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Lorentz equations of motion and a theory of connections in a principal bundle

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The conjecture that unified nonlinear equations for gravitation and electromagnetism may lead directly to the Lorentz equation of motion for charged particles is discussed assuming a theory of connections on a principal bundle with SL(2,Q) as the structure group. It is shown that the Lorentz equation is a consequence of the field equations when the connection reduces to only one particular component, which may be identified with the electromagnetic potential. The proposed equations generalize, in a nontrivial way, the equations of gravitation and electrodynamics.

INTRODUCTION

It is well known that Einstein's field equations of gravitation imply the equations of motion of test particles[1] and, on the contrary, that Maxwell's field equations of electrodynamics do not imply the corresponding equations of motion. In this case, it is necessary to postulate the Lorentz force equation, or derive it from some variational principle. This fact is related to the nonlinearity of gravitation and linearity of electrodynamics.[2]

If a unified theory of gravitation and electrodynamics is constructed with nonlinear field equations, it should be possible to derive the Lorentz equation of motion from the field equations of the unified theory.

Within the Einstein-Maxwell theory, the desired equations of motion were obtained by Infeld and Wallace.[3] In this theory, if we properly choose the stress-energy tensor of electromagnetism T_e, the conservation of the total stress-energy tensor T implies the Lorentz force equation. This should be no surprise because the T_e of electromagnetism is constructed precisely to conserve the energy and momentum of a system of electromagnetic fields and electric charges moving according to the Lorentz force equation. In other words, the structure of T_e "assumes" the Lorentz equation. In the Einstein-Maxwell theory, apart from the gravitational-geometric postulates, we have to postulate separately the exact form of the stress-energy tensor of electromagnetism which contains the assumption of motion due to the Lorentz force. With this postulate the motion of charged particles is determined by conservation, even in flat space, without use of Einstein's equation.[4]

Nevertheless, it may be claimed that the Einstein-Maxwell theory is not a truly geometrically unified theory. Einstein[5] himself was unsatisfied by the nongeometrical character of T and spent his later years looking for a satisfactory unified theory.

Nowadays, the need for theories of weak and strong interactions revives the idea of a geometrically unified theory. The Einstein-Maxwell theory is incomplete, in the sense that it does not provide a geometrical structure capable of representing additional interacting fields.

Most of the work done on the motion of charged particles,[6] including the Infeld and Wallace calculation, accounts for the motion within such state of affairs. Our work is in line with the ideas expressed by Einstein for a unified theory. Our purpose is not to exhaustively study the equations of motion of matter under the ponderomotive forces of some unified theory, but rather, noticing similarities between the Lorentz force expression and the expression for the force on spinning particles in general relativity,[7] to assume this fact is no mere coincidence. This leads us to represent gravitation and electromagnetism by the same geometrical object. By requiring the prediction of the correct equation of motion, including the Lorentz force, at least in a certain limit, we are able to narrow down possible theories.

In the so-called "already" unified field theory, it is required that the curvature tensor satisfies the conditions of Rainich[8] and Misner and Wheeler,[9] which are equivalent to the existence of an electromagnetic stress-energy tensor. With these conditions the Lorentz equation follows in the same way as in the Einstein-Maxwell theory.

In Weyl's unified theory,[10] the equations of motion are subject to the objection, first raised by Einstein,[11] that they imply that the frequencies of the atomic spectral lines should depend on the location and past histories of the atoms.

In the Kaluza theory,[12] the Lorentz equations are obtained from the geodesic equation by interpreting certain components of the connection as the electromagnetic field tensor and the component of the four-velocity along the direction of the Killing vector as the electric charge. We can object that, since it is known that the connection may be made zero along any given curve, the electromagnetic tensor can be made zero in properly chosen coordinates. The physical meaning of this coordinate system is not compatible with known experimental facts of electromagnetism.

Derived equations of motion have been discussed by Johnson,[13] within the Einstein theory of the antisymmetric field.[14] In the nongravitational limit, the electrodynamics of this theory is not the conventional Maxwell theory,[15] although the resultant equations are compatible with the experimentally known interaction of charged particles over laboratory distances.

The theory discussed here gives the Lorentz equation of motion without any of the problems indicated for the other theories. Apart from this feature, the theory offers room for describing other interactions. From a geometrical point of view, we have the simplicity of representing all interactions by a single geometric object. From a physical point of view, spin and charge will play similar roles, offering the opportunity to understand why they are discrete quantities, conserved under external rotations and a particular internal rotation, respectively.

Continuing our line of thought, we propose that electromagnetism enters, neither as part of the metric nor as the stress-energy tensor of matter, but a part of a geometric connection, as originally intended by Weyl. We propose that the gravitational field also be represented by the connection. In our approach we look for the possibility of enlarging the structure group of the theory to incorporate both gravitation and electromagnetism as part of a unified connection, making the theory clearly different from the Einstein-Maxwell, the Weyl, and Kaluza, and the antisymmetric field theories. The physical motivation for this speculation arises from the fact that the gravitational curvature enters in the equations of motion for spinning bodies in the same manner as the electromagnetic curvature in the Lorentz equations for charged particles and the fact that a theory of gravitation that incorporates naturally the equivalence principle should be represented by a connection, not necessarily by a metric. A well-known example of a nonmetric theory of gravitation is the Newton-Cartan theory.[16,17] In any theory of gravitation incorporating the equivalence principle, the motion of idealized test particles defines a set of curves, called "free falls," and a physical parameter on each curve.[18,19] By requiring these curves to be geodesics and the parameter to be the affine parameter, a connection is naturally defined. It should be clear that a connection plays a fundamental role in gravitation. A connection derived from a metric is not a universal feature of gravitation.

It was known to Infeld and van der Waerden,[20–22] when they introduced an spinorial connection, that there appeared arbitrary components which admitted the interpretation of electromagnetic potentials, because they obeyed the necessary field equations. To admit this interpretation we further require that the Lorentz force equation be a consequence of the field equations. Otherwise, the equation of motion, necessarily implied by the theory, is contradictory with the experimentally well-established motion of charged particles, and the theory should be rejected.

If we desire to represent interactions by a connection, we have to consider the way of selecting the group associated with the connection. There are two approaches to select the group: (1) take a group related to space-time, and (2) take a group related to some other area of physics, i.e., particle fields. In the second case we would have to consider how the group is related to the space-time geometry and in the first case we would have to consider the physical interpretation of the group.

Flat space-time is related to the Lorentz group. In many physical theories we deal with representations of this group, i.e., spinors representing some matter fields. To approach unification it is desirable to work with the group $SL(2,C)$, for example, rather than the Lorentz group itself $SO(3,1)$. A gravitation theory related to $SL(2,C)$ is discussed by Carmelli.[23] In order to generalize this group the simplest thing, apparently, is the group $U(1)\times SL(2,C)$, which is the group that preserves the metric associated to a tetrad induced from a spinor base. It is also known that a theory of gravitation can be expressed using tetrads. In our case we would substitute a spinor base for the tetrad. This spinor base would be a physical representation of matter fields.

A first attempt,[24] using $U(1)\times SL(2,C)$ as the group, leads to a negative result, because the equations depend on the commutators of the gravitational and electromagnetic parts, which are zero. This result also follows from this paper, as will be seen later. This indicates that it is not possible, without contradictions, to consider that the $U(1)$ part represents electromagnetism as suggested by Infeld and van der Waerden. This really means that to obtain the correct motion we must expand the group chosen. In compensation for the extra complications, we reach a geometrical structure capable of incorporating additional interactions.

Then we are faced with the problem of selecting a simple group, somehow related to $SL(2,C)$, to express the more general theory, and specifying a prescription of going back from the new group to $SL(2,C)$, from which we obtain a description of the behavior of matter fields.

In particular, we think of the group $SL(2,Q)$, which is the largest simple subgroup of a universal Clifford algebra associated to flat space. This group is known not to preserve the metric. But, if we think of general relativity as linked to transformations which do not necessarily preserve the Lorentz metric, it would be in the same spirit to use such a group. Instead of general coordinate transformations whose physical meaning is associated to a change of observers, we have transformations belonging to the group $SL(2,Q)$ whose physical meaning should be associated to a change of spinors related to observers. Representations of this group would be linked to matter fields. If we restrict to the complex field C, instead of the quarternions Q, we would get spinor representations which are used in many other areas of physics. Since $SL(2,Q)$ is larger than $SL(2,C)$ it gives us an opportunity to associate the extra generators with interactions apart from gravitation and electromagnetism. The generator that plays the role of electromagnetic generator must be consistent with its use in other equations of physics. The physical meaning of the remaining generators should be identified.

The method used to find the equations of motion gives the equation for a test particle with multipole structure. The next step in this direction would appear to be to find the motion of real particles in which the action of the total field on the particle is taken into consideration. A way to accomplish this would be to expand the field equation and the conservation equation in terms of a small parameter in a form similar to the Einstein-Infeld-Hoffmann (EIH) method.[25] In the zeroth order, the particle moves as a test particle, in the first order the field produced by the particle produces a correction to the motion, and successive orders produce more correction terms.

Nevertheless, our purpose here is not really to describe exhaustively the motion of classical matter under the equations of the theory, but rather, as indicated above, to present a geometrical theory and to require that it be compatible with the Lorentz motion. A further requirement would be compatibility with modern ideas in quantum field theory. In this sense it is better to postpone the discussion of a classical approximation until other aspects of the theory are more developed. In particular, at the present situation, it appears more urgent to exploit the opportunity provided by the theory to give a geometrical interpretation of the source in terms of fundamental geometric field objects. This direction could provide a link with quantum theories, as will be discussed in a future paper.

Once the geometrical nonphenomenological structure of the source term J is known, the exact equations of motion for the fields describing matter, including particles, would be the conservation law for J. This relation, being an integrability condition on the field equations, includes all self-reaction terms of the matter on itself. There should be no worries about infinities produced by self-reaction terms. A physical system would be represented by matter fields and interaction fields which are solutions to the set of simultaneous equations. When a perturbation is performed on the equations, for example, to obtain linearity of the equations, the splitting of the equations into equations of different order bring in the concepts of field produced by the source, force produced by the field, and, therefore, self-reaction terms. We should not look at self-reaction as a fundamental problem but, rather, as one introduced by this method of solution.

A classical particle itself may not be the appropriate idealization of the physical world. The theory should provide relations between interaction fields (gravitation, electromagnetism, etc.) and matter fields (masses, charges, spinors, etc.) and specify how these fields evolve. A modern definition of particle and its properties should rest on the fundamental geometrical fields. It would be desirable that both classical and quantum aspects of the physical particle could be obtained from the geometrical theory. Nevertheless, I believe that this is work for future papers.

I. SELECTION OF THE STRUCTURE GROUP

To introduce the new group, we turn again to the process that lead us to $SL_1(2,C)$ in the first place, but looking closely to find a way to generalize it. In essence, the group was obtained by introducing a two-dimensional spinor space and requiring invariance of the induced space-time metric under the group transformation. Fundamentally, the process of finding a square-root operator of the relativistic D'Alembertian operator leads to spinors and the group $SL_1(2,C)$, essentially the square root of the Lorentz group SO(3,1).

The D'Alembertian operator is obtained from a metric g,

$$P\cdot P=g^{\mu\nu}P_\mu P_\nu \ , \tag{1.1}$$

where P is the momentum operator. The metric may be given by a scalar product in the space-time manifold M:

$$g(u,v)=\text{scalar product}, \quad u,v\in TM \ . \tag{1.2}$$

In order to take the square root, instead of the previous procedure, we may introduce geometric elements V such that for each vector $v\in X$, where X is a flat space, and set

$$V^2=-g(v,v) \ . \tag{1.3}$$

These V represent the vectors v and therefore should form a vector space. In addition, we could define a product by the required equation (1.3), turning the vector space into an algebra. An algebra of this type, containing isomorphic copies of X and R, is known as the geometric or Clifford algebra[26] of X.

In our case, the vector space of interest, X, is the tangent space at a given point of the curved space-time manifold M. This flat four-dimensional space has the Lorentz metric. The universal Clifford algebra associated to this space with signature $(-1,1,1,1)$ is the real algebra of 2×2 matrices over the quarternion field Q.[27] The highest-dimensional simple subgroup is $SL(2,Q)$. The subgroup that preserves the metric is $SL_1(2,C)$, which may be obtained from $SL(2,Q)$ by some restriction. It is very suggestive that a natural nontrivial generalization would be to use $SL(2,Q)$ as the structure group of the bundle.

We are then lead to a theory that deals with the evolution of the elements of the simple group $SL(2,Q)$. Therefore, we may construct a principal fiber bundle by taking this group as the fiber and the four-dimensional space-time manifold M as the base space. To establish the evolution of the elements of the fiber, we need to take derivatives and, consequently, we need a connection on the principal bundle. The given connection has a fundamental unifying role; we naturally define its curvature Ω and may take as field equations the simplest natural geometric expression in terms of the dual $^*\Omega$,

$$D\,^*\Omega=J \ , \tag{1.4}$$

where the current J is an $sl(2,Q)$-valued three-form.

Now we have, for any two-form valued on the Lie algebra $sl(2,Q)$,

$$DDX=[X,\Omega] \tag{1.5}$$

and, in particular,

$$DD\,^*\Omega=[^*\Omega,\Omega] \ , \tag{1.6}$$

obtaining an integrability condition on J as in the case of general relativity for T:

$$DJ=0 \ . \tag{1.7}$$

In gravitation, the integrability conditions, T conservation, imply equations of motion for particles. If a multipole structure is assumed, we get the geodesic equation for monopoles and an equation for spinning particles which depends on the curvature tensor.[28,29] Now, the integrability condition Eq. (1.7) also implies equations of motion, particularly when J is expressed in multipole terms. Since J is an $sl(2,Q)$-valued three-form, the multipole structure associated to it must keep these geometrical aspects. The hope is that the equations of motion depend on the electromagnetic part of the curvature tensor; that is, on the Maxwell tensor F.

II. GEOMETRICAL FORMULATION OF THE EQUATIONS OF MOTION

By a classical equation of motion of a point particle we mean a differential equation determining the evolution of the tangent vector to the world line of the particle.

We could idealize a test particle by associating an orthonormal tetrad to it, the timelike vector being the four-velocity and the three spacelike vectors related to the rotating properties of the particle. At two different events in the world line, the corresponding tetrads should differ by a Lorentz transformation, at most. The evolution of the tetrad along the world line is determined by a Lorentz transformation that evolves as a function of the parameter on the world line. Of course, the timelike vector of the tetrad must be required to coincide with the tangent to the world line at all points in the world line.

Consider the principal fiber bundle constructed by taking the Lorentz group as fiber and the space-time manifold as base space. If we are given a curve in this bundle, we have precisely an element of the group evolving as function of the curve parameter. It is clear that the given curve determines, by projection, a curve in the base space of the bundle. The projected curve may be taken as the path of a particle in space-time. If we represent the evolution of the particle by the original curve in the principal bundle, we can obtain the space-time path, but we would have more information, associating an element of the fiber to each point of the path of the particle.

On the other hand, we have introduced in the preceding section a principal bundle with a larger group. Nevertheless, the integrability conditions determine equations for a curve in this bundle, if we associate a classical multipole structure to the current J. If we are able to solve for this curve, we would have information about the path of the particle in space-time and also information about the evolution of internal elements representing the particle. Again, this would provide more complete information than just the space-time path of the particle.

In order to clarify the geometrical aspects involved, we shall designate by E and E', respectively, the fiber bundles with the groups SL(2,Q) and SO(3,1) as fibers, over the space-time manifold M as base space. Here we take the view that the most fundamental expression of the connection field is through the introduction of groups and morphisms among them.

If we have a mapping from SL(2,Q) to SO(3,1), we may define a curve in E' from the knowledge of a curve in E. We could think that the curve in E represents the evolution of a complete idealized observer. This observer carries a complete basis of the fiber that allows him to measure external (base space) magnitudes and internal (fiber space) magnitudes. The curve in E' would represent an observer carrying a space-time tetrad whose evolution is determined by the curve. A classical measurement may be interpreted as the measurement of only the projected path in the base space (space-time). This path is determined by the curve in the larger space, the principal bundle E. In other words, classically we only notice the "shadow" of the particle. We should point out that similar projections are used normally in physics, for example, when we work with the field of complex numbers and restrict ourselves to the field of real numbers.

It is clear, then, that we could restrict the algebra sl(2,Q) to the complex field, leading to the even subalgebra $\mathrm{sl}_1(2,C)$, from which we could pass to the Lorentz group SO(3,1) using the known homomorphism, permitting us to find an equation for the timelike vector of the space-time tetrad; that is, a classical equation of motion for the particle.

To make things more definite, we assume there is a mapping $\mu:E\rightarrow E'$ that allows one to pass from the space of SL(2,Q) variables to the space of SO(3,1) variables. We repeat that we take the point of view that fundamental equations should relate group elements; that is, points in the principal bundles. We shall require, for physical reasons, that μ leave invariant the projection point on the base space M. We have then

$$\mu:(E,M,\pi,G)\rightarrow(E',M,\pi',G') , \tag{2.1}$$

where $G=\mathrm{SL}(2,Q)$, $G'=\mathrm{SO}(3,1)$, and π,π' are the projection mappings of the bundles.

If we have a curve in E,

$$c:R\rightarrow E , \tag{2.2}$$

$$c(\lambda)\in E,\quad \lambda\in R , \tag{2.3}$$

we have its corresponding tangent vector field:

$$\dot{c}(\lambda)_{c(\lambda)}\in TE_{c(\lambda)} . \tag{2.4}$$

Knowledge of $\dot{c}$ determines the curve c up to a constant of integration. It follows that the mapping μ defines a curve in E' by composition:

$$c'=\mu\circ c , \tag{2.5}$$

$$c'(\lambda)\in E',\quad \lambda\in R . \tag{2.6}$$

In order to obtain an equation for the tangent vector to c', we must use the differential of the mapping μ, indicated by μ_*:

$$\mu_*:TE\rightarrow TE' , \tag{2.7}$$

$$\dot{c}'(\lambda)=\mu_*\dot{c}(\lambda) \tag{2.8}$$

The equation for the tangent vector $\dot{c}$ should arise from the integrability conditions of the source as indicated before. This equation should relate to an element of the algebra sl(2,Q), as seen from the field equations, which determine that J is an sl(2,Q)-valued three-form. The result is an equation for the tangent vector to a curve a, in the adjoint bundle of E, denoted by A, where the fiber is the vector space sl(2,Q). To obtain a curve in E, we should exponentiate the algebra. On the other hand, what we really need is an equation for the tangent of a curve in a bundle A' with the algebra so(3,1) as fiber.

Now, the mapping μ_* restricts one naturally to a mapping from bundle A to bundle A'. Consider the vertical subspace of the tangent space at a point $e\in E$, which we shall indicate as T^vE_e. Restrict μ_* to this space,

$$\mu_{*e}:T^vE_e\rightarrow T^vE'_{\mu\circ e} . \tag{2.9}$$

For $m\in U\subset M$, $g\in G$, $g'\in G'$, a trivialization t of E,

$$t:\pi^{-1}(U)\rightarrow U\times G , \tag{2.10}$$

$$t(e)=(\pi(e),\phi(e))=(m,g)\ , \tag{2.11}$$

induces a trivialization of E',

$$t':\pi'^{\,-1}(U)\rightarrow U\times G'\ , \tag{2.12}$$

$$t'(\mu(e))=(\pi'(\mu(e)),\phi'(\mu(e)))=(m,g')\ , \tag{2.13}$$

where $g'=\phi'\circ\mu\circ\phi^{-1}(g)$. In particular, for a trivialization at the point (m,I), we obtain

$$\mu_{*m,I}:TE_{m,I}\rightarrow TE'_{m,I}\ . \tag{2.14}$$

Since the tangent space of a Lie group space, at the identity, is isomorphic to the Lie algebra of the group, we obtain

$$\mu_{*m,I}:A_m\rightarrow A'_m\ , \tag{2.15}$$

which can be written as the bundle mapping

$$\mu_{*I}:A\rightarrow A'\ . \tag{2.16}$$

To specify the mapping μ, it is convenient to do it in two steps. First, we pass from E to a bundle E'' with $SL_1(2,C)$ as structure group and then we pass from E'' to E'.

It is known that any element of a universal geometric algebra may be uniquely expressed as the sum of an even part and an odd part. We may introduce an equivalence relation in $SL(2,Q)$ by defining equivalent points if

$$\exp[\text{even}(a)]=\exp[\text{even}(b)] \tag{2.17}$$

for points $\exp(a)$, $\exp(b)\in SL(2,Q)$. Using the homeomorphism, from $\pi^{-1}(U)$ to $U\times G$, we may define another equivalence relation R, in E, as follows: points are equivalent if they are equivalent in G and project to the same point in the base space. We can construct, then, the quotient of the bundle E by this equivalence relation R and show that it is a principle bundle:

$$E/R=E''\ . \tag{2.18}$$

There is a natural projection from E to the equivalence classes E'',

$$p:E\rightarrow E''\ , \tag{2.19}$$

which preserves the projection on the base space $\pi(e)$:

$$\pi(p^{-1}(e''))=m,\quad e''\in E''\ . \tag{2.20}$$

We shall define the mappings π'' by

$$\pi'':E''\rightarrow M\ , \tag{2.21}$$

$$\pi''=\pi\circ p^{-1}\ . \tag{2.22}$$

Then π'' is a projection on M:

$$\pi''(e'')=\pi''(p(e))=\pi(e)=m\in M\ . \tag{2.23}$$

Consider the following diagram. Because of the isomorphism between G and $\pi^{-1}(m)=E_m$,

$$\begin{array}{ccc} E_m & \xrightarrow{\ \phi\ } & G \\ p\downarrow & & \downarrow p_f\ , \\ E''_m & \xrightarrow{\ \phi''\ } & G'' \end{array}$$

where p_f indicates projection to the equivalence classes in the fiber, we can define the mapping ϕ'' so that the diagram commutes:

$$\phi'':\pi''^{\,-1}(m)\rightarrow SL_1(2,C)\ , \tag{2.24}$$

$$\phi''\circ p=p_f\circ\phi\ . \tag{2.25}$$

Now we have a trivialization on E'' by defining t'':

$$t'':\pi''^{\,-1}(U)\rightarrow U\times G''\ , \tag{2.26}$$

$$t''(e'')=(\pi''(e''),\phi''(e''))=(m,g'')\ . \tag{2.27}$$

The equivalence classes may be characterized by the pair $m\in M$ and $g''\in G''$.

The group G'' may be considered to act on E'' by the definition

$$t''(e''\cdot g'')=(\pi''(e''),\phi''(e'')g'')\ . \tag{2.28}$$

It is clear that

$$\pi''(e''\cdot g'')=\pi''(e'')\ , \tag{2.29}$$

and E'' is a principal fiber bundle over M with $SL_1(2,C)$ as structure group.

To pass to SO(3,1) we notice that we have the following group quotients:

$$SL(2,C)=SL_1(2,C)/U(1)\ , \tag{2.30}$$

$$SO(3,1)=SL(2,C)/(I,-I)\ . \tag{2.31}$$

Because of the isomorphism between $\pi''^{\,-1}(m)$ and $SL_1(2,C)$, we can construct similar quotients in E''. We have

$$E''/(U(1)\times(I,-I))=E'\ , \tag{2.32}$$

and can define a natural projection homomorphism onto the equivalent classes:

$$h:E''\rightarrow E'\ . \tag{2.33}$$

In a similar way as was done before for E'', we can define a projection π',

$$\pi':E'\rightarrow M\ , \tag{2.34}$$

$$\pi'=\pi''\circ h^{-1}\ , \tag{2.35}$$

and a mapping ϕ',

$$\phi':E'\rightarrow G'\ , \tag{2.36}$$

$$\phi'\circ h=h_f\circ\phi''\ , \tag{2.37}$$

where h_f is the corresponding group homomorphism of the fibers. With the trivialization t',

$$t':\pi'^{\,-1}(U)\rightarrow U\times G'\ , \tag{2.38}$$

$$t'(e')=(\pi'(e'),\phi'(e'))\ , \tag{2.39}$$

the group action may be defined by

$$t'(e'\cdot g')=(\pi'(e'),\phi'(e')g')\ , \qquad (2.40)$$

and E' may be recognized as a principal fiber bundle over μ with SO(3,1) as structure group.

We see that the mapping μ needed to go from the fundamental equation obtained from multipole analysis to the equation for a Lorentz rotation is the composition of the two previous mappings:

$$\mu=h\circ p\ . \qquad (2.41)$$

Now we use a definite trivialization to write explicit expressions for the mapping and its differential. The action of p may be simply stated, on $e=\exp(a)$, by

$$p(e)=\exp[\text{even}(a)]=\exp[\text{co}(a)]\ , \qquad (2.42)$$

where co indicates taking the complex part. It is clear that the differential mapping p_{f*}, acting on the tangent space to the group space near the identity, means to take the complex part of the Lie algebra. For simplicity we omit the f in p_f since there should be no confusion. We have

$$p_{*I}:\text{sl}(2,Q)\rightarrow \text{sl}_1(2,C)\ , \qquad (2.43)$$

$$p_{*I}(a)=\text{co}(a),\quad a\in \text{sl}(2,Q)\ . \qquad (2.44)$$

Consider now the homomorphism h_f and its differential, omitting the f as before:

$$h_{*I}:\text{sl}_1(2,C)\rightarrow \text{so}(3,1)\ , \qquad (2.45)$$

$$h^{\beta}_{\alpha}(e')=\tfrac{1}{2}\text{tr}(e'^{\dagger}\sigma_\alpha e'\sigma^\beta)\ , \qquad (2.46)$$

$$h^{\beta}_{\alpha*b}(a)=\tfrac{1}{2}\text{tr}(b^{\dagger}\sigma_\alpha a\sigma^\beta+a^{\dagger}\sigma_\alpha b\sigma^\beta)\ , \qquad (2.47)$$

$$h^{\beta}_{\alpha*I}(a)=\tfrac{1}{2}\text{tr}(\sigma_\alpha a\sigma_\beta+a^{\dagger}\sigma_\alpha\sigma^\beta)\ . \qquad (2.48)$$

Then the composition mapping μ may be written, with the chosen trivialization, as

$$\mu_{*I}(a)=\tfrac{1}{2}\text{tr}[\sigma_\alpha \text{co}(a)\sigma^\beta+\text{co}(a^{\dagger})\sigma_\alpha\sigma^\beta]\ . \qquad (2.49)$$

This mapping allows us to obtain an equation for the tangent vector to the curve a' in A', from the knowledge of an equation for the tangent vector to a in A, as discussed in detail in a later section.

III. EQUATIONS OF MOTION

To obtain the equations of motion, we shall give to $^*J$ a multipole structure in terms of delta functions. Using a method given by Tulczyjew,[30] we shall obtain equations relating the multipole terms starting from the conservation equation:

$$DJ=0\ . \qquad (3.1)$$

The multipole structure is, along a curve $x'(s)$ with tangent vector ξ^μ,

$$^*J^{\mu A}{}_B=\int \tau^{\mu' A'}{}_{B'}\delta^{\mu AB'}_{\mu' A' B}(x-x')ds-\int \nabla_\alpha[\tau^{\alpha'\mu' A'}{}_{B'}\delta^{\alpha\mu AB'}_{\alpha'\mu' A' B}(x-x')]ds+\cdots$$
$$+\sum_n \frac{(-1)^n}{n!}\int \nabla_{\alpha_1}\nabla_{\alpha_2}\cdots\nabla_{\alpha_n}[\tau^{\alpha_1'\cdots\alpha_n' A'}{}_{B'}\delta^{\alpha\cdots AB'}_{\alpha'\cdots A'B}(x-x')]ds\ . \qquad (3.2)$$

We decompose the first two multipole terms as

$$\tau^{\mu A}{}_B=\xi^\mu m^A{}_B+m^A{}_B{}^\mu\ , \qquad (3.3)$$

$$\tau^{\mu\nu A}{}_B=\xi^\mu\xi^\nu\eta^A{}_B+\xi^\mu\eta_1{}^A{}_B{}^\nu+\xi^\nu\eta_2{}^A{}_B{}^\mu+\eta^A{}_B{}^{\mu\nu}\ , \qquad (3.4)$$

where the following relations hold:

$$\xi^\mu\xi_\mu=1\ , \qquad (3.5)$$

$$m^A{}_B=\tau^A{}_B{}^\mu\xi_\mu\ , \qquad (3.6)$$

$$\eta^A{}_B=\tau^{\mu\nu A}{}_B\xi_\mu\xi_\nu\ , \qquad (3.7)$$

$$\eta_1{}^A{}_B{}^\nu=\tau^{\mu\nu A}{}_B\xi_\mu-\xi^\nu\eta^A{}_B\ , \qquad (3.8)$$

$$\eta_2{}^A{}_B{}^\mu=\tau^{\mu\nu A}{}_B\xi_\nu-\xi^\mu\eta^A{}_B\ , \qquad (3.9)$$

so that m, η_1, and η_2 are orthogonal to ξ^μ.

The method to get the equations of motion is based on the following lemmas, which are still valid in the present context.

Lemma 1,

$$\int_{-\infty}^{\infty} ds\ \nabla_\mu[\xi^{\mu'}a_{B'}^{A'\alpha'\beta'\cdots}\ \delta^{\mu B' A\alpha\beta\cdots}_{\mu' B A'\alpha'\beta'\cdots}(x-x')]=\int_{-\infty}^{\infty} ds\frac{D}{ds}(a_{B'}^{A'\alpha'\beta'\cdots})\delta^{B' A\alpha\beta\cdots}_{BA'\alpha'\beta'\cdots}(x-x')\ . \qquad (3.10)$$

An expression of δ's has meaning after integration over its argument. Contracting with arbitrary $\psi_A{}^B{}_{\alpha\beta}$ and integrating, the expression can be verified.

Lemma 2. If we have

$$N^{AB\cdots}=\sum_{k=0}^{n}\int ds\nabla_{\mu_1}\nabla_{\mu_2}\cdots\nabla_{\mu_k}[\nu^{\mu_1'\mu_2'\cdots\mu_k' A'B'\cdots}\ \delta^{\mu_1\cdots A\cdots}_{\mu_1'\cdots A'\cdots}(x-x')]\ , \qquad (3.11)$$

$$\int N^{AB\cdots}K_{AB\cdots}d^4x=0\ , \qquad (3.12)$$

where the expressions ν are symmetric in the μ indices and orthogonal ξ^μ, then the vanishing of the ν is a necessary condition,

$$\nu^{\mu_1\mu_2\cdots\mu_k AB\cdots}=0\ . \qquad (3.13)$$

To verify this lemma, we contract with arbitrary $K_{AB\cdots}$ and integrate by parts,

$$\sum(-1)^k \nabla_\mu K v^{\mu\nu}=0 , \tag{3.14}$$

and then go over the rest system and make use of the arbitrariness of K, assuming that all space derivatives of order lower than k vanish.

Using lemma 1, the conservation equation

$$\nabla_\mu {}^*J^{\mu A}{}_B=0 \tag{3.15}$$

may be transformed to obtain

$$\int\left[\left[\frac{D}{ds}(m^A{}_B)-[\Omega_{\mu\nu},S^{\mu\nu}]^A{}_B\right]\delta(x-x')+\nabla_\mu\left[m^A{}_B-\frac{D}{ds}(N^A{}_B{}^\mu)\delta(x-x')\right]-\nabla_\mu\nabla_\nu[\eta^A{}_B{}^{(\mu\nu)}\delta(x-x')]\right]ds=0 , \tag{3.16}$$

where

$$N^A{}_B{}^\mu=\xi^\mu\eta^A{}_B+\eta_1{}^A{}_B{}^\mu+\eta_2{}^A{}_B{}^\mu , \tag{3.17}$$

$$S^A{}_B{}^{\mu\nu}=\eta_1{}^A{}_B{}^{[\mu}\xi^{\nu]} . \tag{3.18}$$

From Eq. (3.16) we obtain the multipole equations, using lemma 2:

$$\eta^A{}_B{}^{(\mu\nu)}=0 ,$$
$$m^A{}_B{}^\mu-\frac{D}{ds}N^A{}_B{}^\mu=0 , \tag{3.19}$$

$$\frac{D}{ds}m^A{}_B-[\Omega_{\mu\nu},S^{\mu\nu}]^A{}_B=0 . \tag{3.20}$$

The last equation determines the evolution of an element of sl(2,Q) in terms of a parameter s associated with a given curve x^μ in the base space-time manifold M. On the other hand, the object of the calculation is to obtain, precisely, the curve x in M. If this curve is not known *a priori,* Eq. (3.19) is not sufficient to determine the complete evolution of $m^A{}_B$. Additional information is needed. If we impose the physical requirement of identifying the tangent to the curve x with the timelike vector of the tetrad induced by $m^A{}_B$, Eq. (3.19) expresses the evolution along the integral curve tangent to the timelike vector of the tetrad. In particular, this requirement is sufficient to obtain an equation for the evolution of the timelike vector along its own direction. The integral curve determined in this form is the space-time curve desired, indicating the motion of the particle.

IV. LORENTZ EQUATIONS AT MOTION FOR CHARGED PARTICLES

Since E is a principal bundle, there is a natural action of the structure group G on the bundle by right multiplication. In particular, a curve in the Lie algebra of the structure group induces a vertical curve on E.

It follows that a tangent vector $\dot{a}$ to a curve in A maps into a vertical tangent vector on E' by right action of the algebra at some point $e'\in E'$:

$$\dot{c}'_{c'}=c'\dot{a}' . \tag{4.1}$$

The equation we have obtained in Sec. III for the tangent vector $\dot{a}$, Eq. (3.20), is of the type

$$\dot{a}=[\Omega,S] , \tag{4.2}$$

where Ω is the curvature and S is a tensor defined in Eq. (3.18).

The mapping μ maps the curve c onto a curve $\mu\circ c$ in E'. The differential μ_* defines an equation for the tangent vector to c' in E',

$$\dot{a}'=\mu_{*I}\dot{a}=\mu_*([\Omega,S]) , \tag{4.3}$$

$$\dot{c}'=c'\mu_{*I}([\Omega,S]) . \tag{4.4}$$

In other words, the mapping μ_* allows us to find an equation for a curve in E'.

We are interested only in the evolution of the four-velocity of the particle associated with the tetrad. Furthermore, we should make the tangent vector ξ to the path in space-time, correspond to the timelike form of the tetrad, θ^0. We have

$$D_\xi\xi=D_\xi\theta^0=\dot{\theta}^0 . \tag{4.5}$$

Using Eqs. (4.4) and (2.49) we obtain

$$\dot{\theta}^\alpha_\beta=\tfrac{1}{2}\mathrm{tr}[(\mathrm{co}a)^\dagger\sigma_\beta\sigma^\alpha+\sigma_\beta(\mathrm{co}a)\sigma^\alpha] . \tag{4.6}$$

Considering the equation for the tangent vector, Eq. (4.2), we get

$$\dot{\theta}^0_\beta=\tfrac{1}{2}\mathrm{tr}(\mathrm{co}[\Omega,S]^\dagger\sigma_\beta+\sigma_\beta\mathrm{co}[\Omega,S]) . \tag{4.7}$$

In order to obtain the Lorentz equation of motion, the commutator $[\Omega,S]$ must satisfy certain requirements. For this purpose we expand the curvature Ω in terms of the generators of the Lie algebra sl(2,Q). We shall single out one generator as the one related to electromagnetism and indicate it by the symbol E. The curvature two-form associated to E will be identified with the electromagnetic tensor $F_{\mu\nu}$. Additionally, we express the tensor S in terms of its defining multipole terms $\eta^A{}_B{}^\mu$ as indicated in Sec. III. The explicit calculation of Eq. (4.7) leads to various terms. We shall assume that the terms not related to E are small and therefore we shall keep only the electromagnetic terms of interest. We have

$$\dot{\theta}^0=\tfrac{1}{2}\theta^\beta F_{\mu\nu}\xi^\nu\mathrm{tr}(\mathrm{co}[E,\eta^\mu]^\dagger\sigma_\beta+\sigma_\beta\mathrm{co}[E,\eta^\mu])$$
$$+\text{other terms} . \tag{4.8}$$

In the Appendix it is found that there are two equivalent possibilities for the electromagnetic generator if, for simplicity, we impose the requirement that it should be one element of the chosen basis of the Lie algebra. Choosing then, for the ansatz,

$$E=\begin{bmatrix}0 & j\\ -j & 0\end{bmatrix} , \tag{4.9}$$

a direct calculation leads to

$$[E,\eta^p]=g\delta_a^p\sigma_a-i\eta^{p\,32}\ , \tag{4.10}$$

where the following conditions are required for Eqs. (A18)–(A20), in terms of a constant q:

$$\eta^{p\,30}=\delta_1^p q\ , \tag{4.11}$$

$$\eta^{p\,23}=\delta_1^p q\ , \tag{4.12}$$

$$\eta^{p\,21}=-\delta_3^p q\ . \tag{4.13}$$

Then it follows that, substituting Eq. (4.10) in Eq. (4.8),

$$\dot\theta^0=\tfrac{1}{2}\theta^\beta F_{\mu\nu}\xi^\nu \mathrm{tr}[\,\mathrm{co}(q\delta_a^\mu\sigma^a-\delta_p^\mu\eta^{p\,30}i)\sigma_\beta + \sigma_\beta \mathrm{co}(q\delta_a^m\sigma^a-\delta_p^\mu\eta^{p\,32}i)]+\cdots\ , \tag{4.14}$$

$$\dot\theta^0=\tfrac{1}{2}qF_{\mu\nu}\xi^\nu\theta^\beta \mathrm{tr}(\delta_a^\mu\sigma^a\sigma_\beta+\delta_a^\mu\sigma_\beta\sigma^a)+\cdots \tag{4.15}$$

$$=qF_{\mu\nu}\xi^\nu\theta^\beta(\delta_\nu^\mu-\delta_0^\mu\delta_\beta^0)=q\theta^\beta F_{\beta\nu}\xi^\nu+\cdots\ , \tag{4.16}$$

which is equivalent to

$$\left[\frac{D\xi}{ds}\right]_\mu=qF_{\mu\nu}\xi^\nu+\cdots\ , \tag{4.17}$$

or, in other words, to the Lorentz equation for a particle with electric charge q. Thus we have obtained, in this approximation, the correct equation of motion for charged particles from the conservation law for J.

V. INDUCED METRIC AND CONNECTION

We shall designate by frame e' in M an ordered basis (e_0',e_1',e_2',e_3') for TM_m, the tangent space to M at $m\in M$, at all points $m\in U$ for a neighborhood $U\subset M$. The matrix formed by taking the vectors of the basis as columns is an element of the group SO(3,1). We see that a frame is a section in the principal bundle E'. A plain theory of gravitation is compatible with SO(3,1) as the structure group.[31,32]

The mapping μ, previously defined, maps a given point in E to a point in E'. A section in E is mapped by μ into a section in E'. It follows that a section in E canonically defines a frame. Furthermore, using θ, the dual basis of e, the given section in E canonically defines a metric in M by

$$g=\tilde\theta\eta\theta\ , \tag{5.1}$$

where η is the Minkowski metric. In component form,

$$g_{\mu\nu}=\eta_{\alpha\beta}\theta_\mu^\alpha\theta_\nu^\beta\ . \tag{5.2}$$

With the same trivializations used in the preceding section, we can write, for a spinor basis $\phi\in \mathrm{SL}(2,C)$,

$$\theta_\mu^\alpha=\tfrac{1}{2}\mathrm{tr}(\phi^\dagger\sigma_\mu\phi\sigma^\alpha)\ , \tag{5.3}$$

and in component form,

$$g_{\mu\nu}=\eta_{\alpha\beta}\phi^{\bar A}{}_{\bar M}\sigma^\alpha_{A\bar B}\phi^B_N\sigma_\mu^{N\bar M}\phi^{\bar C}_{\bar X}\sigma^\beta_{\bar C D}\phi^D_Y\sigma_\nu^{Y\bar X} \tag{5.4}$$

$$=\epsilon_{AC}\epsilon_{\bar B\bar D}\gamma_\mu^{A\bar B}\gamma_\nu^{C\bar D}\ . \tag{5.5}$$

The elements ϕ are obtained from a given point in the original bundle E by the action of the mapping p as described in Sec. II. We see that a section in E determines the metric in M in this canonical way.

In addition, a connection ω in E also induces a connection in E'. We shall recall a definition for a connection in a fiber bundle.[33] A connection is a Lie-algebra-valued one-form ω on a principal bundle, such that

$$\omega(\sigma(a))=a\ , \tag{5.6}$$

where $a\in A$, the Lie algebra of the group G, and σ is the natural vertical vector induced by the algebra. Furthermore, it is required that

$$\omega(R_{g*}v)=\mathrm{ad}(g^{-1})\omega(v)\ , \tag{5.7}$$

where $v\in TE$, R_g is right multiplication by g, and ad is the adjoint map. If we define the induced connection in E'' by

$$\omega'':TE''\to \mathrm{sl}_1(2,C)\ , \tag{5.8}$$

$$\omega''(p_*v)=p_{*I}(\omega(v))\ , \tag{5.9}$$

in terms of the given connection ω and the projection mapping p, we have, in some trivialization,

$$p(m,g)=(m,g'')\ , \tag{5.10}$$

where

$$g''\in \mathrm{SL}_1(2,C)\ . \tag{5.11}$$

We can define a vector field σ'' over the subspace of E determined by $G''\supset G$, by the restriction of σ to this subspace. Then the connection may be expressed in terms of the even part of the algebra:

$$\omega''(\sigma''(p_{*I}q))=\omega''(p_{*I}\sigma(q))\ , \tag{5.12}$$

where $q\in \mathrm{sl}(2,Q)$ and $p_{*I}q\in \mathrm{sl}_1(2,C)$. The commutation is possible because $\sigma''(p_*q)$ is complex (not quarternionic).

It follows from the definition of ω'' that

$$\omega''(\sigma''(p_{*I}q))=p_{*I}\omega(\sigma(q))=p_{*I}q\ , \tag{5.13}$$

and ω'' satisfies one of the requirements to be a connection. Furthermore,

$$\begin{aligned}\omega''(R_{g''*I}p_*v)&=\omega''(p_*R_{g''*}v)=p_{*I}\omega(R_g''v)\\&=p_{*I}g''^{-1}\omega(v)g''=g^{-1}p_{*I}\omega''(v)g''\\&=g''^{-1}\omega(p_*v)g''=\mathrm{ad}(g''^{-1})\omega(p_*v)\ ,\end{aligned} \tag{5.14}$$

and ω'' satisfies the second requirement to be a connection on E''.

In the particular trivialization chosen, the definition of ω'' gives

$$\omega''(\mathrm{co}v)=\mathrm{co}[\omega(v)] \tag{5.15}$$

and it follows, in component form,

$$\omega_\nu''^{\mu}=\mathrm{co}(\omega^A{}_C)\gamma_\nu^{C\bar B}\gamma^\mu_{A\bar B}+\mathrm{co}(\omega^{\bar B}{}_{\bar C})\gamma_\nu^{A\bar C}\gamma^\mu_{A\bar B}\ , \tag{5.16}$$

which coincides with a definition for an $\mathrm{sl}_1(2,C)$ connection used in previous work.[34] It is known that a connection of this type satisfies

$$D''\gamma=0\ , \tag{5.17}$$

which, in turn, implies

$$D''g=0\ . \tag{5.18}$$

The metric and affine structures canonically induced from the frame in M are compatible.

CONCLUSIONS

We have shown that it is possible to obtain the Lorentz equation of motion from the conservation law implied by the field equation of the unified theory under consideration. This theory was constructed by identifying the gravitational and electromagnetic fields with a connection on the principal fiber bundle with SL$(2,Q)$ as structure group and space-time as the base manifold.

In addition, the field equations give in the proper limit, when the structure group is reduced to a U(1) subgroup, the Maxwell field equations for electromagnetism. Also, when the group is reduced to SL$(2,C)$, we obtain a theory of gravitation which includes as a solution the Schwarzschild space-time and therefore leads to Newtonian motion under a $1/r$ gravitational potential for weak fields.

It appears possible to find additional predictions related to the rich algebraic and geometric structure of the principal fiber bundle, which ties together in a nontrivial way the gravitational electromagnetic and other interactions.

APPENDIX

We shall look for a possible generator Z that leads to a Lorentz equation of motion. One possibility is that the commutator in Eq. (4.8) contains the Pauli matrices in a manner similar to

$$[Z,\eta^p]=\delta^p_a\sigma_a\ . \tag{A1}$$

For simplicity, let us assume that Z is one element of a basis for the algebra, taken of the form $k_\alpha\kappa_\beta$, where k_α is basis for the quarternions and κ_α a basis for the matrices; that is,

$$k_0=1,\ \ k_1=i,\ \ k_2=j,\ \ k_3=k\ , \tag{A2}$$

$$\kappa_0=\sigma_0,\ \ \kappa_1=\sigma_1,\ \ \kappa_2=i\sigma_2,\ \ \kappa_3=\sigma_3\ . \tag{A3}$$

We have the following three possibilities.

Case 1:

$$Z=k_0\kappa_b=\kappa_b\ , \tag{A4}$$

$$\begin{aligned}[][\kappa_b,\eta^{p\nu\mu}k_\mu\kappa_\nu]&=\eta^{p\mu\nu}k_\mu[\kappa_b,\kappa_\nu]\\ &=\eta^{p\mu n}k_\mu\epsilon_{bnr}\tilde{\kappa}_r\ .\end{aligned} \tag{A5}$$

Because of the presence of ϵ, we see, by inspection, that one of the κ matrices does not appear in the final expression. This case is not possible because it does not give the desired result.

Case 2:

$$Z=k_a\kappa_0\ , \tag{A6}$$

$$\begin{aligned}[][k_a\kappa_0,\eta^{p\mu\nu}k_\mu\kappa_\nu]&=\eta^{p\mu\nu}\kappa_\nu[k_a,k_\mu]\\ &=\eta^{pm\nu}\kappa_\nu\epsilon_{amr}k_r\ .\end{aligned} \tag{A7}$$

This case is not possible because, since the η are real, only the σ_2 matrix appears.

Case 3:

$$Z=k_a\kappa_b\ , \tag{A8}$$

$$[k_a\kappa_b,\eta^{p\mu\nu}k_\mu\kappa_\nu]=\eta^{p\mu\nu}[k_a\kappa_b,k_\mu\kappa_\nu]\ . \tag{A9}$$

It can be determined that, if $b=1$, κ_1 does not appear. Also, if $b=3$, κ_3 does not appear. Therefore $b=2$ and, in this case, for $a=1$, the expression gives zero. Then we have only two choices. Choice $b=2$, $a=2$,

$$[Z,\eta^p]=\eta^{p\,30}\sigma_2+\eta^{p\,23}\sigma_1-\eta^{p\,21}\sigma_3\ , \tag{A10}$$

$$\eta^{230}=1,\ \ \eta^{123}=1,\ \ \eta^{321}=-1 \tag{A11}$$

Choice $b=2$, $a=3$,

$$[Z,\eta]=-\eta\sigma_2+\eta\sigma_1-\eta\sigma_3\ , \tag{A12}$$

$$\eta^{220}=-1,\ \ \eta^{133}=1,\ \ \eta^{331}=-1\ . \tag{A13}$$

We notice that the two possibilities are

$$Z=k_2\kappa_2\ , \tag{A14}$$

$$Z=k_3\kappa_2\ . \tag{A15}$$

Consider now that Z is given by Eq. (A14). The commutator gives

$$\begin{aligned}[][Z,\eta^p]&=\eta^{p\mu\nu}[k_2\kappa_2,k_\mu\kappa_\nu]\\ &=\nu^{pm\,0}\kappa_2\epsilon_{2m\,1}i+\eta^{p\,2m}(\epsilon_{2m\,1}\kappa_1+\epsilon_{2m\,3}\kappa_3)\\ &\quad-\eta^{pm\,2}\epsilon_{2m\,1}i\ ,\end{aligned} \tag{A16}$$

and the most general η^p corresponds to

$$\eta^{p\,30}=q\delta^p_2\ , \tag{A17}$$

$$\eta^{p\,23}=q\delta^p_1\ , \tag{A18}$$

$$\eta^{p\,21}=-q\delta^p_3\ , \tag{A19}$$

giving for the commutator,

$$[Z,\eta^p]=q\sigma^p-i\eta^{p\,32}\ . \tag{A20}$$

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