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Can black holes nucleate vacuum phase transitions?

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The decay of false-vacuum states through a first-order phase transition in the presence of a Schwarzschild black hole is studied in the zero-temperature limit. The equations of motion for a thin-wall bubble which forms in a spherically symmetric fashion around a Schwarzschild—de Sitter (or —anti—de Sitter) black hole are derived. The Euclidean action for these bubble solutions is calculated and is found to be smaller, by up to a factor of roughly 2, than the O(4)-symmetric action in which no black hole is present. It thus appears that a black hole can act as an effective nucleation center for a first-order vacuum phase transition. It will then be more difficult (require more "fine-tuning" of the potential) than was previously believed to achieve supercooling of a false-vacuum state, provided that appropriate-mass black holes are present.

I. INTRODUCTION

Since the widespread adoption of gauge field theories with spontaneous symmetry breaking to describe elementary-particle interactions, it has become commonplace to assume that the Universe, in the course of its expansion and cooling, has undergone a number of vacuum phase transitions. These phase transitions involve a downward change in the value of the vacuum energy density as the symmetries of the fundamental gauge group are spontaneously broken.

The basic theory of the decay of the false vacuum was first developed by Voloshin *et al.*[1] and Coleman.[2] The gravitational effects of nonzero vacuum energy densities were first considered by Coleman and De Luccia[3] (and extended by Parke[4] to arbitrary vacuum energy densities), and have been thoroughly studied since, especially since the development of inflationary cosmologies.[5–7] Essentially all of these studies have, however, made the simplifying assumption of (Euclidean) O(4) symmetry, thus restricting the geometry to be of the de Sitter, Minkowski, or anti—de Sitter forms.

In this paper I examine whether an inhomogeneity in the spacetime geometry, namely, a nonrotating, uncharged black hole, can act as a nucleation site for bubbles of the lower-energy true vacuum. The initial state is taken to be described by either the Schwarzschild—de Sitter, Schwarzschild, or Schwarzschild—anti—de Sitter metric, depending on whether the initial (false-) vacuum energy density is, respectively, positive, zero, or negative. A bubble of the lower-energy true vacuum is assumed to form in a spherically symmetric fashion around the black hole and subsequently expands, converting the false vacuum to true.

The decay of the false vacuum in the presence of black holes is studied in the thin-bubble-wall approximation in the zero-temperature limit. The rate of formation of bubbles of true vacuum in a first-order phase transition has the form[2]

$$\Gamma = A \exp(-B/\hbar)[1+O(\hbar)] , \tag{1}$$

where Γ is the rate of bubble formation per unit four-volume, A is a coefficient with dimensions of $(\text{length})^{-4}$, and B is the difference between the Euclidean action for the bubble solution and the Euclidean action of the spacetime with no bubble (geometrical units have been chosen so that $G=c=1$). This paper is concerned only with calculating the exponential coefficient B. No attempt is made to evaluate A for the Schwarzschild—de Sitter geometry; such a calculation would necessarily involve examining a particular model field theory with spontaneous symmetry breaking. The effect of the presence of black holes on the exponential coefficient B, on the other hand, can be determined without restriction to any particular theory.

Without an explicit choice of model and calculation of the change in the value of A caused by the black-hole geometry, it is impossible to determine the precise numerical effect of black holes on the vacuum decay rate. However, whatever the change in the value of A is, it will be overwhelmed by the change in $\exp(-B/\hbar)$ when $B/\hbar$ is large enough (i.e., in a first-order phase transition with extreme supercooling). For the remainder of this paper, I will rather glibly refer to decreased values of B as directly implying an increase in the decay rate of the false vacuum when, of course, knowledge of the value of A is needed to make such a statement precise.

In the earlier works cited above, the O(4) symmetry of the bubble solutions allowed one to easily calculate the total Euclidean action, and then perform the variation to find the equations of motion for the bubble wall. In the present case, with the black hole present, it proves easier to first solve the equations of motion (in the thin-wall approximation) directly to obtain the bubble solution, and then calculate the Euclidean action by directly integrating the solution.

Black holes as nucleation sites for vacuum phase transitions have previously been considered by Hawking[8] and Moss.[9] In their studies the black hole acted primarily as a source of thermal radiation to support a semiopaque bubble of false vacuum surrounded by a true vacuum against collapse. In the present case, attention is focused instead on the role of the classical curved geometry of the Schwarzschild (or Schwarzschild—de Sitter or Schwarzschild—anti—de Sitter) metric and its effect on first-order phase transitions in the zero-temperature limit. The additional outward pressure which would exist on the bubble wall due to its partial opacity to the Hawking radiation, and the finite-temperature corrections to the Higgs potential are ignored. A criterion for determining when it is appropriate to ignore the nonzero temperature of the black hole will be developed in Sec. IV.

The primary result obtained is that the Euclidean action for a bubble solution in the presence of a black hole is always less than the equivalent O(4)-symmetric bubble action with no black hole. For fixed values of the false- and true-vacuum energy densities, there is a maximum-mass black hole for which solutions to the Euclidean equations of motion exist; the smallest action is obtained when the black hole has this maximum mass. The action can be as small as $\frac{4}{3}(1-3^{-1/2})\approx 0.5635$ times the O(4)-symmetric action. Thus, a black hole can act in much the same manner as an impurity does in an ordinary material phase transition; it can rapidly nucleate bubbles of the new phase, greatly hastening the phase transition. The effects of black-hole bubble nucleation will be greatest when the exponential coefficient $B/\hbar$ is large and hence the vacuum decay rate small, i.e., in phase transitions in which supercooling is expected to take place. Possible quickening of the phase transition by nucleation may thus have important consequences for model cosmologies which involve extended periods of expansion in a supercooled false-vacuum state, as some (particularly "old") inflationary models do. There may also be interesting implications for any theory which predicts that we are living in a (long-lived) metastable false-vacuum state today.

The rest of the paper is organized as follows. In Sec. II the equations of motion are derived for a spherically symmetric thin bubble wall, centered on a nonrotating, uncharged black hole, with arbitrary values of the vacuum energy density inside and outside the bubble. In Sec. III the Euclidean action for the black-hole bubble solutions is calculated and compared with the O(4) (no-black-hole) action. Finally, in Sec. IV, the range of validity of the approximations made in this calculation is estimated (e.g., when is ignoring the temperature of the black hole reasonable?), and the possible importance of these results is briefly discussed.

II. BLACK-HOLE BUBBLE-WALL MOTIONS

In the thin-wall approximation,[2] the Higgs scalar field (or its equivalent) is assumed to vary in a stepwise discontinuous fashion from its false-vacuum to true-vacuum values. On each side of the discontinuity the stress-energy tensor is then simply of the form $-\rho g_{\mu\nu}$, where ρ is the value of the vacuum energy density. The discontinuity itself is idealized as a three-dimensional timelike hypersurface containing a surface stress energy of the form $-\sigma h_{ab}$, where σ is the surface energy density and h_{ab} is the intrinsic metric of the three-surface. All of the potential complexities of spontaneous symmetry breaking are thus reduced in this approximation to three numbers: the false- and true-vacuum energy densities, and the surface energy density.

In this section the equations of motion for a spherically symmetric infinitesimally thin bubble wall separating two different (arbitrary) values of the vacuum energy density and centered on a nonrotating, uncharged black hole are developed, using the well-known formalism for dealing with surface layers in general relativity which was developed by Israel.[10] The treatment is similar to that of Berezin, Kuzmin, and Tkachev;[11] unlike their work, however, I shall require that the Schwarzschild mass parameter have the same value inside and outside the bubble. This restriction guarantees that the bubble itself is always formed with a total mass of zero (that bubbles are formed with precisely zero mass, and the manner in which this is inherent in the Coleman–de Luccia bubble solutions, was emphasized by Weinberg[12]). The calculations in this section are all performed with Lorentzian metric signature.

The spacetime inside and outside the bubble wall is described by some combination of Schwarzschild–de Sitter (if the vacuum energy density is positive); Schwarzschild (if the vacuum energy density is exactly zero); and Schwarzschild–anti–de Sitter (if the vacuum density is negative) metrics. Inside the bubble, the spacetime metric is given by one of these three metrics with vacuum energy density ρ_1:

$$ds^2=-\left[1-\frac{2M}{r}-\frac{8\pi\rho_1 r^2}{3}\right]dT^2+\left[1-\frac{2M}{r}-\frac{8\pi\rho_1 r^2}{3}\right]^{-1}dr^2+r^2d\Omega^2\ , \qquad (2)$$

where $d\Omega^2$ is the two-sphere metric, $d\Omega^2 \equiv d\theta^2+\sin^2\theta d\phi^2$. Outside, the metric will have the same form but with a vacuum energy density now equal to ρ_2:

$$ds^2=-\left[1-\frac{2M}{\tilde r}-\frac{8\pi\rho_2 \tilde r^2}{3}\right]dt^2+\left[1-\frac{2M}{\tilde r}-\frac{8\pi\rho_2 \tilde r^2}{3}\right]^{-1}d\tilde r^2+\tilde r^2d\Omega^2\ . \qquad (3)$$

The solution of the Einstein equations for the infinitesimally thin bubble wall consists of two parts: first, matching the intrinsic metrics induced on the wall from each side; and second, relating the difference in the interior and exterior extrinsic curvatures to the bubble-wall surface stress energy. A spherically symmetric bubble wall will follow a path $R(\tau)$, where τ is the proper-time coordinate of the bubble wall and R is the curvature radial coordinate, i.e., the square root of the area of the bubble wall at proper time τ divided by 4π. Equality of the induced two-sphere metrics on the surface $R(\tau)$ then requires that, on the surface, $r=\tilde r=R(\tau)$. The full three-dimensional induced intrinsic metrics will then be equivalent if the interior time coordinate T and the exterior time coordinate t are related to the proper time of the wall, τ, by

$$\frac{dT}{d\tau}=\left\{\left[1-\frac{2M}{R}-\frac{8\pi\rho_1R^2}{3}\right]^{-1}\times\left[1+\left[1-\frac{2M}{R}-\frac{8\pi\rho_1R^2}{3}\right]^{-1}\dot R^2\right]\right\}^{1/2}\ , \qquad (4)$$

$$\frac{dt}{d\tau}=\left\{\left[1-\frac{2M}{R}-\frac{8\pi\rho_2R^2}{3}\right]^{-1}\times\left[1+\left[1-\frac{2M}{R}-\frac{8\pi\rho_2R^2}{3}\right]^{-1}\dot{R}^2\right]\right\}^{1/2}, \quad (5)$$

where an overdot denotes differentiation with respect to proper time.

The extrinsic curvature tensor of the three-surface must be calculated in both the interior and exterior geometries. Let e_a^μ be an orthonormal triad constructed on the surface; its components in the (T or t,r,θ,ϕ) coordinate system are chosen to be

$$e_1^\mu=u^\mu,\quad e_2^\mu=\left[0,0,\frac{1}{R},0\right],\quad e_3^\mu=\left[0,0,0,\frac{1}{R\sin\theta}\right], \quad (6)$$

where u^μ is the four-velocity of the surface. The outward-pointing unit normal to the three-surface has components

$$n_\alpha^-=(-\dot{R},u^T,0,0)\ , \quad (7)$$

in the interior, and

$$n_\alpha^+=(-\dot{R},u^t,0,0)\ , \quad (8)$$

in the exterior. The extrinsic curvature tensor is then defined by

$$K_{ab}^{\pm}=-e_a^\mu e_b^\nu n_{\mu;\nu}^{\pm}\ . \quad (9)$$

The Einstein equations then reduce to

$$\gamma_{ab}-h_{ab}\gamma=8\pi S_{ab}\ , \quad (10)$$

where γ_{ab} is the difference between the exterior and interior extrinsic curvatures, $\gamma_{ab}\equiv K_{ab}^+-K_{ab}^-$, γ is the trace of γ_{ab}, S_{ab} is the surface stress-energy tensor, and h_{ab} is the three-metric of the surface.

In the present case, with a spherically symmetric surface of discontinuity and a surface stress-energy tensor of the form $-\sigma h_{ab}$, the Einstein equations [Eq. (10)] reduce to just one equation of motion for the surface:

$$\left[1-\frac{2M}{R}-\frac{8\pi\rho_1R^2}{3}+\dot{R}^2\right]^{1/2}-\left[1-\frac{2M}{R}-\frac{8\pi\rho_2R^2}{3}+\dot{R}^2\right]^{1/2}=4\pi\sigma R\ . \quad (11)$$

Solutions to Eq. (11) describe bubble walls which, classically, either plunge inward from infinity to a turning point at some minimum radius, and then return to infinity, or bubbles which begin at zero radius, increase in size until a turning point is reached, and then collapse back to zero radius. The second class of solutions is not of interest in the current application, as collapsing bubbles cannot complete a phase transition. Quantum mechanically, only half of the classical bubble solution is relevant. In a first-order phase transition, bubbles of the new phase appear via quantum tunneling at the turning-point radius of the classical solutions; thereafter they expand outward toward infinity.

A common procedure at this point is to solve Eq. (11) for $\dot{R}$ by squaring; one must, however, be alert to the possibility of introducing spurious solutions by this procedure.[13] Thus, before proceeding further by solving for $\dot{R}$, it is worthwhile to examine Eq. (11) directly to determine for what range of the parameters (ρ_1,ρ_2,σ,M,R) solutions exist. *A priori*, the only restriction placed on the three densities is that the surface density σ should be positive (this is because the surface density is generated by the derivative terms in the scalar field stress-energy tensor; such terms have positive-definite sign, unlike the potential terms which contribute to ρ_1 and ρ_2). An acceptable solution must also have non-negative values for R and M. The first condition which the vacuum energy densities must satisfy is fairly obvious: if the left-hand side of Eq. (11) is to be positive for at least some values of R, then clearly $\rho_2>\rho_1$. This condition is not satisfied in the situation studied by Moss,[9] wherein a bubble of the (higher energy density) false vacuum is surrounded by the lower-energy-density true vacuum. The resolution of this apparent puzzle is that the Moss-type bubbles are supported against gravitational collapse by nongravitational forces: namely, the radiation pressure supplied by the hot central black hole. These additional forces are ignored in the present study. Further examination of Eq. (11) shows that there will be not acceptable solutions for $\rho_2>0$ (decay of Schwarzschild–de Sitter into any lower-energy-density space) unless

$$\rho_2\geq\rho_1+6\pi\sigma^2\ . \quad (12)$$

If $\rho_2>0$ and the equality in Eq. (12) is satisfied, then the minimum radius of the bubble is equal to the radius of the exterior de Sitter universe; violation of Eq. (12) would lead to a bubble with a minimum radius greater than the size of the external de Sitter space. If $\rho_2\leq0$ (decay of Schwarzschild–anti–de Sitter or Schwarzschild into Schwarzschild–anti–de Sitter space) then there will be no acceptable solutions unless

$$\left[\left[\frac{-\rho_2}{6\pi\sigma^2}\right]^{1/2}+1\right]^2\geq-\frac{\rho_1}{6\pi\sigma^2}\ . \quad (13)$$

When the equality in Eq. (13) is satisfied, the minimum radius of the bubble is infinity.

The restrictions on possible decays of the false vacuum imposed by Eqs. (12) and (13) are depicted graphically in Fig. 1. Several aspects of this figure have been previously noted, particularly the existence of the point at coordinates $(-1,0)$ limiting possible decays of Minkowski space into anti–de Sitter space (see Refs. 3 and 12) and the curve in the third quadrant representing Eq. (13) (see Ref. 4). The constraint described by Eq. (12), or equivalently the straight line proceeding diagonally upward from the point $(-1,0)$ has, however, escaped the notice of several workers in this area. The papers of both Coleman and De Luccia,[3] and of Parke,[4] state that the minimum (critical) bubble size is always less than or equal to the size of the exterior de Sitter universe when $\rho_2>0$. Direct examination of the bubble-wall equation of motion, *before* squar-

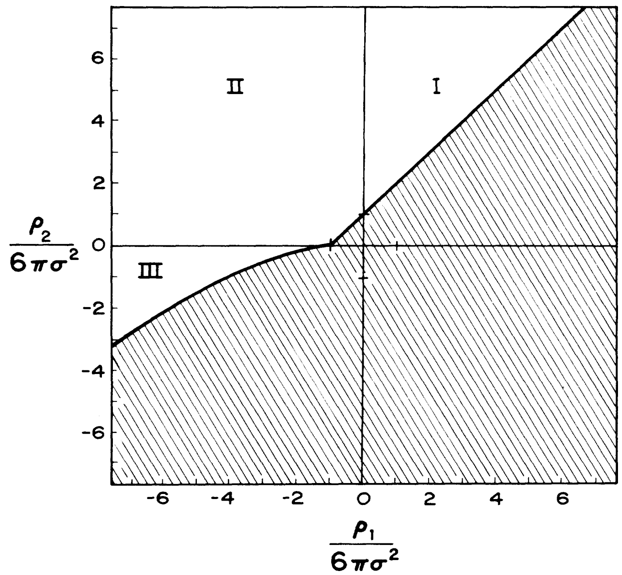

FIG. 1. Possible false-vacuum decays by the formation of thin-wall zero-temperature bubbles. The horizontal axis is the energy density of the true vacuum, the vertical axis is the energy density of the false vacuum. Both are rendered dimensionless by dividing by $6\pi\sigma^2$. The shaded region is forbidden; no solutions to the equations of motion exist in that region. The upper right quadrant, labeled I, represents decays of de Sitter space into de Sitter space; the upper left quadrant labeled II, represents decays of de Sitter space into anti–de Sitter space; and the lower left quadrant, labeled III, represents decays of anti–de Sitter space into anti–de Sitter space. The boundary of the forbidden region is given by Eqs. (12) and (13).

ing [Eq. (11)] clearly shows that this is incorrect. It should be emphasized that the restrictions on vacuum decay illustrated in Fig. 1 apply only to the thin-wall, zero-temperature approximation. It is interesting to note that the mass of the black hole does not enter into these restrictions on the vacuum energy densities in any interesting way.[14]

If Eq. (11) is now solved for $\dot{R}$, one obtains

$$\dot{R}^2=\beta^2R^2-1+\frac{2M}{R}\ , \tag{14}$$

where

$$\beta^2\equiv\left[\frac{\rho_2-\rho_1-6\pi\sigma^2}{3\sigma}\right]^2+\frac{8\pi\rho_2}{3}\ . \tag{15}$$

It should be remembered that since Eq. (11) was squared twice to yield Eq. (14), it contains extraneous solutions which do not satisfy Eqs. (12) or (13).

Comparing the evolution described by Eq. (14) to the previously known bubble solutions, for which $M=0$, there are several interesting changes. First, for small values of M, there are two zeros of the right-hand side of Eq. (14), and hence two radii at which bubbles can form. The outer radius at which $\dot{R}=0$ will be called R_0, and the inner radius R_1. In the limit as $M\rightarrow 0$, R_0 goes to the previously known value,[4] β^{-1}, and R_1 approaches zero: $R_1\sim 2M\rightarrow 0$. Bubbles formed at the outer radius expand forever, while those formed at the inner radius collapse into the central black hole. For any nonzero value of M, the outer radius is less than in the zero-mass case. As the mass of the black hole is increased, R_0 decreases and R_1 increases, approaching each other, until they meet when the roots of the cubic equation coalesce; for still larger values of M there are no solutions of Eq. (14) for $\dot{R}=0$ and $R>0$. Thus, bubbles can only form at rest and subsequently expand if

$$M\leq M_c\equiv 3^{-3/2}\beta^{-1}\ . \tag{16}$$

If equality holds in Eq. (16) then the bubble forms at $R=3M_c$, and stays there, neither expanding nor collapsing (it is, however, in unstable equilibrium).

III. EUCLIDEAN ACTION CALCULATION

In this section the Euclidean action will be calculated for the Euclidean metric signature equivalents of the solutions found in the last section, and compared to the Euclidean actions for the bubble solutions in the absence of a central black hole.

The Euclidean signature equivalents of the solutions found in the last section are easily found. The Euclidean Schwarzschild–de Sitter-type metrics are related to the Lorentzian metrics by simply making the replacement $\tilde{T}=iT$ in Eq. (2), and $\tilde{t}=it$ in Eq. (3). If the Euclidean time coordinate, $\tilde{T}$ (or $\tilde{t}$), is then identified with period $2\pi\kappa^{-1}$, where κ is the surface gravity of the event horizon, then the metric is completely regular at that horizon. Note, however, that if $\rho>0$, and $M\neq 0$, then there will exist two event horizons with different surface gravities; no choice of period for $\tilde{T}$ (or $\tilde{t}$) will allow the metric to be regular at both horizons. The horizon for which the period does not match the surface gravity will be a conical singularity.

The Euclidean signature equivalents of Eqs. (4) and (5), relating $\tilde{T}$ and $\tilde{t}$ to the Euclidean proper time, $\tilde{\tau}=i\tau$, and the equation of motion for the bubble wall, Eq. (14), are obtained by simply substituting $-i\tilde{T}$, $-i\tilde{t}$, and $-i\tilde{\tau}$ for T, t, and τ, respectively. Redefining $\dot{R}$ as $\dot{R}=dR/d\tilde{\tau}$, the signs of the terms in Eqs. (4), (5), and (14) containing $\dot{R}^2$ changes, since $\dot{R}=dR/d\tilde{\tau}=-idR/d\tau$ now. This change of sign for $\dot{R}^2$ changes the character of the motion described by Eq. (14). Bubble solutions in the Euclidean sector do not expand forever or collapse to zero radius (unless $M=0$). Instead, they oscillate between R_0 and R_1. Again, as the mass approaches the critical value given by Eq. (16), the inner and outer radii approach each other, until in the limiting case the bubble wall just sits at $R=3M$. In the Euclidean case, however, this constant radius solution is in stable equilibrium.

The exponential decay coefficient, B, is simply the difference between the Euclidean action of a solution to the Einstein equations containing a bubble and the action for the same spacetime (asymptotically) in the absence of a bubble, i.e.,

$$B\equiv S_E(\Phi)-S_E(\Phi_+)\ , \tag{17}$$

where $S_E(\Phi)$ represents the Euclidean action for a bubble solution, in which the Higgs scalar field varies between its true-vacuum value at $r=0$ and the false-vacuum value at infinity, and $S_E(\Phi_+)$ represents the Euclidean action of a purely false-vacuum spacetime with the Higgs field (and hence the vacuum energy density) everywhere equal to their false-vacuum values.

The Euclidean action for a solution of Einstein's equations with stress-energy tensor proportional to the metric tensor is given by

$$S_E=\int\left[\rho-\frac{R}{16\pi}\right]dV\ , \tag{18}$$

where R is now (in this equation only) the Ricci scalar curvature, not the radius of the bubble wall, and the integration is performed over the full four-volume of the Euclidean space.

It is natural to divide the integration to evaluate the Euclidean action into three parts: inside the bubble wall $[0<r<R(\tilde{\tau})]$, the wall itself $[r=R(\tilde{\tau})]$, and outside the bubble wall $[R(\tilde{\tau})<r<\infty]$.

After replacing the Ricci scalar curvature by its value obtained from the Einstein equations, $R=32\pi\rho$, and performing the trivial θ and ϕ integrations in Eq. (18), the interior action for the bubble has the form

$$S_E^{\rm int}(\Phi)=-4\pi\rho_1\int_{-\tilde{T}_0}^{\tilde{T}_0}\int_{{}_1r_+}^{R(\tilde{T})}r^2dr\,d\tilde{T}\ , \tag{19}$$

where ${}_1r_+$ is the interior (black-hole) horizon radius for the true-vacuum metric with energy density ρ_1. The bubble is assumed to form at its minimum radius R_1 at time $-\tilde{T}_0$, expand to its maximum radius R_0 at time $\tilde{T}=0$, and then recontract to R_1 at time $\tilde{T}_0$. The actual Euclidean solution, of course, oscillates between R_0 and R_1 for all values of $\tilde{T}$ from $-\infty$ to $+\infty$; in order to yield a finite action the integration is performed over only one period of the oscillation [i.e., the Euclidean time coordinate is identified with a period $(2\tilde{T}_0)$ determined by the equation of motion for the bubble wall]. The integration is cut off at the radius of the event horizon because the Euclidean Schwarzschild–de Sitter-type spaces actually end at the event horizon. Since the Euclidean time coordinate is being identified with a period determined by the motion of the bubble wall rather than the surface gravity of the black hole, there will in general be a conical singularity at $r={}_1r_+$; this mild singularity has no effect on the evaluation of the action integrals.

The integral over the interior region in the absence of a bubble has essentially the same form as Eq. (18):

$$S_E^{\rm int}(\Phi_+)=-4\pi\rho_2\int_{-\tilde{t}_0}^{\tilde{t}_0}\int_{{}_2r_+}^{R(\tilde{t})}r^2dr\,d\tilde{t}\ , \tag{20}$$

where now ${}_2r_+$ is the interior (black-hole) event horizon radius for the false-vacuum metric with density ρ_2, and $2\tilde{t}_0$ is the period of the bubble-wall motion in the exterior, false-vacuum metric, time coordinate. Since $R(\tilde{T})$ and $R(\tilde{t})$ are at this point not known in explicit form, it is easiest to proceed by converting the time integrations into integrations over the radius of the bubble wall, e.g, $d\tilde{T}=(d\tilde{T}/dR)dR$, since the derivatives $d\tilde{T}/dR$ and $d\tilde{t}/dR$ are known from the Euclidean versions of Eq. (14) for $dR/d\tilde{\tau}$, and Eqs. (4) and (5) for $d\tilde{T}/d\tilde{\tau}$ and $d\tilde{t}/d\tilde{\tau}$. The integral in Eq. (18) then can be rewritten as

$$S_E^{\rm int}(\Phi)=-\frac{8\pi\rho_1}{3}\int_{R_1}^{R_0}(R^3-{}_1r_+{}^3)\left[\frac{d\tilde{T}}{dR}\right]dR=\frac{8\pi\rho_1}{3}\left[\frac{\rho_2-\rho_1+6\pi\sigma^2}{3\sigma}\right]\int_{R_1}^{R_0}\frac{(R^3-{}_1r_+{}^3)R^{5/2}dR}{\left[R-2M-\dfrac{8\pi\rho_1R^2}{3}\right](R-2M-\beta^2R^3)^{1/2}}\ , \tag{21}$$

and that of Eq. (20) as

$$S_E^{\rm int}(\Phi_+)=-\frac{8\pi\rho_2}{3}\int_{R_1}^{R_0}(R^3-{}_2r_+{}^3)\left[\frac{d\tilde{t}}{dR}\right]dR=\frac{8\pi\rho_2}{3}\left[\frac{\rho_2-\rho_1-6\pi\sigma^2}{3\sigma}\right]\int_{R_1}^{R_0}\frac{(R^3-{}_2r_+{}^3)R^{5/2}dR}{\left[R-2M-\dfrac{8\pi\rho_2R^2}{3}\right](R-2M-\beta^2R^3)^{1/2}}\ . \tag{22}$$

The Euclidean action of the bubble wall itself is of the form

$$S_E^{\rm wall}=\int\left[\sigma-\frac{\gamma}{8\pi}\right]dA\ , \tag{23}$$

where the integration is over the three-dimensional hypersurface of the bubble wall. Equation (23) is somewhat different than the expressions in the earlier works of Coleman, De Luccia, and Parke. In their papers, an expression was given for the action of the bubble wall (including the effects of gravitation) which did not include the second term in Eq. (23); this made it appear that gravity has no effect on the action of a boundary surface layer. Equation (23) is in fact the correct total action for the bubble wall; the apparent lack of a gravitational term in Refs. 3 and 4 is due to those authors having performed an integration by parts of the action *before* separating the total action into interior, wall, and exterior contributions. As a result, an interior boundary term was included in the wall action which exactly cancels the gravitational contribution of the wall. Their final results are correct; it is only the splitting of the action into interior and wall contributions in Refs. 3 and 4 which is misleading. Replacing γ by its value determined from Eq. (10), $12\pi\sigma$, performing the integration over the two-sphere, and simplifying the integral by converting the integral over proper time into one over the radius of the wall yields

$$S_E^{\text{wall}}=4\pi\sigma\int_{R_1}^{R_0}R^2\left[\frac{d\tilde{\tau}}{dR}\right]dR$$

$$=-4\pi\sigma\int_{R_1}^{R_0}\frac{R^{5/2}dR}{(R-2M-\beta^2R^3)^{1/2}}\ . \tag{24}$$

The exterior integral $[r>R(\tilde{\tau})]$ is trivially disposed of, as the spacetime geometry and stress energy outside the bubble wall are identical to that of the purely false-vacuum spacetime. Thus the two integrated actions in Eq. (16) will be equal for $R(\tilde{\tau})<r<\infty$, and there will be no contribution to B from outside the bubble wall.

Finding the value of the exponential decay rate constant, B, is thus reduced to performing the integration in Eqs. (21), (22), and (24). While it is conceivable that these integrals can be explicitly written in terms of expressions involving elliptic integrals (the wall action integral certainly can be; however, the resulting expression is approximately one journal page long, and still contains a number of elliptic integrals), it is far easier to evaluate them directly numerically for the cases of interest.

One special case in which the integrals can be easily evaluated is the critical case when equality holds in Eq. (16), $M=M_c$, and the bubble wall is stationary at $R=3M$. In this case the integrals in Eqs. (19), (20), and (24) are trivial; all one needs to know are the periods $\tilde{T}_0$, $\tilde{t}_0$, and $\tilde{\tau}_0$. These are easily evaluated by expanding the integrals around the critical solution to lowest order in $\epsilon\equiv M_c-M$. In this limiting case, when the mass of the central black hole is as large as it can be, the integrals have the values

$$S_E^{\text{int}}(\Phi)=-24\pi^2\rho_1\left[\frac{\rho_2-\rho_1+6\pi\sigma^2}{\sigma}\right]\left[\frac{27M_c{}^3-{}_1r_+{}^3}{1-72\pi\rho_1M_c{}^2}\right]M_c{}^2\ , \tag{25}$$

$$S_E^{\text{int}}(\Phi_+)=-24\pi^2\rho_2\left[\frac{\rho_2-\rho_1-6\pi\sigma^2}{\sigma}\right]\left[\frac{27M_c{}^3-{}_2r_+{}^3}{1-72\pi\rho_2M_c{}^2}\right]M_c{}^2\ , \tag{26}$$

$$S_E^{\text{wall}}=-108\pi^2\sigma M_c{}^3\ , \tag{27}$$

where M_c is given in terms of ρ_1, ρ_2, and σ by Eqs. (15) and (16).

If $M=0$, then the bubble solutions (and the Euclidean metrics) have O(4) symmetry and the integrals are easily expressed in terms of elementary functions. Let the decay coefficient for $M=0$ be called B_1; if ρ_1 and ρ_2 are both nonzero its value is[4]

$$B_1=\frac{(\rho_2-\rho_1)^2+6\pi\sigma^2(\rho_2+\rho_1)-3\sigma\beta(\rho_2-\rho_1)}{48\sigma\rho_1\rho_2\beta}\ , \tag{28}$$

where β was defined in Eq. (15). If either ρ_1 or ρ_2 is zero, then the above expression reduces to[3]

$$B_1=\frac{27\pi^2\sigma^4}{\rho_2(\rho_2+6\pi\sigma^2)^2}\quad(\rho_1=0)\ , \tag{29}$$

or

$$B_1=\frac{-27\pi^2\sigma^4}{\rho_1(\rho_1+6\pi\sigma^2)^2}\quad(\rho_2=0)\ . \tag{30}$$

The expressions given in Eqs. (21), (22), and (24), and their values in the $M=M_c$ case, given by Eqs. (25), (26), and (29), will yield the value of the decay coefficient B for any given values of the four parameters ρ_1, ρ_2, σ, and M.

The ratio B/B_1 is dimensionless and independent of the value of $\hbar$; it then can only depend on dimensionless combinations of the four-dimensional parameters ρ_1, ρ_2, σ, and M. One convenient choice is to let B/B_1 be a function of $\rho_1/6\pi\sigma^2$, $\rho_2/6\pi\sigma^2$, and $M\sigma$. There is thus a three-dimensional parameter space on which B/B_1 is defined.

In the remainder of this paper I will only discuss the values of B/B_1 in the two special cases treated by Coleman and De Luccia: the first is history, namely, that an early Universe positive vacuum energy density false vacuum has decayed into the zero energy density vacuum of today, i.e., that $\rho_2>0$ and $\rho_1=0$; the second is catastrophe, namely, the uncomfortable possibility that we are living in a false-vacuum state today, so that $\rho_2=0$, and $\rho_1<0$.

In the first case a Schwarzschild—de Sitter spacetime decays into an ordinary Schwarzschild spacetime. The ratio of the decay coefficient, B, to the decay coefficient in the absence of any black hole, B_1, is shown in Fig. 2 for several different values of the de Sitter (false-) vacuum energy density, and for a full range of possible Schwarzschild masses. The value of B/B_1 is seen to be less than 1 for all values of M, and is in fact minimized in the extreme case when $M=M_c$. The reduction in the value of B is largest when the decay is most inhibited; as ρ_2 approaches $6\pi\sigma^2$, B_1 becomes large, so that the rate of bubble formation becomes very small. It is precisely in this case, however, that a nonzero mass makes the largest difference in the value of B, and hence the largest difference in the rate of bubble formation. Since the minimum values of B (for fixed $\rho_2/6\pi\sigma^2$) always occur for $M=M_c$, in Fig. 3 B/B_1 is plotted for a wide range of values of $\rho_2/6\pi\sigma^2$ for $M=M_c$. The limiting values of B/B_1 for $M=M_c$, as $\rho_2\rightarrow6\pi\sigma^2$ or $\rho_2\rightarrow\infty$, are obtainable by evaluating the limits of the integrals in Eqs. (25), (26), and (27) in those cases. As $\rho_2\rightarrow6\pi\sigma^2$, B/B_1 approaches $\frac{4}{3}(1-3^{-1/2})\approx0.5635$, while as $\rho_2\rightarrow\infty$, B/B_1 approaches $88/3^{9/2}=0.6272$. The largest effects are again seen to occur when ρ_2 is near its critical value, $6\pi\sigma^2$ (slow decay;

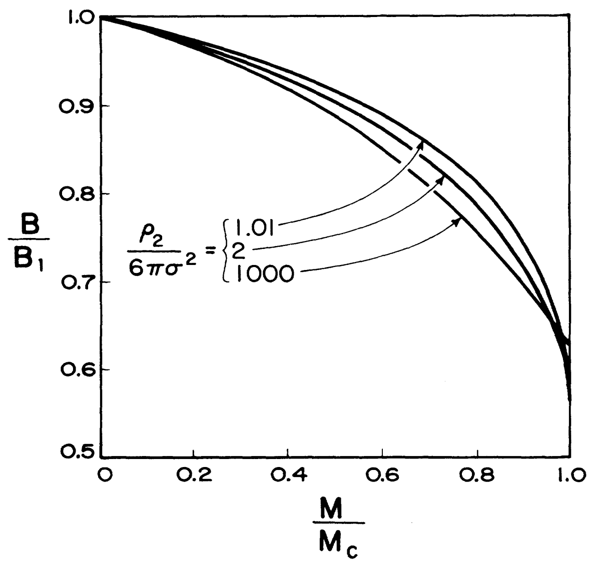

FIG. 2. The ratio B/B_1 for the decay of a positive energy density false vacuum into a zero-energy-density true vacuum as a function of the black-hole mass for several values of the false-vacuum energy density. For any fixed value of the false-vacuum energy density, the minimum value of B/B_1 occurs when $M=M_c$.

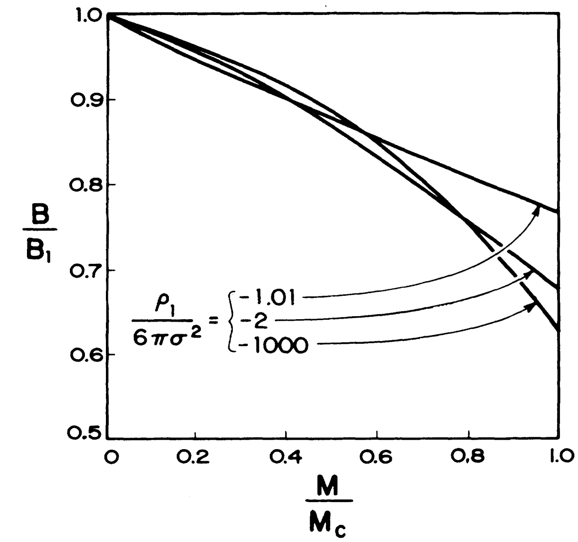

FIG. 4. The ratio B/B_1 for the decay of a zero energy density false vacuum into a negative energy density true vacuum as a function of the black-hole mass for several different values of the true-vacuum energy density. Again, for any fixed value of the true-vacuum energy density, the minimum value of B/B_1 occurs when $M=M_c$.

minimum bubble radius nearly equal to size of de Sitter universe).

In the second case, our present-day Minkowski (or ordinary Schwarzschild) spacetime is destined to decay into a negative energy density Schwarzschild—anti—de Sitter spacetime. The value of the ratio of the exponential decay coefficients, B/B_1, is shown for several different values of $\rho_1/6\pi\sigma^2$, and a full range of possible masses, $M_c \geq M \geq 0$, in Fig. 4. As in the first case, it appears that the value of B/B_1 in the presence of any mass black hole is less than in the zero-mass case. Again, the maximum reduction in the value of the B for a fixed difference in vacuum energies (value of $\rho_1/6\pi\sigma^2$) occurs when the black-hole mass is maximized: $M=M_c$. Figure 5 shows the value of B/B_1 for $M=M_c$ and a wide range of values of $\rho_1/6\pi\sigma^2$. The asymptotic values for B/B_1, with $M=M_c$, as $\rho_1 \to -6\pi\sigma^2$ for $\rho_1 \to -\infty$ are, respectively, $4/3^{3/2} \approx 0.7698$ and $88/3^{9/2} \approx 0.6272$. In this case the largest changes in the value of B/B_1 occur when the coefficient B_1 is small (or, equivalently, $|\rho_1|/6\pi\sigma^2$ is large). It is also interesting that the reduction of the value of B for $M=M_c$ is always less in the second case than in the first; in fact, the minimum value of B/B_1 in the second case (obtained when $\rho_1 \to -\infty$) is equal to the maximum value in the first case (obtained when $\rho_2 \to \infty$).

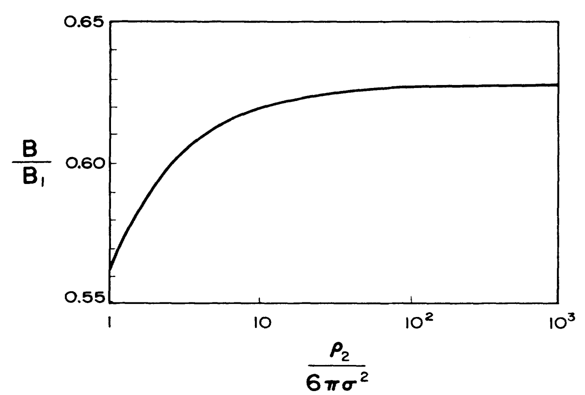

FIG. 3. The ratio B/B_1 for the decay of a positive energy density false vacuum into a zero energy density true vacuum for $M=M_c$ as a function of the energy density of the false vacuum. This curve shows, for a given value of ρ_2, the maximum decrease in B/B_1 caused by nucleation around black holes.

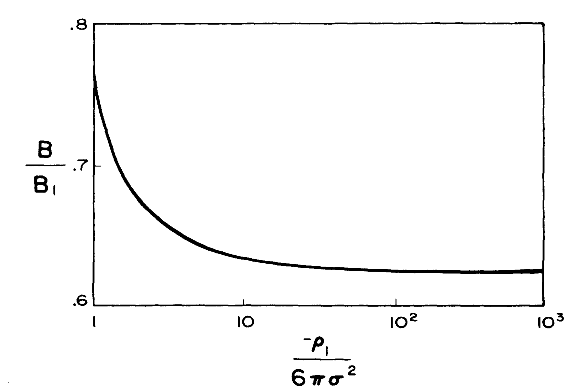

FIG. 5. The ratio B/B_1 for the decay of a zero energy density false vacuum into a negative energy density true vacuum for $M=M_c$ as a function of the energy density of the true vacuum. This curve shows, for a given value of ρ_1, the maximum decrease in B/B_1 caused by nucleation around black holes.

IV. DISCUSSION

An issue which has been ignored in the mathematics of the previous two sections is the physical size of a subcritical-mass black hole ($M \leq M_c$); two important questions need to be addressed. First, what is the maximum-mass black hole around which a bubble can nucleate in a realistic field theory with spontaneous symmetry breaking (in particular, is it greater than the Planck mass)? Second, when can the nonzero temperature of the black hole be safely ignored in calculating the decay rate of the false vacuum?

In the case of Schwarzschild–de Sitter space decaying to Schwarzschild space, the maximum (critical) mass is given by

$$M_c = 3^{-1/2} \frac{\sigma}{\rho_2 + 6\pi\sigma^2} \quad (\rho_1 = 0) . \tag{31}$$

This critical mass will be large when $\rho_2/6\pi\sigma^2$ is small (only slightly larger than 1). This is also when B_1 will be large, and supercooling would be expected. As $\rho_2/6\pi\sigma^2$ approaches 1, the critical mass approaches

$$M_c \rightarrow (72\pi\rho_2)^{-1/2} . \tag{32}$$

This is the largest-mass black hole which a de Sitter universe with energy density ρ_2 can contain; when $M = M_c = (72\pi\rho_2)^{-1/2}$, the black-hole event horizon and the cosmological event horizon are coincident. In this case, then, bubbles can form around any black holes which can fit into the false-vacuum de Sitter universe. Any primordial black holes present in the early Universe could then be potential nucleation sites for bubbles in a strongly first-order vacuum phase transition. And, of course, as long as the false-vacuum energy density is much less than the Planck density, this critical mass will be much greater than the Planck mass.

In the case of the possible decay of our present vacuum state, in which Schwarzschild space would decay to Schwarzschild–anti-de Sitter space, the critical mass is given by

$$M_c = 3^{-1/2} \frac{-\sigma}{\rho_1 + 6\pi\sigma^2} \quad (\rho_2 = 0) . \tag{33}$$

This will be large when $|\rho_1|/6\pi\sigma^2$ is only slightly greater than unity, which again implies a large value for B_1 and expected supercooling (slow decay rate). As $\rho_1 \rightarrow -6\pi\sigma^2$, the critical mass diverges to infinity. It is then possible that if we are currently living in a false-vacuum state, arbitrarily large (astrophysically sized—solar mass or larger) black holes could possibly nucleate bubbles of the negative energy density true vacuum. The longer lived the false vacuum is (i.e., the closer $|\rho_1|/6\pi\sigma^2$ is to unity), the larger M_c will be.

If, instead, $\rho_2/6\pi\sigma^2$ or $|\rho_1|/6\pi\sigma^2$ is large, then it is not possible to estimate M_c without making some assumption about the underlying field theory and its spontaneous symmetry breaking, to fix the ratio of ρ to σ. These cases are intrinsically less interesting, since $B_1/\hbar$ and $B/\hbar$ are expected to be small, the vacuum decay rate more rapid (no supercooling), and governed more strongly by the value of the unknown coefficient A. The simplest field theories with spontaneous symmetry breaking have essentially one energy scale E, which defines the dynamics of the symmetry breaking (there is certainly more than one scale in an absolute sense; however, for many grand unified and other theories, all of the relevant energy scales are within roughly an order of magnitude of each other). In such a theory, the difference in the energy densities of the false and true vacua is of order $E^4/\hbar^3$, and the bubble-wall surface energy density will be of order $E^3/\hbar^2$. The critical mass is then of order

$$M_c \sim \sigma/\rho \sim \hbar/E , \tag{34}$$

or roughly equal to the Compton wavelength of the Higgs particles. Again, as long as the energy scale is below the Planck scale, the black-hole masses will be large compared to the Planck mass, and thus can be treated semiclassically.

When the nonzero temperature of the black hole be safely ignored in calculating the decay rate of the false vacuum? Since the area near the horizon, where a bubble of true vacuum will first form, is significantly heated by the Hawking effect, it is probable that the effect of including the finite-temperature effects will be to lessen the decay rate, since high temperatures can stabilize the false vacuum.

It is clear immediately that there are some situations where the finite-temperature effects may be ignored: for instance, in the decay to anti–de Sitter space described above in which the black holes may have stellar masses. A second example is in the supercooled decay of Schwarzschild–de Sitter space described above. As M_c approaches the value given in Eq. (32), the temperatures of both the cosmological and black-hole event horizons approach zero. The zero-temperature approximation is then at least justified for the range of black-hole masses ($M \lesssim M_c$) which cause the largest decrease in the value of B is precisely the cases of most interest: when the O(4)-symmetric theory predicts a slow vacuum decay rate and substantial supercooling.

Now consider again the general case, where $|\rho_2|/6\pi\sigma^2$ or $\rho_2/6\pi\sigma^2$ may be large. The largest temperature effects will occur when the radius of the bubble is smallest (compared to the radius of the black-hole event horizon); i.e., in the $M = M_c$, $R = 3M$ limit. The zero-temperature approximation will be justified if the difference between the false- and true-vacuum energy densities, $\rho_2 - \rho_1$, is much larger than the quantum stress energy associated with the "hot" black hole at $r = 3M$. A rough estimate of the quantum stress energy in this case may be obtained from the work of Howard and Candelas,[15,16] which calculates the expectation value of the stress-energy tensor for a conformally coupled massless scalar field in the Hartle-Hawking vacuum state in the Schwarzschild geometry. The roughness of the estimate lies in my applying their rigorous results to the case of a Schwarzschild–de Sitter (or –anti–de Sitter) black hole. The quantum energy density measured by a static observer at $r = 3M$ is approximately

$$\rho \sim \frac{\hbar}{36\,864\pi^2 M^4} . \tag{35}$$

The actual quantum energy density will also depend on the number of fundamental fields present; in a typical grand unified theory there are $\sim 10^2$ fundamental fields; if each contributes roughly as much to the quantum energy density as a conformally coupled massless scalar field does, then the total quantum stress energy density, might be

$$\rho_H \sim 10^{-4}\hbar/M^4 \ . \tag{36}$$

Equation (36) is given as an upper limit since many of the fundamental fields may be massive and not contribute substantially to ρ_H, depending on the particular parameters of the field theory and the phase transition. The zero-temperature approximation will be justified if $\rho_2-\rho_1$ is much larger than ρ_H as given in Eq. (36). For the "one energy scale" theories described above, with $\rho_2-\rho_1 \simeq E^4/\hbar^3$, $\rho_2-\rho_1$ will be much larger than ρ_H if $1 \gg 10^{-4}$. Naively it then appears that the zero-temperature approximation will always be valid; however, the actual energy density and critical mass in such a theory might very well differ from the crude estimate of Eq. (34) by several orders of magnitude; in which case, a detailed examination of the particular theory will be needed to determine the limits of applicability of the zero-temperature approximation.

In the absence of a calculated value for the decay coefficient A, the precise effect of a black hole on the vacuum decay rate cannot be determined. The only new dimensional parameter is the mass of the black hole. As was shown above, in some cases the natural mass scale is roughly equal to the Compton wavelength of the Higgs field, and thus no new length scale is introduced (at least, in terms of the order of magnitude of the scale). In any case, the range of possible masses is determined by Eqs. (15) and (16); the black-hole masses are simple algebraic functions of the energy scales already defined in the O(4)-symmetric problem. The change in the value of A might then be reasonably expected to be at most an order of magnitude or so. The most confidence can be placed in the results for the case of a very slow vacuum decay, where $|\rho_1|/6\pi\sigma^2$ or $\rho_2/6\pi\sigma^2$ is close to unity. In that case, $B/\hbar$ is likely to be a very large number, and the effect of reducing B by a factor of 0.5 to 0.75 owing to the existence of a black hole will be large.

As an example, consider a theory in which the current zero energy density vacuum state in which we live is a false-vacuum state, and the energy scale of the symmetry breaking is of order E. Then, on simple dimensional grounds, the coefficient will be of order $(E/\hbar)^4$. Suppose that the bubble nucleation rate is such that our false-vacuum state is just today on the verge of decaying, *ignoring the effect of black holes as bubble nucleation sites* (i.e., using B_1 to determine the decay rate); then the product of the bubble nucleation rate and the four-volume of the observable Universe must be about equal to one:

$$\Gamma_1 V \simeq A T_0{}^4 \exp(-B/\hbar) \simeq 1 \ , \tag{37}$$

where the approximate four-volume of the observable Universe is set equal to the Hubble time to the fourth power. This limits the value of B_1:

$$B_1 \lesssim 4\hbar \ln\left[\frac{ET_0}{\hbar}\right] . \tag{38}$$

If it is now assumed that black holes are present with some distribution of masses, and the possibility of nucleating a bubble around a black hole is considered, then B can be smaller than the limiting value in Eq. (38) by at least a factor of about 0.77. Assuming (perhaps naively?) that the value of A is not significantly changed by the existence of black holes, this implies a bubble formation rate which can be considerably larger than the original no-black-hole rate

$$\Gamma/\Gamma_1 = \exp[(B_1-B)/\hbar] \simeq \left[\frac{ET_0}{\hbar}\right] . \tag{39}$$

If $E \sim 100$ GeV, then $\Gamma/\Gamma_1 \approx 10^{44}$. The time required to create one bubble within our past light cone is reduced from about 20×10^9 yr to

$$T_{\text{bubble}} = \frac{\Gamma_1}{\Gamma} T_0 \simeq 10^{-26} \text{ sec} \ . \tag{40}$$

If $E \sim 10^{19}$ GeV, then $\Gamma/\Gamma_1 \approx 10^{61}$; in this case the time to create one bubble within our past light cone would be reduced to the order of the Planck time: 10^{-43} sec.

Whatever the change in the coefficient A is, it is obviously not likely to be comparable to the change in $\exp(-B/\hbar)$ in this sort of case. Thus it is possible to conclude with confidence that black holes can speed up at least some strongly first-order phase transitions.

In conclusion, it has been shown that black holes can act as nucleation sites for the creation of bubbles of true vacuum in a first-order phase transition, and that the vacuum decay rate can be significantly increased by the existence of appropriate mass black holes. The largest effect is in the case of the decay of a Schwarzschild—de Sitter spacetime to a Schwarzschild spacetime, as might occur in the early Universe; in this case an extreme supercooling may be much more difficult to achieve than in the absence of black holes. Another case of interest is the possible decay of our present vacuum, which, if unstable, certainly seems to be in a long-lived supercooled state. Again, the presence of black holes can greatly accelerate the decay of the false vacuum; in this case, astrophysical-sized black holes of stellar mass or even larger may potentially play an important role. Precise predictions of the change in the decay rate will require applying the results of this work to specific model theories with spontaneous symmetry breaking.

ACKNOWLEDGMENT

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