 L)\tau(\Lambda)]\underset{\Lambda\rightarrow\infty}{\sim}\ln[\exp(\Lambda^2 L)\widetilde{A}(\Lambda)]\,. \tag{63}$$

The expansion in inverse powers of Λ for $\exp(\Lambda^2 L)\widetilde{A}(\Lambda)$ is obtained from (59) and (61):

$$\exp(\Lambda^2 L)\widetilde{A}(\Lambda)=1+\sum_{l=1}^{\infty}\Lambda^{-1}a_l\,, \tag{64}$$

where the first five terms of the expansion are

$$a_1=\Delta\sum_n q_{1n}\gamma_{n-1}^2+\Delta\sum_n q_{0n}\gamma_{n-1}^1-\tfrac{1}{2}\epsilon_1\Delta^2\sum_{\substack{n_i n_j\\ i<j}}\sum q_{1n_i}q_{1n_i}^{*}q_{1n_j}\gamma_{n_j-1}^2$$
$$-\tfrac{1}{2}\epsilon_1\Delta^2\sum_{\substack{n_i n_j\\ i<j}}\sum q_{1n_i}q_{1n_i}^{*}q_{0n_j}\gamma_{n_j-1}^1-\tfrac{1}{2}\epsilon_1\Delta^2\sum_{\substack{n_i n_j\\ i<j}}\sum q_{1n_i}\gamma_{n_i-1}^2 q_{1n_j}q_{1n_j}^{*}$$
$$-\tfrac{1}{2}\epsilon_1\Delta^2\sum_{\substack{n_i n_j\\ i<j}}\sum q_{0n_i}\gamma_{n_i-1}^1 q_{1n_j}q_{1n_j}^{*}+\Delta^2\sum_{\substack{n_i n_j\\ i<j}}\sum q_{1n_i}\gamma_{n_i-1}^1 q_{0n_j}\gamma_{n_j-1}^1+\Delta^2\sum_{\substack{n_i n_j\\ i<j}}\sum q_{0n_i}\gamma_{n_i-1}^1 q_{1n_j}\gamma_{n_j-1}^1\,,$$

$$a_2=\Delta\sum_n q_{1n}\gamma_{n-1}^3+\Delta\sum_n q_{0n}\gamma_{n-1}^2-\tfrac{1}{2}\epsilon_1\Delta^2\sum_{\substack{n_i n_j\\ i<j}}\sum q_{1n_i}q_{1n_i}^{*}q_{1n_j}\gamma_{n_j-1}^3$$
$$-\tfrac{1}{2}\epsilon_1\Delta^2\sum_{\substack{n_i n_j\\ i<j}}\sum q_{1n_i}q_{1n_i}^{*}q_{0n_j}\gamma_{n_j-1}^2-\tfrac{1}{2}\epsilon_1\Delta^2\sum_{\substack{n_i n_j\\ i<j}}\sum q_{1n_i}\gamma_{n_i-1}^3 q_{1n_j}q_{1n_j}^{*}$$
$$-\tfrac{1}{2}\epsilon_1\Delta^2\sum_{\substack{n_i n_j\\ i<j}}\sum q_{0n_i}\gamma_{n_i-1}^2 q_{1n_j}q_{1n_j}^{*}+\Delta^2\sum_{\substack{n_i n_j\\ i<j}}\sum q_{1n_i}\gamma_{n_i-1}^2 q_{1n_j}\gamma_{n_j-1}^2+\Delta^2\sum_{\substack{n_i n_j\\ i<j}}\sum q_{0n_i}\gamma_{n_i+1}^1 q_{0n_j}\gamma_{n_j-1}^1\,,$$

$$a_3=\Delta\sum_n q_{1n}\gamma_{n-1}^4+\Delta\sum_n q_{0n}\gamma_{n-1}^3-\tfrac{1}{2}\epsilon_1\Delta^2\sum_{\substack{n_i n_j\\ i<j}}\sum q_{1n_i}q_{1n_i}^{*}q_{1n_j}\gamma_{n_j-1}^4$$
$$-\tfrac{1}{2}\epsilon_1\Delta^2\sum_{\substack{n_i n_j\\ i<j}}\sum q_{1n_i}q_{1n_j}^{*}q_{0n_j}\gamma_{n_j-1}^3-\tfrac{1}{2}\epsilon_1\Delta^2\sum_{\substack{n_i n_j\\ i<j}}\sum q_{1n_i}\gamma_{n_i-1}^4 q_{1n_j}q_{1n_j}^{*}$$
$$-\tfrac{1}{2}\epsilon_1\Delta^2\sum_{\substack{n_i n_j\\ i<j}}\sum q_{0n_i}\gamma_{n_i-1}^3 q_{1n_j}q_{1n_j}^{*}+\Delta^2\sum_{\substack{n_i n_j\\ i<j}}\sum q_{1n_i}\gamma_{n_i-1}^2 q_{0n_j}\gamma_{n_j-1}^2+\Delta^2\sum_{\substack{n_i n_j\\ i<j}}\sum q_{0n_i}\gamma_{n_i-1}^2 q_{1n_j}\gamma_{n_j-1}^2\,, \tag{65}$$

$$a_4=\Delta\sum_n q_{1n}\gamma_{n-1}^5+\Delta\sum_n q_{0n}\gamma_{n-1}^4-\tfrac{1}{2}\epsilon_1\Delta^2\sum_{\substack{n_in_j\\ i<j}}\sum q_{1n_i}q_{1n_i}^*q_{1n_j}\gamma_{n_j-1}^5$$
$$-\tfrac{1}{2}\epsilon_1\Delta^2\sum_{\substack{n_in_j\\ i<j}}\sum q_{1n_i}q_{1n_i}^*q_{0n_j}\gamma_{n_j-1}^4-\tfrac{1}{2}\epsilon_1\Delta^2\sum_{\substack{n_in_j\\ i<j}}\sum q_{1n_i}\gamma_{n_i-1}^5q_{1n_j}q_{1n_j}^*$$
$$-\tfrac{1}{2}\epsilon_1\Delta^2\sum_{\substack{n_in_j\\ i<j}}\sum q_{0n_i}\gamma_{n_i-1}^4q_{1n_j}q_{1n_j}^*+\Delta^2\sum_{\substack{n_in_j\\ i<j}}\sum q_{1n_i}\gamma_{n_i-1}^3q_{1n_j}\gamma_{n_j-1}^3+\Delta^2\sum_{\substack{n_in_j\\ i<j}}\sum q_{0n_i}\gamma_{n_i-1}^2q_{0n_j}\gamma_{n_j-1}^2\,,$$
$$a_5=\Delta\sum_n q_{1n}\gamma_{n-1}^6+\Delta\sum_n q_{0n}\gamma_{n-1}^5-\tfrac{1}{2}\epsilon_1\Delta^2\sum_{\substack{n_in_j\\ i<j}}\sum q_{1n_i}q_{1n_i}^*q_{1n_j}\gamma_{n_j-1}^6$$
$$-\tfrac{1}{2}\epsilon_1\Delta^2\sum_{\substack{n_in_j\\ i<j}}\sum q_{1n_i}q_{1n_i}^*q_{0n_j}\gamma_{n_j-1}^5-\tfrac{1}{2}\epsilon_1\Delta^2\sum_{\substack{n_in_j\\ i<j}}\sum q_{1n_i}\gamma_{n_i-1}^6q_{1n_j}q_{1n_j}^*$$
$$-\tfrac{1}{2}\epsilon_1\Delta^2\sum_{\substack{n_in_j\\ i<j}}\sum q_{0n_i}\gamma_{n_i-1}^5q_{1n_j}q_{1n_j}^*+\Delta^2\sum_{\substack{n_in_j\\ i<j}}\sum q_{1n_i}\gamma_{n_i-1}^3q_{0n_j}\gamma_{n_j-1}^3+\Delta^2\sum_{\substack{n_in_j\\ i<j}}\sum q_{0n_i}\gamma_{n_i-1}^3q_{1n_j}\gamma_{n_j-1}^3\,.$$

Taking the logarithm of (64), the first five coefficients of the expansion (52) are seen to be

$$\begin{aligned}&c_1=a_1,\quad c_2=a_2-\frac{a_1^2}{2},\quad c_3=a_3-a_1a_2+\tfrac{1}{3}a_1^3\,,\\ &c_4=a_4-a_1a_3+a_1^2a_1-\tfrac{1}{2}a_2^2-\tfrac{1}{4}a_1^4\,,\\ &c_5=a_5-a_1a_4+a_1^2a_3-6a_1^3a_2-a_2a_3+a_2^2a_1+\tfrac{1}{5}a_1^5\,.\end{aligned}\tag{66}$$

Now to make a transition to the continuous limit we consider as before $\Delta\to0$, $N\to\infty$, $\Delta N=L=$const, which then yields that the above c_i's are the renormalized versions of the classical integrals of motion:

$$\begin{aligned}&D^{(1)}=\frac{i}{2}\int_{-\infty}^{\infty}dx(\epsilon_1q_0q_1^*+\epsilon_0q_1q_0^*)\,,\\ &D^{(2)}+\frac{i}{2}\int_{-\infty}^{\infty}dx\left[\epsilon_0q_0q_0^*+\tfrac{1}{4}\epsilon_1^2q_1^2q_1^{*2}+i\frac{\epsilon_1}{4}(q_1^*q_{1x}-q_1q_{1x}^*)\right]\,,\\ &D^{(3)}=-\tfrac{1}{4}\int_{-\infty}^{\infty}dx(\epsilon_0q_0^*q_{1x}-\epsilon_1q_0q_{1x}^*)\,,\\ &D^{(4)}=\frac{i}{8}\int_{-\infty}^{\infty}dx\left[\epsilon_1q_{1x}^*q_{1x}+i\epsilon_0(q_0^*q_{0x}-q_0q_{0x}^*)+(\epsilon_1q_1^*q_0+\epsilon_0q_0^*q_1)^2-\frac{i}{4}\epsilon_1^2q_1q_1^*(q_1^*q_{1x}-q_1q_{1x}^*)\right]\,,\\ &H=\tfrac{1}{2}\int_{-\infty}^{\infty}dx\left[-(\epsilon_0q_{1x}q_{0x}^*+\epsilon_1q_{1x}^*q_{0x})+\frac{3i\epsilon_1}{5}|q_1^2|(\epsilon_0q_{1x}q_0^*-\epsilon_1q_0q_{1x}^*)+V_0V_1^5\right]\,,\end{aligned}\tag{67}$$

where

$$V_0=\epsilon_1q_0q_1^*+\epsilon_0q_1q_0^*\,,$$
$$V_1=\frac{\epsilon_1^2}{2}|q_1^4|-2\epsilon_0|q_0^2|\,.$$

VI. EXPLICIT FORM OF EXCITATION SPECTRUM

Similar to Eqs. (38) and (39) we observe that the eigenvalue $E(\Lambda_j\Lambda_j)$ corresponding to the multipseudoparticle states is given by

$$E(\Lambda_i\Lambda_j)=e^{-\Lambda_i^2L'}\left[2\prod_{i\neq j}\frac{1}{\alpha(\Lambda_i\Lambda_j)}\right]\,,\tag{68}$$

where

$$\alpha(\Lambda_i\Lambda_j)=\frac{2(\Lambda_i-\Lambda_j)}{1+2(\Lambda_i-\Lambda_j)}\,.$$

To ascertain the nature of the eigenmomenta and energy, we consider the $\ln[e^{\Lambda_i^2L'}E(\Lambda_i\Lambda_j)]$ and expand in inverse powers of Λ where the coefficient of Λ^{-4} gives us the eigenmomenta and the coefficient of Λ^{-5} gives us the eigenenergy. Thus, for a one-particle state,

$$\begin{aligned}&P_1=\frac{\Lambda_1^3}{2}-\tfrac{3}{8}\Lambda_1^2+\tfrac{1}{8}\Lambda_1\,,\\ &E_1=\frac{\Lambda_1^4}{2}-\frac{\Lambda_1^3}{2}+\frac{3\Lambda_1^2}{8}-\frac{1}{160}\,.\end{aligned}\tag{69}$$

For a two-particle state

$$P_2=\tfrac{1}{2}(\Lambda_1^3+\Lambda_2^3)-\tfrac{3}{8}(\Lambda_1^2+\Lambda_2^2)+\tfrac{1}{8}(\Lambda_1+\Lambda_2)\ ,$$

$$E_2=\tfrac{1}{2}(\Lambda_1^4+\Lambda_2^4)-\tfrac{1}{2}(\Lambda_1^3+\Lambda_2^3)+\tfrac{3}{8}(\Lambda_1^2+\Lambda_2^2)-\tfrac{1}{80}\ . \tag{70}$$

VII. GROUND-STATE ENERGY

From Eq. (44) we observe that if $\rho(K)$ is the density of states then the number of particle is given by

$$\int_{-K_F}^{K_F}\rho(K)dK=\frac{N}{L}\ , \tag{71}$$

where the ground-state energy is given as

$$E_0=\frac{1}{L}\int_{-K_F}^{K_F}K^4\rho(K)dK\ . \tag{72}$$

It is interesting to note that our Eq. (44) is similar to that of a usual nonlinear Schrödinger equation except that the inhomogeneous term is proportional to Λ rather than a constant. The explicit solution is rendered difficult by the finite limits of integration in (72). Also there may be some divergence difficulty due to the first-order lattice (terms only up to Δ) approximation of the continuous model. We hope to formulate the exact lattice equivalent model in a future communication and then discuss the numerical solution of the integral equation for the Fermi momenta K_F and the density of states $\rho(K)$. Only in that situation is it possible to get an actual expression for $\rho(K)$.

VIII. DISCUSSIONS

In our above computation we have presented the QISM for the extended version of DNLSE. The original DNLSE being nonultralocal, it cannot be treated in the usual formalism and still no method exists to quantize it. The analysis of Eqs. (44) and (48) for the study of string configuration is under way and will be the subject matter of the next communication. At this point it is interesting to note that there is no way to compare the reproduced spectrum with the actual excitations in the derivative nonlinear Schrödinger system because even up till now there is no method for quantizing nonlocal theories.

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