Non-Gaussian multiphoton squeezed states

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The centered distribution functions and their moments are constructed for the generalized multiphoton squeezed states recently defined by the authors. It is shown that such states are non-Gaussian, and exhibit higher-order squeezing, depending on the choice of either the squeezing parameter or the generalized boson number.

INTRODUCTION

The concept of squeezing of a quantum field has recently been given a great deal of interest, in view both of the possible reduction of noise of quantum origin in systems such as linear or nonlinear amplifiers, and of the possible enhancement of the efficiency of the information content in quantum communication channels.[1]

Indeed a squeezed state is characterized by fluctuations in one of the state variables smaller than those in a coherent state (of course at the cost of increased fluctuations in the conjugate variables), so that it is purely quantum mechanical and has no classical counterpart. The quadratic deviation, which gives a measure of the field fluctuations, has customarily been assumed as the optimal indicator of squeezing, but recently Hong and Mandel[2] have generalized the concept of squeezing to higher-(even) order moments of the quantum field.

However, there seems to be an inherent difficulty in producing states which exhibit a sensible amount of squeezing of higher order while showing the correct dynamical behavior, and the construction of the corresponding Hamiltonians is far from trivial.[3]

In Ref. 4 we introduced a new set of generalized multiboson states, which were further extended in Ref. 5. The aim of this paper is to perform a thorough discussion of such multiboson states, showing that they are not Gaussian, and allow squeezing of different orders for different choices of the squeezing parameter.

Such states are interesting in physical applications because they allow a detailed analysis of the quantum noise problem, leading to unexpected bonds and relations among the physical observables (see Ref. 5 for a discussion of one of such relations, which appears to be especially relevant in the applications in that it connects the maximum amount of squeezing with the number of bosons). Moreover they evolve coherently under the action of both the harmonic-oscillator Hamiltonian and more complicated Hamiltonians of the form discussed in Ref. 3, which are simple in terms of generalized Bose operators, but extremely complex as functions of the canonical variables.

Under the latter point of view they generalize to a wide class of self-adjoint Hamiltonians, which are entire functions in the Heisenberg operators, the coherent-state representation of quantum-mechanical propagators with polynomial Hamiltonians discussed by Daubechies and Klauder.[6]

In Sec. I the definition of multiboson squeezed states is briefly reviewed. Section II is devoted to the general setting of the scheme whereby the corresponding centered distribution functions and their moments can be computed. In Sec. III explicit compact formulas are given for the distribution moments of any order. Section IV presents a detailed numerical analysis of these formulas as well as a brief discussion of their physical meaning. Some of the analytical details are given in the Appendix. Our final section gives a few conclusive comments.

I. MANY-PHOTON SQUEEZED STATES

In Ref. 1 a new type of multiphoton squeezing operator was constructed resorting to the generalized Bose operators of Brandt and Greenberg:[7] $b_{(k)}$ and $b_{(k)}^{\dagger}$.

The latter satisfy the commutation relations

$$[b_{(k)}, b_{(k)}^{\dagger}] = 1 , \qquad (1.1)$$

$$[N, b_{(k)}] = -k b_{(k)} , \qquad (1.2)$$

where $N = a^{\dagger} a$ is the usual number operator.

Equations (1.1) and (1.2) lead to interpreting $b_{(k)}$ and $b_{(k)}^{\dagger}$ as annihilation and creation operators of k photons simultaneously, even though it should be noted that $b_{(1)} = a$, but $b_{(k)} \neq a^k$, for $k \geq 2$.

From (1.1) and (1.2) one can derive the normal-ordered representation

$$b_{(k)} = \sum_{j=0}^{\infty} \alpha_j^{(k)} (a^{\dagger})^j a^{j+k} , \qquad (1.3)$$

where

$$\alpha_j^{(k)} = \sum_{l=0}^{j} \frac{(-)^{j-l}}{(j-l)!} \left[\frac{1+[[l/k]]}{l!(l+k)!} \right]^{1/2} e^{i\phi_l} . \qquad (1.4)$$

In (1.4) $[[x]]$ denotes the maximum integer $\leq x$, whereas the phases ϕ_m, $m=0,\ldots,j$ are arbitrary real numbers.

In the Fock space $b_{(k)}$ and $b_{(k)}^\dagger$ operate as

$$b_{(k)}|n\rangle=([[n/k]])^{1/2}|n-k\rangle, \quad [[n/k]]\geq 1, \quad (1.5)$$

$$b_{(k)}^\dagger|n\rangle=([[n/k]]+1)^{1/2}|n+k\rangle, \quad [[n/k]]\geq 0, \quad (1.6)$$

where $n\geq 0$, $k\geq 1$.

The multiphoton squeezing operator reads

$$S_{(k)}(z;w)=\exp(iwN+zb_{(k)}^\dagger-z^*b_{(k)}),$$

$$z\in\mathbb{C}, \quad w\in\mathbb{R} \quad (1.7)$$

and the corresponding squeezed vacuum is given by

$$S_{(k)}(z;w)|0\rangle=e^{-i\theta_{(k)}}e^{-|\zeta|^2/2}e^{\zeta b_{(k)}^\dagger}|0\rangle, \quad (1.8)$$

where

$$\theta_{(k)}=\frac{kw-\sin(kw)}{(kw)^2}, \quad (1.9)$$

$$\zeta=\frac{\sin(\frac{1}{2}kw)}{\frac{1}{2}kw}e^{ikw/2}z. \quad (1.10)$$

One can notice from (1.8) how the operator N in (1.7) has only the effects of multiplying the squeezed vacuum by a (irrelevant) phase factor $e^{-i\theta_{(k)}}$, and of rescaling the variable z to $\zeta\in\mathbb{C}$. We introduce then a rescaled squeezing operator

$$S_{(k)}(\zeta)=\exp(\zeta b_{(k)}^\dagger-\zeta^*b_{(k)}) \quad (1.11)$$

whose corresponding squeezed state is

$$|\alpha;\zeta\rangle_{(k)}=D(\alpha)S_{(k)}(\zeta)|0\rangle \quad (1.12)$$

with

$$D(\alpha)=\exp(\alpha a^\dagger-\alpha^*a), \quad \alpha\in\mathbb{C} \quad (1.13)$$

denoting the usual displacement operator.

It was shown in Ref. 1 that the states $|\alpha;\zeta\rangle_{(k)}$ are indeed squeezed states, in the sense that for $k=2$ one can obtain for the uncertainties $\Delta Q,\Delta P$ of the canonical coordinates $\hat{Q}$ and $\hat{P}$ values separately lower than those of the nonsqueezed coherent state ($\zeta=0$).

II. MOMENTS OF THE SQUEEZED-STATE DISTRIBUTION FUNCTION

We adopt the following notations. Corresponding to the position and momentum operators $\hat{Q},\hat{P}$ we define the rescaled canonical variables

$$\hat{q}=a_0\hat{Q}, \quad \hat{p}=\frac{\hat{P}}{a_0\hbar}, \quad (2.1)$$

respectively, where

$$a_0=\frac{\mu\omega}{\hbar}. \quad (2.2)$$

μ and ω denote, respectively, mass and proper frequency of the oscillator; $\hat{q}$ and $\hat{p}$ have the commutation relation

$$[\hat{q},\hat{p}]=i. \quad (2.3)$$

We introduce the complete orthonormal sets of states for both $\hat{q}$ and $\hat{p}$:

$$\hat{q}|q\rangle=q|q\rangle, \quad \int dq|q\rangle\langle q|=\hat{1}, \quad (2.4)$$

$$\hat{p}|p\rangle=p|p\rangle, \quad \int dp|p\rangle\langle p|=\hat{1}, \quad (2.5)$$

with the transformation matrix

$$\langle p|q\rangle=(2\pi)^{-1/2}e^{-ipq}. \quad (2.6)$$

Moreover we define in the usual way creation and annihilation operators as

$$a=\frac{1}{\sqrt{2}}(\hat{q}+i\hat{p}), \quad a^\dagger=\frac{1}{\sqrt{2}}(\hat{q}-i\hat{p}). \quad (2.7)$$

The coherent states for the harmonic oscillator are defined as

$$|\alpha\rangle_\omega=D(\alpha)|\omega\rangle, \quad (2.8)$$

$$D(\alpha)=\exp(\alpha a^\dagger-\alpha^*a), \quad (2.9)$$

where $|\omega\rangle$ is a fixed vector of unit norm in the Fock space, and $D(\alpha)$ is the unitary displacement operator of (1.13).

For $|\omega\rangle\equiv|0\rangle$ (2.8) gives the usual Glauber coherent states.[8] The so-called squeezed states as utilized by Nieto, Mandel, Caves, and others,[1,9] correspond to choosing

$$|\omega\rangle=S(z)|0\rangle=\exp[\tfrac{1}{2}(za^{\dagger 2}-z^*a^2)]|0\rangle. \quad (2.10)$$

The former are minimum-uncertainty wave packets with $\Delta p=\Delta q=1/\sqrt{2}$; the latter have uncertainties which may be made separately lower than $1/\sqrt{2}$. Both have Gaussian distributions in position as well as in momentum. In general the probability distributions are defined by

$$\mathscr{D}_{\alpha,\omega}(q)=|\langle q|\alpha\rangle_\omega|^2=|\langle q|D(\alpha)|\omega\rangle|^2, \quad (2.11)$$

$$\mathscr{P}_{\alpha,\omega}(p)=|\langle p|\alpha\rangle_\omega|^2=|\langle p|D(\alpha)|\omega\rangle|^2. \quad (2.12)$$

Upon noticing that, because of (2.9), (2.4), and (2.5) one can write

$$D^\dagger(\alpha)|q\rangle=\exp[-i\bar{p}(q-\tfrac{1}{2}\bar{q})]|q-\bar{q}\rangle, \quad (2.13)$$

$$D^\dagger(\alpha)|p\rangle=\exp[i\bar{q}(p-\tfrac{1}{2}\bar{p})]|p-\bar{p}\rangle, \quad (2.14)$$

where

$$\bar{q}=\sqrt{2}\,\mathrm{Re}\alpha \quad (2.15)$$

and

$$\bar{p}=\sqrt{2}\,\mathrm{Im}\alpha. \quad (2.16)$$

$\bar{q}$ and $\bar{p}$ indeed coincide with the quantum average values in the coherent state $|\alpha\rangle_\omega$ provided

$$\langle\omega|a|\omega\rangle=0; \quad (2.17)$$

namely, the expansion of $|\omega\rangle$ in the Fock space does not contain single-particle terms.

By (2.13) and (2.14) one can rewrite (2.11) and (2.12) in the form

$$\mathscr{D}_{\alpha,\omega}(q)=|\langle q-\bar{q}|\omega\rangle|^2=\mathscr{D}_{0,\omega}(q-\bar{q}), \quad (2.18)$$

$$\mathscr{P}_{\alpha,\omega}(p)=|\langle p-\bar{p}|\omega\rangle|^2=\mathscr{P}_{0,\omega}(p-\bar{p}). \quad (2.19)$$

In other words the $D(\alpha)$'s merely produce an overall translation of the distribution functions without changing their shapes, which depend on $|\omega\rangle$ only.

Clearly,

$$\int dq \, \mathcal{2}_{0,\omega}(q-\bar{q}) = \int dp \, \mathcal{P}_{0,\omega}(p-\bar{p}) = 1 \ . \tag{2.20}$$

The Fourier transform of such distribution functions gives the generating function for the corresponding moments. One obtains, in a straightforward manner,

$$\tilde{\mathcal{2}}_{\omega}(x) = \mathcal{F}[\mathcal{2}_{0,\omega}](x) = \langle \omega \, | \, e^{ix\hat{q}} \, | \, \omega \rangle \ , \tag{2.21}$$

$$\tilde{\mathcal{P}}_{\omega}(y) = \mathcal{F}[\mathcal{P}_{0,\omega}](y) = \langle \omega \, | \, e^{iy\hat{p}} \, | \, \omega \rangle \ . \tag{2.22}$$

The mth moments are then

$$\begin{aligned} \chi_{\omega}^{(m)} &= {}_{\omega}\langle \alpha \, | \, (\hat{q} - \bar{q})^m \, | \, \alpha \rangle_{\omega} \\ &= \int dq \, \mathcal{2}_{\alpha,\omega}(q)(q-\bar{q})^m \\ &= (-i)^m \frac{d^m}{dx^m} \bigg|_{x=0} \tilde{\mathcal{2}}_{\omega}(x) = \langle \omega \, | \, \hat{q}^{\,m} \, | \, \omega \rangle \end{aligned} \tag{2.23}$$

and

$$\begin{aligned} \pi_{\omega}^{(m)} &= {}_{\omega}\langle \alpha \, | \, (\hat{p} - \bar{p})^m \, | \, \alpha \rangle_{\omega} \\ &= \int dp \, \mathcal{P}_{\alpha,\omega}(p)(p-\bar{p})^m \\ &= (-i)^m \frac{d^m}{dy^m} \bigg|_{y=0} \tilde{\mathcal{P}}_{\omega}(y) = \langle \omega \, | \, \hat{p}^{\,m} \, | \, \omega \rangle \ . \end{aligned} \tag{2.24}$$

Upon writing

$$|\omega\rangle = S \, |0\rangle \ , \tag{2.25}$$

where S is a unitary operator,

$$S = \exp(-is), \quad s^{\dagger} = s \ , \tag{2.26}$$

(2.23) and (2.24) are reconducted to the elegant compact form, expressing them as vacuum expectation values

$$\chi_{\omega}^{(m)} = \langle 0 \, | \, (\mathrm{Ad}e^{is})\hat{q}^{\,m} \, | \, 0 \rangle \ , \tag{2.27}$$

$$\pi_{\omega}^{(m)} = \langle 0 \, | \, (\mathrm{Ad}e^{is})\hat{p}^{\,m} \, | \, 0 \rangle \ . \tag{2.28}$$

III. EXPLICIT FORM OF THE MOMENTS FOR THE MANY-PHOTON SQUEEZED STATES

When the operator S in (2.26) is identified with (1.8), the distribution functions $\mathcal{2}(q)$ and $\mathcal{P}(p)$ become, respectively,

$$\begin{aligned} \mathcal{2}_{(k,\zeta)}(q) &= e^{-|\zeta|^2} \left| \sum_{n=0}^{\infty} \frac{\zeta^n}{\sqrt{n!}} \langle q \, | \, kn \rangle \right|^2 \\ &= \frac{1}{\sqrt{\pi}} e^{-(q^2 + |\zeta|^2)} \\ &\quad \times \left| \sum_{n=0}^{\infty} \zeta^n \frac{H_{kn}(q)}{[(kn)!n!2^{kn}]^{1/2}} \right|^2 \end{aligned} \tag{3.1}$$

and

$$\mathcal{P}_{(k,\zeta)}(p) \equiv \mathcal{2}_{(k,(-i)^k\zeta)}(p) \ , \tag{3.2}$$

where the usual position representation of the Fock states for the harmonic oscillator has been used:

$$\langle q \, | \, m \rangle = (\sqrt{\pi}m!2^m)^{-1/2}e^{-q^2/2}H_m(q) \ , \tag{3.3}$$

$H_m(q)$ denoting the Hermite polynomials.

By (2.21) one has, therefore,

$$\tilde{\mathcal{2}}_{(k,\zeta)}(x) = e^{-(\rho^2 + x^2/4)} \left[\sum_{n=0}^{\infty} \frac{\rho^{2n}}{n!} L_{kn}(\tfrac{1}{2}x^2) + 2 \sum_{n=1}^{\infty} \sum_{m=0}^{n-1} \rho^{n+m}\cos[(n-m)\phi] \left[\frac{(2km)!!}{(2kn)!!n!m!} \right]^{1/2} (ix)^{k(n-m)} L_{km}^{k(n-m)}(\tfrac{1}{2}x^2) \right] \ , \tag{3.4}$$

where we have set $\zeta = \rho e^{i\phi}$, $\rho = |\zeta|$, and $L_m^{n(y)}$ denotes the Laguerre polynomials $[L_m(y) = L_m^0(y)]$.

Resorting to (2.23) the moments $\chi_{(k,\zeta)}^{(n)}$ can now be computed. After a lengthy and nontrivial algebraic manipulation, some details of which are reported in the Appendix, one gets, for the even-order moments and for even k ($n = 2N, k = 2K$),

$$\chi_{(2K,\zeta)}^{(2N)} = 2^{-N}(2N-1)!! \left[\sum_{m=0}^{N} \Lambda_{N,2K}^{(0)}(m)\rho^{2m} + 2N! \sum_{t=1}^{[[N/K]]} \frac{\sqrt{(2Kt)!}}{(N+Kt)!}(2^K\rho)^t\cos(t\phi) \sum_{l=0}^{N-Kt} \Lambda_{N,2K}^{(t)}(l)e^{-\rho^2}\rho^{2l}\frac{d^l}{d\rho^{2l}}F_{2K,t}(\rho^2) \right] \ , \tag{3.5}$$

where

$$\Lambda_{N,h}^{(t)}(l) = \sum_{m=l}^{N-[[ht/2]]} \sum_{v=m}^{N-[[ht/2]]} \frac{2^v}{v!} \left[\begin{matrix} N + [[ht/2]] \\ v + ht \end{matrix} \right] h^m S_v^{(m)} \mathscr{S}_m^{(l)} \tag{3.6}$$

$S_v^{(m)}$ and $\mathscr{S}_m^{(l)}$ denote the Stirling numbers of the first and second kind, respectively,[10] and the auxiliary functions $F_{h,t}(x)$ are given by

$$F_{h,t}(x) = \sum_{s=0}^{\infty} \frac{x^s}{s!} \left[\left[\begin{matrix} h(s+t) \\ hs \end{matrix} \right] \frac{s!}{(s+t)!} \right]^{1/2} \ . \tag{3.7}$$

Notice that we focused our attention only on even moments, the odd-order moments being less significant in view of the comparison, we are interested in, with the Gaussian unsqueezed distribution.

On the other hand, the even-order moments corresponding to odd k ($k = 2K + 1$) are given by

$$\chi^{(2N)}_{(2K+1,\zeta)}=2^{-N}(2N-1)!!\left[\sum_{m=0}^{N}\Lambda^{(0)}_{N,2K+1}(m)\rho^{2m}+2N!\sum_{t=1}^{[[N/(2K+1)]]}\frac{\sqrt{2t(2K+1)!}}{[n+t(2K+1)]!}(2^{2K+1}\rho^2)^t\cos(2t\phi)\right.$$

$$\left.\times\sum_{l=0}^{N-t(2K+1)}\Lambda^{(2t)}_{N,2K+1}(l)e^{-\rho^2}\rho^{2l}\frac{d^l}{d\rho^{2l}}F_{2K+1,2t}(\rho^2)\right].$$

$$(3.8)$$

Because of (3.2) the moments $\pi^{(2N)}_{(k,\zeta)}$ can be straightforwardly obtained from Eq. (3.5), in which ϕ is replaced by $(\phi-k\pi/2)$, and (3.8). From the algebraic steps leading to Eq. (3.5), one can prove the following general feature (see the Appendix for details): only even-order ($n=2N$) moments corresponding to even k ($k=2K$) such that $n\geq k$ ($N\geq K$) can be squeezed over a nonempty domain of the complex ζ plane. For distributions corresponding to odd k ($k=2K+1$) only higher-order moments ($2N\geq 2K+1$) can be possibly squeezed.

It should be noted that given an arbitrary distribution with even moments M_{2m}, the Schwartz inequalities, which read, in general,
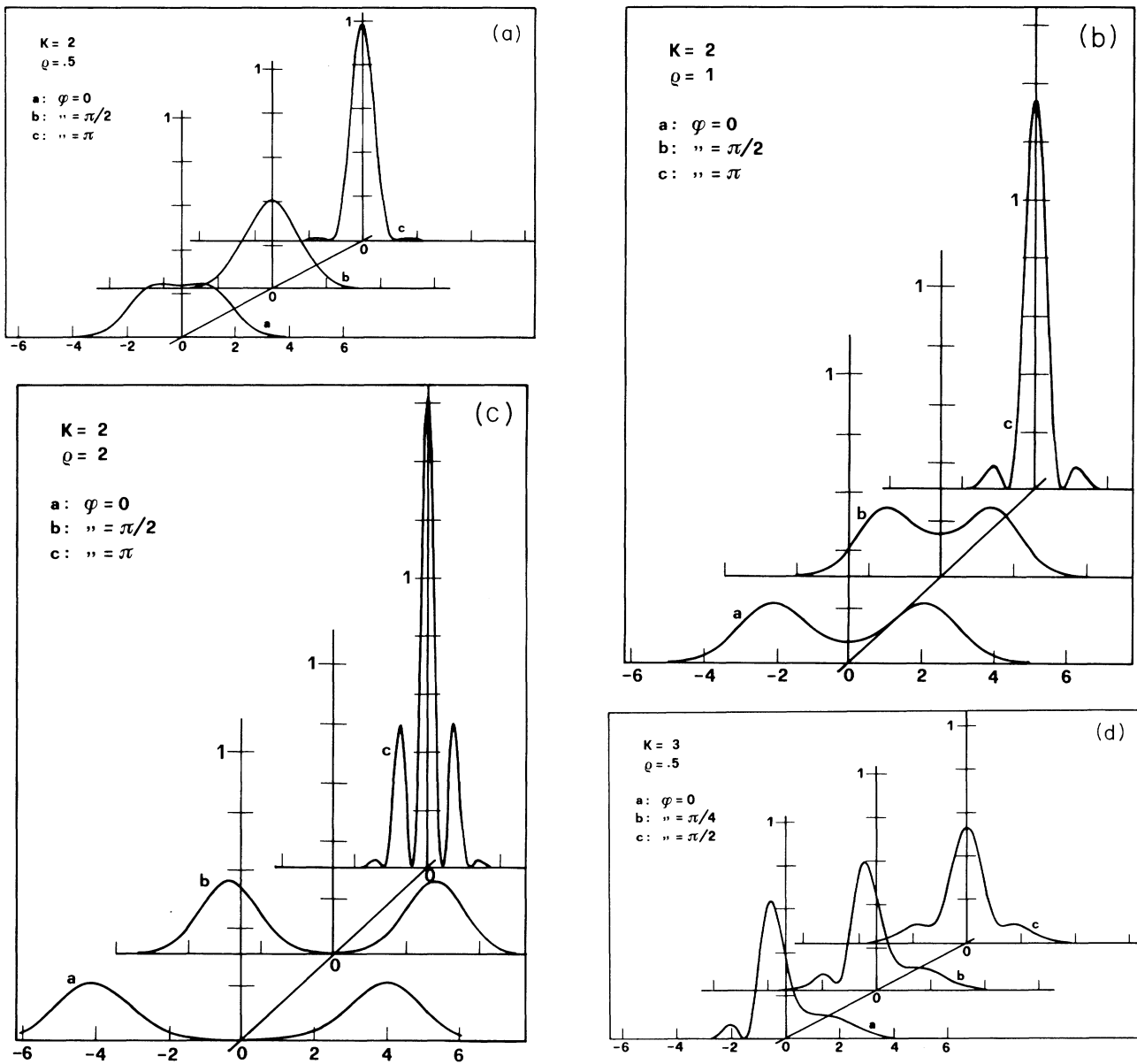

FIG. 1. The probability distribution function $\mathcal{P}(k,\zeta)$ vs the dimensionless position q, for different values of k and different choices of $\zeta=\rho e^{i\phi}$.

$$M_{2n} \leq (M_{2(n+1)})^{n/(n+1)}, \quad n \geq 0 , \tag{3.9}$$

do not limit the possibility of independently squeezing all the moments of the distribution itself, but constrain only the amount of squeezing for each moment.

IV. NUMERICAL RESULTS

Figures 1(a)—1(h) represent $\mathscr{D}_{(k,\zeta)}(q)$ for various choices of k and for different values of $\zeta = \rho e^{i\phi}$. One notices how such distributions exhibit a sensible deviation from the Gaussian behavior. It is worth pointing out that here this happens even though the number operator has sharp eigenvalues and the Fock space is not smeared as in the framework of bilinear quantum field theories.[11]

The functions corresponding to even k are symmetric under the exchange $q \rightarrow -q$, whereas there is no symmetry for odd k except for $\phi = \pi/2$. A characteristic feature of the functions $\mathscr{D}_{(k,\zeta)}(q)$ is that they show an increasing number of zeros when ρ is increased at fixed ϕ, for any k. The same effect, i.e., a richer structure corresponding to a larger number of nodes, appears when k is increased keeping ζ fixed. In particular for $k=2$ the second moment appears to increase with ρ for $\phi = 0, \pi/2$, whereas it decreases for $\phi = \pi$; for $k=4$ the second moment is always increasing.

We are thus led to analyze in more detail the behavior of the moments. For low-order n $(=2,3,4)$ and arbitrary k the formulas (3.5) and (3.8) of previous section have the following explicit forms:

$$\chi^{(2)}_{(k,\zeta)} = \frac{1}{2}\left[1 + 2k\,|\zeta|^2 + \delta_{k,2}2\sqrt{2}\,\mathrm{Re}\zeta e^{-|\zeta|^2}\sum_{l=0}^{\infty}\frac{|\zeta|^{2l}}{l!}\sqrt{2l+1}\,\right] , \tag{4.1}$$

$$\chi^{(3)}_{(k,\zeta)} = (\tfrac{3}{2})^{1/2}\delta_{k,3}\mathrm{Re}\zeta e^{-|\zeta|^2}\sum_{l=0}^{\infty}\frac{|\zeta|^{2l}}{l!}\sqrt{(3l+1)(3l+2)} , \tag{4.2}$$
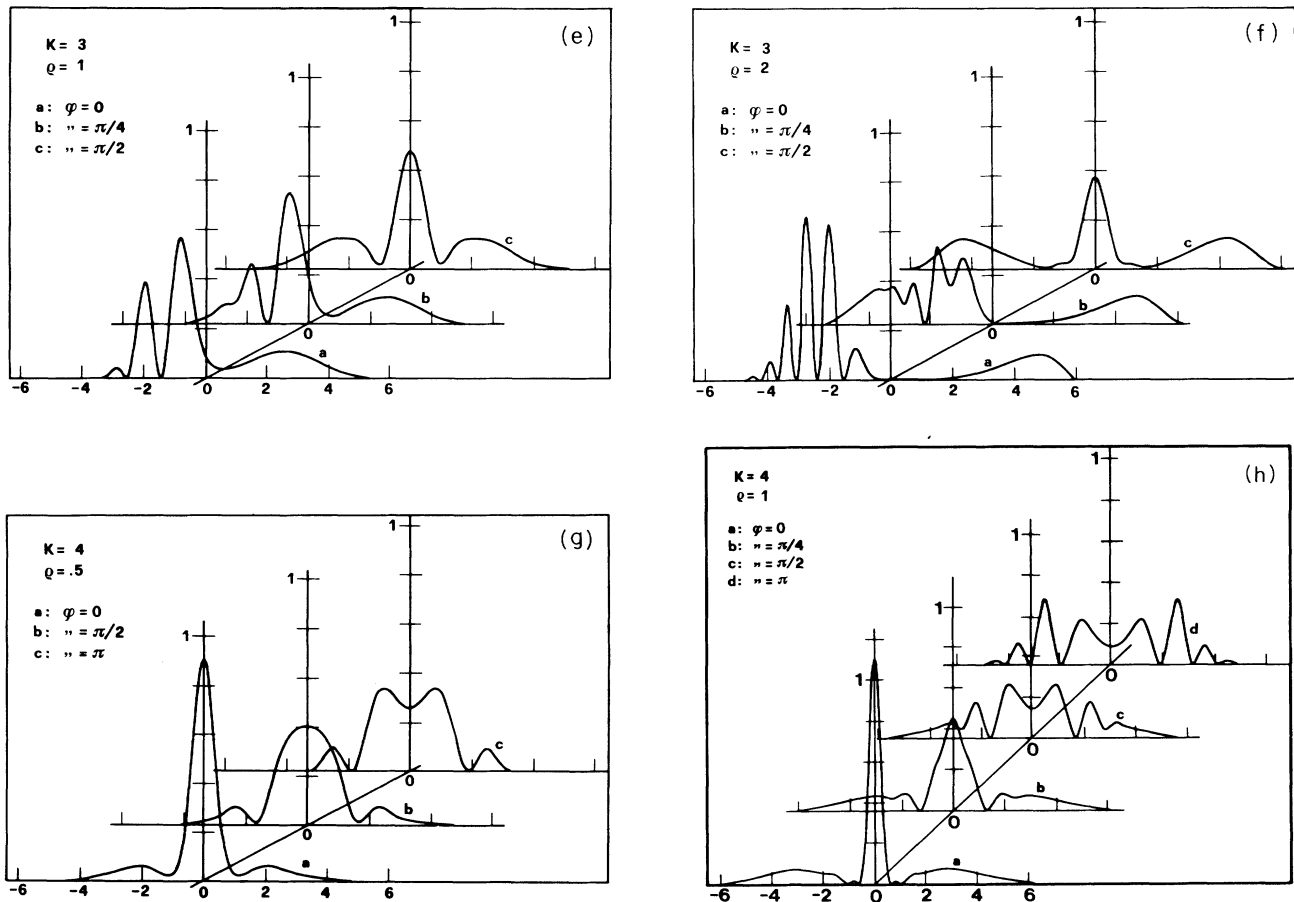

FIG. 1. (Continued).

$$\chi^{(4)}_{(k,\zeta)} = \frac{1}{4} \left[3[1+2k(k+1)]\,|\zeta|^2 + 2k^2\,|\zeta|^4 \right.$$

$$+\delta_{k,2}4e^{-|\zeta|^2}\left[[(\mathrm{Re}\zeta)^2-(\mathrm{Im}\zeta)^2] \sum_{l=0}^{\infty}\frac{|\zeta|^{2l}}{l!}\sqrt{(2l+1)(2l+3)}\right.$$

$$\left. +4\sqrt{2}\mathrm{Re}\zeta\,|\zeta|^2\sum_{l=0}^{\infty}\frac{|\zeta|^{2l}}{l!}\sqrt{2l+3}+3\sqrt{2}\mathrm{Re}\zeta\sum_{l=0}^{\infty}\frac{|\zeta|^{2l}}{l!}\sqrt{(2l+1)}\right]$$

$$\left. +\delta_{k,4}4\,\mathrm{Re}\zeta e^{-|\zeta|^2}\sum_{l=0}^{\infty}\frac{|\zeta|^{2l}}{l!}\sqrt{(4l+1)(4l+2)(4l+3)}\right] . \tag{4.3}$$
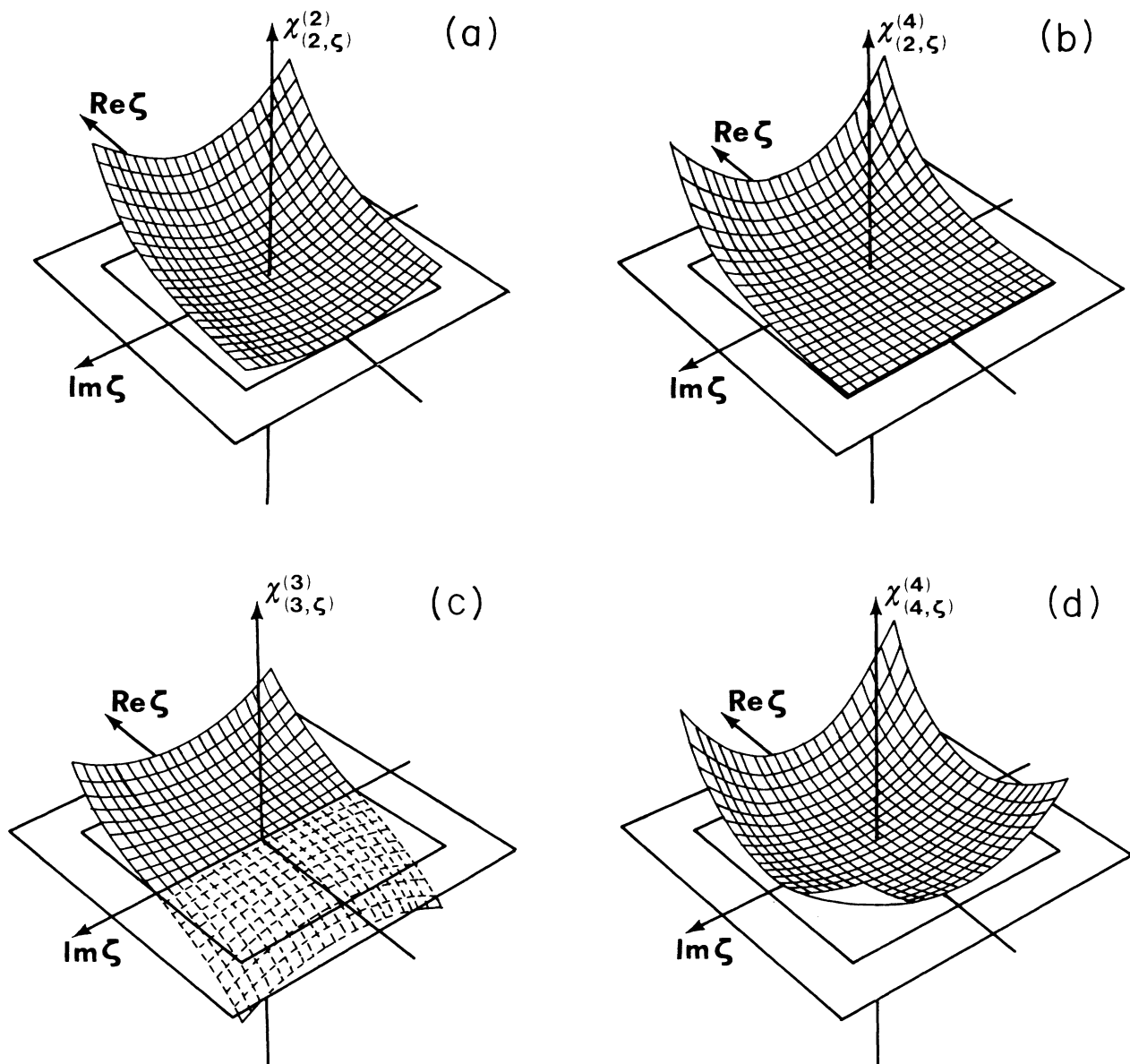

FIG. 2. The nth moments $\chi^{(n)}_{(k,\zeta)}$ of the probability distribution function $\mathscr{D}_{(k,\zeta)}$ vs the complex squeezing parameter ζ for $\mathrm{Re}\zeta$ and $\mathrm{Im}\zeta$ ranging from -2 to 2 and for different values of n and k. The corresponding vertical scale determined by the height of the maximal point of the surface is (a) 28, (b) 881, (c) 62, (d) 2190.
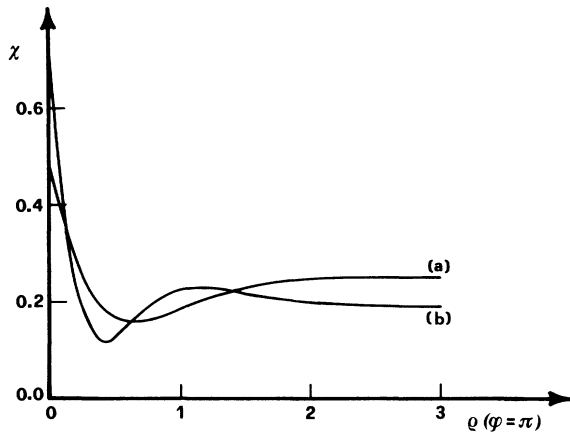
FIG. 3. Second (a) and fourth (b) moments for $k=2$ vs $\rho = |\zeta|$ along the direction of maximum squeezing $(\phi = \pi)$.
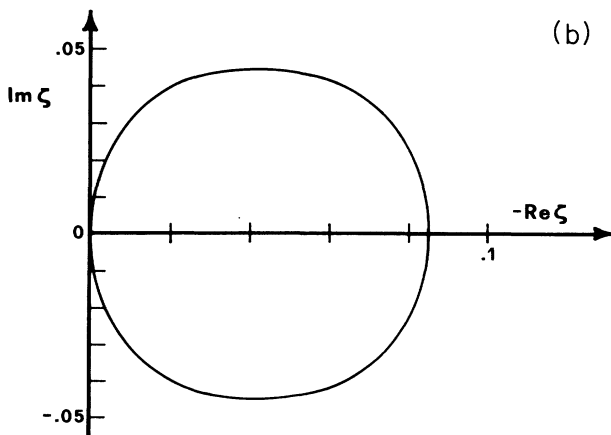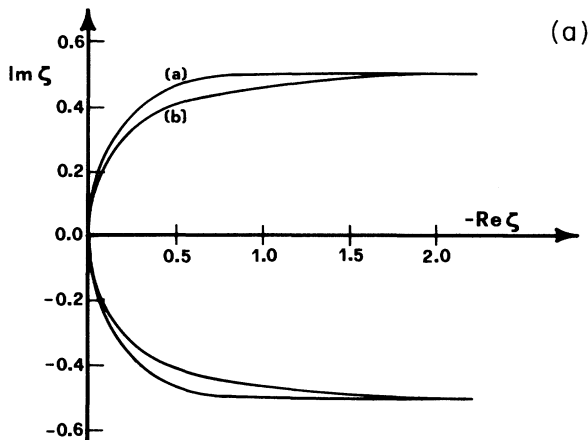

FIG. 4. The boundaries of the squeezing domain for (a) $k=2$, (a) second moment, (b) fourth moment, (b) $k=4$, fourth moment.

Figures 2(a)—2(d) show $\chi^{(2)}_{(2,\zeta)}$, $\chi^{(4)}_{(2,\zeta)}$, $\chi^{(3)}_{(3,\zeta)}$, and $\chi^{(2)}_{(4,\zeta)}$, respectively, in the complex ζ plane.

One can notice how all except $\chi^{(3)}$ are asymptotically concave functions. The maximum squeezing takes always place along the axis $\mathrm{Re}\zeta < 0$ [Fig. 3 shows how the second and fourth moments for $k=2$ vs $\mathrm{Re}\zeta$ do not have a monotonic behavior but exhibit a well-marked minimum, respectively, at $\mathrm{Re}\zeta = -0.64$ (with $\chi^{(2)}_{\min} = 0.1587$) and at $\mathrm{Re}\zeta = -0.42$ (with $\chi^{(4)}_{\min} = 0.1277$); in both cases the squeezing becomes asymptotically constant, with $\chi^{(2)}_{\mathrm{asympt}} = 0.25$, $\chi^{(4)}_{\mathrm{asmpt}} = 0.19$].

By comparison with the Gaussian value $\chi^{(2n)}_{\mathrm{Gauss}} = (2n-1)!!/2^n$ one can check the interesting properties.

(1) For $k=2$, (i) the second moment is squeezed in a domain which contains the whole semiaxis $\mathrm{Re}\zeta < 0$, and is delimited by the two asymptotes $\mathrm{Im}\zeta = \pm \frac{1}{2}$ [as one can check analytically from Eqs. (4.1) and (4.3) resorting to the asymptotic expansion given in Ref. 4], (ii) the same holds for the fourth moment, however—as Fig. 4(a) shows—the corresponding squeezing domain is contained in that of the second moment. In other words, there is a region of the complex ζ plane in which the second moment can be squeezed without squeezing the fourth, and another in which both the second and the fourth moments can be squeezed.

(2) For $k=4$, as the theorem proved in Sec. III states, only the fourth moment can be squeezed, in correspondence to the points in the ζ plane contained in the region represented in Fig. 4(b).

CONCLUSIONS

A thorough analysis of the probability distribution functions corresponding to the multiboson squeezed states introduced in Ref. 4, shows that such states are non-Gaussian.

Indeed they can be squeezed at different orders by selecting in different ways the squeezing parameter ζ or the "number of photons" k. One interesting feature of these states is that, due to Eq. (3.2), the fourth moments for both the q and p distributions can be squeezed simultaneously. On the contrary the second moment squeezing domains for p and q are nonoverlapping (opposite phase) in the complex ζ plane. This raises an interesting set of questions in the perspective of the "inverse quantization" problem. Indeed, using an integral transform connecting the matrix elements between coherent states of a quantum operator to the corresponding classical functions, Dau-bechies, Grossman, and Reignier[12] were able to give a complete characterization of the Hilbert spaces of images, showing that the latter may correspond to non-Gaussian quantum states. We believe that an unexpected insight in the dynamics of nonlinear classical systems, as well as in the problem of their quantization could be gained by constructing explicitly and studying the classical action functions connected to the quantum-mechanical squeezing operator S (see Ref. 13 for a preliminary discussion of some related questions).

It should be noted that the multiboson squeezed states

(1.12) have been further generalized in Ref. 5 by using the realization (1.3) and (1.4) of the Weyl algebra into the nonlinear realization of SU(2) and SU(1,1) due to Holstein and Primakoff, and constructing the generalized coherent states[14] corresponding to these two algebras. We expect that such new states should lead to an even richer structure of moments for the corresponding distributions, and also a more flexible one, in that they depend on the extra parameter σ labeling the representation. Work is in progress on this problem.

APPENDIX

The generating function $\widetilde{\mathscr{D}}_{(k,\zeta)}(x)$ for the moments $\chi^{(n)}_{(k,\zeta)}$ of the position $\hat{q}$ can be derived by Fourier transforming the distribution function $\mathscr{D}_{(k,\zeta)}(q)$ given in Eq. (3.1) and using the identity

$$\int_{-\infty}^{+\infty} dq\, e^{-q^2+ixq} H_{kl}(q) H_{ks}(q) = e^{-x^2/4} 2^{kl} \sqrt{\pi}\, (\tfrac{1}{2}ix)^{k(l-s)} (ks)! L_{ks}^{k(l-s)}(\tfrac{1}{2}x^2)\ . \tag{A1}$$

One obtains Eq. (3.4) which gives, in terms of the auxiliary variable $z=ix$,

$$\widetilde{\mathscr{D}}_{(x,\zeta)}(z) = e^{-(\rho^2-z^2/4)} \left[\sum_{l=0}^{\infty} \frac{\rho^{2l}}{l!} L_{kl}(-\tfrac{1}{2}z^2) + 2\sum_{l=1}^{\infty}\sum_{s=0}^{l-1} \rho^{l+s}\cos[(l-s)\phi] \left[\frac{(2ks)!!}{(2kl)!!l!s!} \right]^{1/2} z^{k(l-s)} L_{ks}^{k(l-s)}(-\tfrac{1}{2}z^2) \right]\ . \tag{A2}$$

By the definition (2.23), which can now be written

$$\chi^{(n)}_{(k,\zeta)} = \frac{d^n}{dz^n}\Bigg|_{z=0} \widetilde{\mathscr{D}}_{(k,\zeta)}(z)\ , \tag{A3}$$

the calculation of the even-order moments is reconducted to the formula ($m \geq 0$)

$$\frac{d^{2N}}{dz^{2N}}\Bigg|_{z=0} [e^{z^2/4} z^{km} L_{ks}^{km}(-\tfrac{1}{2}z^2)] = \sum_{l=0}^{N}\begin{bmatrix}2N\\2l\end{bmatrix}\delta_{2(N-l),km}(km)!2^{-l}(2l-1)!!\sum_{v=0}^{l}\begin{bmatrix}l\\v\end{bmatrix}\begin{bmatrix}k(s+m)\\ks-v\end{bmatrix}2^v$$

$$= \theta(N-[[km/2]])\frac{(2N)!}{(2N-km)!}2^{-N+[[km/2]]}[2(N-[[km/2]])-1]!!$$

$$\times \sum_{v=0}^{N-[[km/2]]}\begin{bmatrix}N-[[km/2]]\\v\end{bmatrix}\begin{bmatrix}k(s+m)\\ks-v\end{bmatrix}2^v\ , \tag{A4}$$

where $\theta(x)$ denotes the Heaviside function equal to 1 for $x \geq 0$, and to 0 for $x < 0$.

Upon inserting (A2) into (A3) and computing the derivative according to (A4), one obtains the moments $\chi^{(n)}_{(k,\zeta)}$.

We describe in some detail the analysis of the "diagonal" contribution D to such moments, coming from the simple sum at the right-hand side of (A2). The off-diagonal contribution corresponding to the double sum leads to a perfectly analogous algebraic manipulation. We have

$$\frac{2^N}{(2N-1)!!}D = e^{-\rho^2}\sum_{l=0}^{\infty}\frac{\rho^{2l}}{l!}\frac{d^{2N}}{dz^{2N}}\Bigg|_{z=0}[e^{z^2/4}L_{kl}(-\tfrac{1}{2}z^2)]$$

$$= e^{-\rho^2}\sum_{m=0}^{N}\frac{2^m}{m!}\begin{bmatrix}N\\m\end{bmatrix}\sum_{l=0}^{\infty}\frac{kl(kl-1)\cdots(kl-m+1)}{l!}\rho^{2l}$$

$$= e^{-\rho^2}\sum_{m=0}^{N}\frac{2^m}{m!}\begin{bmatrix}N\\m\end{bmatrix}\sum_{l=0}^{\infty}\sum_{s=0}^{m}S_m^{(s)}(kl)^s\frac{\rho^{2l}}{l!}$$

$$= e^{-\rho^2}\sum_{m=0}^{N}\frac{2^m}{m!}\begin{bmatrix}N\\m\end{bmatrix}\sum_{l=0}^{\infty}\sum_{s=0}^{m}S_m^{(s)}k^s\sum_{r=0}^{s}\mathscr{S}_s^{(r)}\frac{l(l-1)\cdots(l-r+1)}{l!}\rho^{2l}$$

$$= \sum_{r=0}^{N}\sum_{s=r}^{N}\sum_{m=s}^{N}\frac{2^m}{m!}\begin{bmatrix}N\\m\end{bmatrix}S_m^{(s)}k^s\mathscr{S}_s^{(r)}\rho^{2r}\ , \tag{A5}$$

where the definitions of the Stirling numbers of the first and second kind in terms of generating functions[10] have been used. Equation (A5) gives the diagonal part of (3.5) and (3.6).

The expression for D as given in the second step of (A5) shows that the diagonal contribution D is larger than the corresponding Gaussian moment, except for $q=0$ where they are equal. Thus one can have squeezing if the off-

diagonal contribution is negative and large enough in absolute value as to give a moment smaller than the Gaussian. For even k we notice that (i) the off-diagonal part of $\chi^{(2N)}_{(2K,\zeta)}$ is nonvanishing only if $[[N/K]] \geq 1$, namely, $N \geq K$, and (ii) in the latter case, it has a linear term in ρ, whose coefficient has the sign of $\cos\phi$, whereas the diagonal part does not have any linear term in ρ. The domain

of squeezing in the ζ plane in this case contains then at least a nonempty neighborhood of $\rho = 0$. Analogously one can check that for odd k the off-diagonal part of $\chi^{(2N)}_{(2K+1,\zeta)}$ can be negative only if $[[2N/(2K+1)]] \geq 1$, namely, $2N \geq 2K + 1$. However, it has no linear part in ρ, and the possibility of squeezing has to be checked numerically.

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