Is inflation natural?

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We show that under very general conditions any inhomogeneous cosmological model with a positive cosmological constant that can be described in a synchronous reference system will tend asymptotically in time towards the de Sitter solution, so making the problem of initial conditions less severe. The implications for inflationary scenarios are examined, and it is found that after inflation the Universe stays isotropic and homogeneous for a very long time.

I. INTRODUCTION

The observable Universe today seems to be remarkably homogeneous and isotropic on a very large scale and a good cosmological model has been constructed capable of describing its large-scale properties very nicely. This is the so-called Friedmann-Robertson-Walker (FRW) cosmology. It represents a perfectly homogeneous and isotropic space. Regardless of the nice features of this cosmological model, we have to ask why is the Universe described by such a model. There are two answers to this question. Either the Universe has always been like this, i.e., the initial conditions were such that the Universe was and has remained isotropic and homogeneous, or the Universe started in a less symmetrical phase and evolved through some dynamical process to become a FRW cosmology today. The former is certainly a very unsatisfactory solution, and from a statistical point of view very improbable. In this paper we shall explore the latter possibility.

The early attempts to solve the problem were based on retaining the homogeneity of space as an essential feature of our Universe. One of the reasons for doing so is the fact that all possible geometries describing homogeneous but anisotropic models fall into one of nine classes. These were classified a long time ago and have been extensively studied.[1] These are the so-called Bianchi and Kantowski-Sachs models. Early studies of these models showed that some of them eventually (asymptotically) became FRW cosmologies. However, not all did. Nevertheless, to obtain such models, the constraints imposed on the degrees of freedom of the gravitational field are so severe that only a set of measure zero of all initial conditions can give rise to these models.

To relax homogeneity would make the problem mathematically intractable. It would require the knowledge of the general solution of Einstein's equations together with its boundary and/or initial conditions, clearly a daunting task. However, all is not lost. A way out is provided by the inflationary scenarios.[2] In these models one usually assumes that the Universe becomes dominated by a positive vacuum energy, i.e., a cosmological constant $\Lambda > 0$, and for a period of time expands exponentially at the Hubble rate $H = \sqrt{3\Lambda}$ followed by a reheating period that eventually terminates in a FRW flat, radiation-dominated Universe, after which it can proceed its evolution in the standard way (see Ref. 3, for a review on inflation). If the Universe undergoes a period of exponential expansion of more than about 60 Hubble times it is possible to explain the homogeneity and isotropy of our observable Universe as well as solve the oldness and monopole problems in a natural way.[2]

It has been shown that inflation will take place in almost all Bianchi models with a positive cosmological constant (except perhaps in Bianchi type IX)[4] that do not contain vorticity and that once the inflationary phase begins the process of isotropization due to the cosmological constant is very efficient. The end result of this process is to smooth out anisotropies and eliminate curvature. Furthermore, it has been shown[5] that for a class of Bianchi cosmologies (the orthogonal models) the effect of anisotropies on scales larger than the horizon will take a very long time to act back on the observable Universe. It is worth pointing out that this effect is independent of the Bianchi model and of the initial anisotropy. Despite this result, Bianchi cosmologies are still highly symmetrical, very restrictive models, and probably form a set of measure zero among the set of all possible cosmologies. Attempts have been made in the past to generalize these results to nonhomogeneous models. However, the lack of sufficient exact inhomogeneous solutions to Einstein's equations has made this difficult. Some explicit inhomogeneous examples that present the same properties have appeared in the literature.[6]

Some of this evidence led to the belief that some fundamental principle was underlying the process of isotropization and inflation was indeed a very general process. This would have important consequences for the problem of initial conditions, as it would make our presently observed Universe much more probable. Some time ago the existence of such a principle was conjectured by Gibbons and Hawking and by Hawking and Moss.[7] It states that cosmologies with a positive cosmological constant would approach the de Sitter solution asymptotically in time (several alternative formulations of this conjecture have appeared in the literature); this is the so-called "no-hair" conjecture. Several attempts at proving the conjecture have been made,[8] and a general proof has been obtained for homogeneous cosmologies (Bianchi models).[4] It has also been shown[9] that "general" solutions to Einstein's equations exist which asymptotically approach the de Sitter solution (at least locally).

Early on it was shown that the conjecture, as it stands, was false, as trivial counter examples can be provided, the most obvious one being the closed FRW model which collapses before it enters an inflationary phase. Nevertheless, the number and diversity of models that did obey this principle lead to the general belief that if not this then maybe a weaker version of the conjecture should be true. In this work we would like to present a proof of the no-hair conjecture for a very large class of inhomogeneous and anisotropic cosmological models. We will show that given an arbitrary synchronous space-time, a positive cosmological constant, an energy-momentum tensor satisfying the weak- and strong-energy conditions, and a constraint on the three-curvature of space, then this space-time will evolve towards a de Sitter space-time (at least locally). This result holds in an arbitrary number of dimensions.

We will also present a generalization of earlier work on orthogonal Bianchi models that shows that once inflation takes over and the phase transition is successfully terminated, then the Universe remains homogeneous and isotropic for a very long time. This implies that inflation is a generic feature of almost any universe that contains a positive cosmological constant. Initial conditions of these cosmologies then become almost irrelevant since all the models end up in the same asymptotical state, the de Sitter cosmology.

The paper will be organized as follows. In Sec. II the metric, field equations, and the energy conditions will be given, the conjecture will be formulated and proven, in Sec. III we will generalize our early result on orthogonal Bianchi models to inhomogeneous models. Section IV will contain some comments and conclusions.

II. THE NO-HAIR CONJECTURE

We consider Einstein's equations

$$R_{\mu\nu}=T_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}T-g_{\mu\nu}\Lambda \ . \tag{2.1}$$

Here $g_{\mu\nu}$ is the space-time metric, $T_{\mu\nu}$ the energy-momentum tensor, and $T=T^{\mu}_{\mu}$. We use the sign conventions $(+,-,-,-)$ and the notation of Ref. 10. Greek indices run from 0 to 3 and latin from 1 to 3. The only assumptions we make about the energy-momentum tensor is that it satisfies (i) the dominant energy condition, that $T_{\mu\nu}t^{\mu}t^{\nu}\geq 0$ and $T_{\mu\nu}t^{\nu}$ is nonspacelike for all timelike t^{ν}, and (ii) the strong-energy condition, that $(T_{\mu\nu}-\frac{1}{2}g_{\mu\nu}T)t^{\mu}t^{\nu}\geq 0$ for all timelike t^{μ}. The dominant energy condition is equivalent to demanding that the energy density is non-negative and the energy flow is causal. All known forms of matter satisfy this condition. (For a perfect fluid it reduces to $\rho\geq|p|$.) The strong-energy condition for a perfect fluid reduces to the usual requirement that $\rho+3p\geq 0$, i.e., a large negative energy density or large negative pressures must be present to violate this condition. We choose to work in a synchronous reference system where $g_{00}=1$ and $g_{0i}=0$. We shall also introduce a (positive-definite) spatial metric tensor $h_{ab}\equiv -g_{ab}$ and define $s_{ab}=\dot{h}_{ab}$. Using this, Eq. (2.1) becomes[10]

$$\begin{aligned} R^0_0&=-\tfrac{1}{2}\dot{s}^{\,a}_{\,a}-\tfrac{1}{4}s^b_a s^a_b=T^0_0-\tfrac{1}{2}T-\Lambda \ , \\ R^0_a&=\tfrac{1}{2}(s^b_{a;b}-s^b_{b;a})=T^0_a \ , \\ R^b_a&=-P^b_a-\frac{1}{2\sqrt{h}}\frac{\partial}{\partial t}(\sqrt{h}\,s^b_a)=T^b_a-\tfrac{1}{2}\delta^b_a T-\delta^b_a\Lambda \ . \end{aligned} \tag{2.2}$$

Here P_{ab} is the three-dimensional Ricci tensor calculated using h_{ab} and $h=\det h_{ab}$. ($\sqrt{h}$ can be interpreted as the volume element in three-space.) Let us define the volume expansion by $K\equiv\frac{1}{2}\dot{h}/h=\frac{1}{2}s^a_a$. In what follows we make the assumption that the space is open or flat, i.e., that the scalar spatial curvature $P=P^a_a\leq 0$ for all times. Equation (2.2) then implies

$$-R^0_0=\dot{K}+\tfrac{1}{4}s^b_a s^a_b=-T^0_0+\tfrac{1}{2}T+\Lambda \ , \tag{2.3a}$$

$$-R^a_a=\dot{K}+K^2+P=-T^a_a+\tfrac{3}{2}T+3\Lambda \ . \tag{2.3b}$$

To proceed and solve (2.3) we must first calculate $s^b_a s^a_b$. If we introduce the trace-free part of s_{ab}, $2\sigma_{ab}\equiv(s_{ab}-\frac{1}{3}s^c_c h_{ab})$, we find that

$$\begin{aligned} s^a_b s^b_a=s_{ab}s^{ab}&=\tfrac{1}{3}(s^a_a)^2+4\sigma_{ab}\sigma^{ab} \\ &=\tfrac{4}{3}K^2+4\sigma_{ab}\sigma^{ab} \ . \end{aligned}$$

Substituting this into (2.3a), we find

$$\dot{K}=\Lambda-\tfrac{1}{3}K^2-\sigma_{ab}\sigma^{ab}-(T^0_0-\tfrac{1}{2}T) \ . \tag{2.4}$$

Eliminating $\dot{K}$ using (2.3b) this gives

$$\Lambda-\tfrac{1}{3}K^2=-\tfrac{1}{2}\sigma_{ab}\sigma^{ab}-T^0_0+\frac{P}{2} \ . \tag{2.5}$$

Clearly $\sigma_{ab}\sigma^{ab}$ is non-negative and zero only when $\sigma_{ab}=0$. The strong and dominant energy conditions imply that $T^0_0-\frac{1}{2}T$ and T_{00} are positive, furthermore $P\leq 0$, so from (2.4) and (2.5) we find

$$\dot{K}\leq\Lambda-\tfrac{1}{3}K^2=-\sigma_{ab}\sigma^{ab}-T^0_0+\frac{P}{2}\leq 0 \tag{2.6}$$

so $K^2\geq 3\Lambda$. Also, after integration of the first inequality of (2.6) we find that

$$K\leq\sqrt{3\Lambda}/\tanh\left[\left[\frac{\Lambda}{3}\right]^{1/2}[t+t_0(x_c)]\right] \ ,$$

where t_0 only depends on space. [Here we have chosen the positive square root so $K_0=K(t_0)$ is positive corresponding to an expanding universe.] This implies that asymptotically

$$0\leq K-\sqrt{3\Lambda}\leq 4\sqrt{3\Lambda}\exp\left[-2\left[\frac{\Lambda}{3}\right]^{1/2}(t+t_0)\right] \ ;$$

i.e., the expansion rate K of the volume tends to the de Sitter rate of $\sqrt{3\Lambda}$. From (2.6) it also follows that $\sigma_{ab}\sigma^{ab}$, T_{00}, and $-P/2$ all are suppressed by

$$8\Lambda\exp\left[-2\left[\frac{\Lambda}{3}\right]^{1/2}(t+t_0)\right] \ .$$

This means that locally the Universe undergoes rapid isotropization; indeed, $\sigma_{ab}=0$ asymptotically implies that

$$\dot{h}_{ab}-2\left[\frac{\Lambda}{3}\right]^{1/2}h_{ab}=0 .$$

From this we see that asymptotically h_{ab} becomes de Sitter type,

$$h_{ab}(t,x_c)=\exp\left[2\left[\frac{\Lambda}{3}\right]^{1/2}t\right]\tilde{h}_{ab}(x_c) ,$$

where $\tilde{h}$ only depends on space. Also for consistency we show that the energy-momentum tensor decays exponentially: from the dominant energy condition it follows that for all timelike t^ν,

$$(T_{0\nu}t^\nu)^2 \geq T_{a\nu}t^\nu h^{ab}T_{b\mu}t^\mu \geq 0 . \tag{2.7}$$

Choosing $t^\nu=\delta^{0\nu}$ this shows that

$$T_{00}{}^2 \geq T_{a0}h^{ab}T_{b0} \geq 0 .$$

Since $T_{00}{}^2$ vanishes faster than h^{ab} this forces $T^0{}_a$ to vanish asymptotically. Substituting this back into (2.7) we similarly find that $T_a{}^b$ vanishes so all components of the energy-momentum tensor vanish exponentially fast. Using the above, it follows that inside any given physical volume V_0 space-time rapidly becomes equal to the vacuum de Sitter space-time: because of the exponential expansion, a fluctuation of scale l is rapidly red-shifted and "smoothed" over V_0, all anisotropy as well as the energy-momentum tensor decays exponentially. It is important to keep in mind that although space-time becomes de Sitter type locally there is no reason for this to happen globally.

We have shown that the no-hair conjecture is true for a wide class of spatially open and flat cosmological models. The only requirements are the existence of a synchronous reference frame, a positive cosmological constant, and an energy-momentum tensor that satisfies both the strong- and dominant-energy conditions.

In particular this applies to the case of homogeneous cosmologies, the Bianchi models. Except for Bianchi type IX all these are flat or open, so in the presence of a positive cosmological constant these will all approach de Sitter space, in agreement with previous results.[4]

We also wish to point out that our argument holds in higher dimensions: under the conditions given above an $(n+1)$-dimensional cosmology under the influence of a positive cosmological constant will eventually expand at a rate of $\sqrt{\Lambda/n}$ in each of the n spatial directions. This rules out inflation in Kaluza-Klein-type theories in which P (the sum of the curvature of the internal space and three-space) is negative, since eventually the internal dimensions will expand and become observable. (Although it may be possible to have $P \leq 0$ if one is willing to violate the energy conditions.)

III. INFLATIONARY COSMOLOGY

As we have mentioned in the beginning, the fact that such a large class of cosmologies under the influence of a positive cosmological constant tend to the de Sitter space-time has great importance for the inflationary cosmology scenarios.[11] In these, gravity is coupled to a massless scalar field ϕ with a "flat" potential $V(\phi)$. The cosmological constant is fine-tuned so that the energy density vanishes at the minimum of V corresponding to the absence of a cosmological constant today. The equation of motion for ϕ is

$$\ddot{\phi}+K\dot{\phi}=-V'(\phi) , \tag{3.1}$$

where ϕ is taken to be smooth so that we can neglect gradient terms in (3.1), and K is given by (2.5). We notice that if $P \leq 0$, then K is greater than the Hubble constant $\sqrt{3\Lambda}$ of de Sitter phase. Therefore the "friction force" felt by the field, $K\dot{\phi}$, is greater than in de Sitter phase making the field roll slower over the potential; in the same token a greater K means faster expansion. We shall assume that initially ϕ is stabilized (by, for example, initial conditions or thermal corrections) on the "flat" part of V. Then we find that at least as much inflation is produced in the general case with the Universe being inhomogeneous prior to inflation than in the usual case with FRW cosmology prior to inflation (similar arguments were used in Ref. 12 for the anisotropic case). The resulting universe is highly homogeneous and isotropic on scales much larger than the horizon and after ϕ has returned to its minimum the Universe evolves like the usual FRW model, but without the extra assumption of initial homogeneity or isotropy. We should comment that there is no need to assume that $P \leq 0$ in all of space, in order to have successful inflation. We just need $P \leq 0$ in some region large enough that surface effects can be neglected, then this region will eventually evolve into de Sitter space and may then become our observable Universe. This strongly supports the belief that inflation is a very universal feature largely independent of the initial conditions of the Universe.

We have argued that after inflation any inhomogeneities (anisotropies) have been "pushed" far beyond the horizon and the Universe evolves almost exactly like the FRW model. However, since the horizon volume increases faster than a typical comoving volume in FRW models, then we expect that eventually the inhomogeneities will reappear through the horizon again. Given the number of Hubble times N during which the Universe inflates, we shall now estimate how long it takes before the inhomogeneities reappear. To do this we must study the evolution of the inhomogeneities as described by (2.2).

Let us with t_0 define the time of the onset of inflation, and let t_N be N Hubble times later, i.e., $t_N=t_0+(3/\Lambda)^{1/2}N$, then the metric before and after inflation are related by

$$h_{ab}(t_N,x_c)=e^{2N}h_{ab}(t_0,x_c) .$$

Using this in the usual expression for the Christoffel symbols, we find that after N Hubble times of inflation the spatial Ricci-tensor scales like

$$P_a^b(t_N,x_c)=e^{-2N}P_a^b(t_0,x_c) .$$

This gives us the initial conditions for evolution of the Einstein's equations (2.2) after inflation. Here we assume as usual an effective reheating after inflation so that all the vacuum energy density is converted into radiation energy density. The energy-momentum tensor after inflation then becomes that of a perfect fluid with $p=\gamma p$ ($\gamma=\frac{1}{3}$ for radiation; we are also assuming that there is no cosmic rotation or fluid acceleration). From conservation of the energy-momentum tensor we find that $\rho \propto h^{-(1+\gamma)/2}$, and assuming effective reheating we can find the constant of proportionality

$$\rho=\Lambda\left[\frac{h_N}{h}\right]^{(1+\gamma)/2},$$

where $h_N \equiv h(t_N,x_c)$. Now, let us define new variables $\tilde{h}_{ab}(t,x_c)\equiv e^{-2N}h_{ab}(t,x_c)$, and quantities carrying a tilde are defined as usual but using $\tilde{h}_{ab}$. Then the last equation in (2.2) becomes

$$\frac{1}{2\sqrt{\tilde{h}}\,\partial t}\frac{\partial}{\partial t}(\sqrt{\tilde{h}}\,\tilde{s}_a^{\,b})=-e^{-2N}\tilde{P}_a^{\,b}+\tfrac{1}{2}(1-\gamma)\Lambda\left[\frac{\tilde{h}_N}{\tilde{h}}\right]^{(1+\gamma)/2}\delta_a^b\,. \tag{3.2}$$

Again, to make this equation more transparent it is advantageous to carry out another change of variables: set

$$\mu^{-1}=(1-\gamma)\Lambda h_0{}^{(1+\gamma)/2}$$

and define

$$\bar{h}_{ab}=e^{-2N}\mu\tilde{h}_{ab}\,,$$

$$\bar{t}=e^{-2N}\sqrt{\mu}\,t\,.$$

Using that $\tilde{h}_N=h_0$, then (3.2) becomes

$$\frac{1}{2\sqrt{\bar{h}}}\frac{\partial}{\partial\bar{t}}(\sqrt{\bar{h}}\,\bar{s}_a^{\,b})=-\bar{P}_a^{\,b}+\left[\frac{1}{\bar{h}}\right]^{(1+\gamma)/2}\delta_a^b \tag{3.3a}$$

with initial conditions

$$\bar{h}_{ab}=e^{-2N}\mu h_{ab}(t_0,x_c)\quad\text{at } \bar{t}=e^{-2N}\sqrt{\mu}\,t_N\,,$$

i.e.,

$$\bar{h}_{ab}(\bar{t}=0,x_c)=0\,. \tag{3.3b}$$

We notice that all scales are eliminated in (3.3); therefore, it follows that it will take a time $\bar{t}\simeq 1$ before the curvature term in (3.3) begins to dominate and anisotropy and inhomogeneities become important. This corresponds to a cosmic time

$$t_*=e^{2N}\sqrt{(1-\gamma)\Lambda}\,h_0{}^{(1+\gamma)/4}\,. \tag{3.4}$$

We have here assumed that the Universe is radiation dominated after inflation; however, a period of matter domination will only increase the time t_* at which the anisotropies and inhomogeneities reappear through the horizon. The reason for this is that for matter the volume increases like t^2 as compared to $t^{3/2}$ for radiation, while the horizon volume increases like t^3 in both cases. This shows that if the Universe inflates it will remain homogeneous and isotropic for a very long time. At time t_* the Universe again "remembers" initial conditions and may evolve to become inhomogeneous and anisotropic again on observable scales. This result applies, in particular, to all the open or flat homogeneous cosmologies, the Bianchi-type I—VIII cosmologies. In the presence of a positive cosmological constant these models will inflate and after inflation remain isotropic until time t_*. This is in agreement with previous and more laborious results obtained for some (the orthogonal) Bianchi models.[5]

IV. CONCLUSIONS

We showed that inflation is a very universal feature, a process very likely to have occurred in the early history of the Universe. A modified and slightly different version of the no-hair conjecture has been proven. This states that given any inhomogeneous and anisotropic space-time that can be written in a synchronous gauge, and with a nonpositive three-Ricci scalar, then in the presence of a positive cosmological constant and a fluid whose energy-momentum tensor satisfies the strong- and dominant-energy conditions, this will evolve towards the de Sitter space-time. In particular, we have demonstrated that within our observable Universe inflation is capable of smoothing out all inhomogeneities and anisotropies as well as making the space flat, so explaining the nature of our observable Universe. We have discussed the relevance of this result to the problem of initial conditions, where we have argued that if the conditions of the conjecture are satisfied the initial conditions are not important any more as the final evolution of our model will be independent of them. Clearly this elevates inflation and makes it a much more appealing solution to the question we posed in the Introduction. Inflation provides us, for the first time, with a dynamical mechanism by which the Universe undergoes a transition from a very inhomogeneous and nonregular past to a nice smooth future. Of course, if the Universe has always been a FRW universe or something very close to it, then there is nothing to worry about.

Knowing that inflation will take place in a large class of inhomogeneous space-times, we calculated how effective the process of smoothing was. This generalizes early results on orthogonal Bianchi models that uses the fact that a similar conjecture had been proven that ensures the existence of an inflationary phase to assess the efficiency of the isotropization. We found that the time scale for anisotropies to act back on the FRW model left by inflation goes exponentially with the number of e-folds the Universe inflates. This is much larger than the age of the Universe. Precisely the same happens in the inhomogeneous models. A successful inflationary phase will produce a very smooth flat radiation-dominated FRW model that will stay as a FRW model for a time of order $t_*\simeq e^{2N}\sqrt{\Lambda}$.

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