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Evolution of the constraint equations in general relativity

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We consider the evolution of initial data in general relativity. The Bianchi identity guarantees that data which initially satisfy the constraint equations will always satisfy the constraints. But we often expect only the approximate satisfaction of the constraint equations, for example, in numerical analysis. Here we study the evolution of only approximately good initial data. Under rather general circumstances the evolution drives the data away from good data; and we give a simple example where the traditional methods employed in numerical relativity would be likely to give erroneous results. We also present a minor modification of these traditional methods which would be likely to remedy this difficulty.

I. INTRODUCTION

In recent years great interest in the development of gravitational-wave detectors has grown. As this interest continues, it becomes more important to obtain accurate analysis of strong field sources of gravitational waves. The complexity of Einstein's equations have forced much of this analysis to be numerical; and, by and large, progress in the development of numerical methods in general relativity[1–10] has kept pace with the progress in experimental methods.[11]

A recurring difficulty encountered in numerical relativity is the treatment of the constraint equations. A metric and extrinsic curvature of a spacelike hypersurface embedded in spacetime are the data needed for determining the evolution of the geometry to a nearby hypersurface. But the data may not be specified arbitrarily—they must satisfy four constraints which are consequences of the Gauss-Codazzi equations and the Einstein equations. It has long been known that the Bianchi identity implies that the evolution equations will preserve these constraints. But numerical analysis deals with constraints and evolution only approximately; after some time of numerical evolution of initially good data, it is often found that the constraints are no longer well satisfied. The data are usually modified to resatisfy the constraints, and the evolution then proceeds. It seems reasonable that as long as the data are not allowed to stray too far from the constraints on any one hypersurface, then the numerical evolution ought to approximate well the true physical system being modeled.

We look in detail at this situation and develop techniques for the careful analysis of evolution away from good data. In Sec. II we summarize the initial-value formulation of the constraint equations along with the evolution equations. In Sec. III we use the Bianchi identity to analyze the evolution away from good data. In Sec. IV this technique shows, for at least one particularly simple example, that the traditional numerical approach to relativity is not satisfactory. And, finally, in Sec. V we give a minor modification of the traditional methods which seems to remedy these difficulties. York[12] has given a clear and careful review of the initial-value formulation of general relativity, and we follow his notation for the most part. Geroch[13] describes some useful methods of tensor manipulation which we have found invaluable.

II. THE INITIAL-VALUE FORMALISM AND THE EVOLUTION EQUATIONS

Given a four-dimensional spacetime with the metric, g_{ab}, which satisfies the Einstein equations

$$G_{ab} \equiv \mathscr{R}_{ab} - \tfrac{1}{2} g_{ab} \mathscr{R} = \kappa T_{ab} \ , \tag{1}$$

where $\kappa = 8\pi G$ and $\mathscr{R}_{ab}$ is the four-dimensional Ricci tensor, one can consider a three-dimensional, spacelike hypersurface imbedded in the spacetime. The induced metric on the hypersurface is

$$\gamma_{ab} = g_{ab} + n_a n_b \ , \tag{2}$$

where n^a is the forward-pointing unit vector perpendicular to the hypersurface. The imbedding is further described by the extrinsic curvature K_{ab} of the hypersurface:

$$K_{ab} = -\tfrac{1}{2} \mathscr{L}_n \gamma_{ab} = -\perp \nabla_{(a} n_{b)} \ . \tag{3}$$

The symbol $\perp$ denotes the projection of all following indices perpendicular to n^a, $\mathscr{L}$ is the Lie derivative, and ∇ is the derivative operator associated with g_{ab}.

Four of the ten Einstein equations for the full four-dimensional metric involve only γ_{ab} and K_{ab} and no time derivatives; these are the constraint equations. Solving these equations is known as solving the "initial-value problem" in general relativity. The constraint equations are the Hamiltonian constraint

$$2G_{nn}=R+K^2-K_{ab}K^{ab}=2\kappa\rho \tag{4}$$

and the momentum constraint

$$-\perp G_n{}^a=D_b(K^{ba}-\gamma^{ba}K)=\kappa j^a\ , \tag{5}$$

the symbol D is the derivative operator on the hypersurface associated with γ_{ab}, R is the scalar curvature on the hypersurface, and an index n denotes contraction with the vector n^a. Here and below it is important to remember where these equations come from so we include the appropriate component of the Einstein tensor for each equation. The left-hand equality in each of these expressions is just a consequence of the Gauss-Codazzi equations; the right-hand equality is the Einstein equation for a stress-energy tensor T_{ab} whose components may be projected perpendicular and then parallel to n^a:

$$T_{ab}=\rho n_a n_b+2j_{(a}n_{b)}+\mathscr{S}_{ab}\ , \tag{6}$$

where both j_a and $\mathscr{S}_{ab}$ are perpendicular to n^a. Two more tensor equations involve the time derivative (the Lie derivatives with respect to αn^a) of γ_{ab} and K_{ab} and determine the "evolution" of the tensors:

$$\gamma_{ab}{}^{\cdot}=-2\alpha K_{ab} \tag{7}$$

and

$$K_a{}^{b\cdot}=-\alpha(\perp G_a{}^b-\tfrac{1}{2}\gamma_a{}^b g^{cd}G_{cd})-D_aD^b\alpha +\alpha R_a{}^b+\alpha K K_a{}^b\ . \tag{8}$$

The quantity α is the lapse function, a dot denotes a Lie derivative with respect to the vector $N^a=\alpha n^a$, which is that vector perpendicular to one hypersurface of constant time and pointing to a nearby hypersurface of a later time. The lapse function is not determined by the geometry or the Einstein equations, but it does determine how the entire four-dimensional spacetime is foliated into its constituent hypersurfaces.

Both of these evolution equations, as written above, are identities which follow from the definitions of the Einstein tensor and the extrinsic curvature, and from the Gauss-Codazzi equations. A distinction which will be important later is that Eqs. (7) and (8) always hold whether the Einstein equations are imposed or not. The imposition of the Einstein equations is performed by replacing the Einstein tensor G_{ab} by the appropriate components of the stress-energy tensor T_{ab}; Eq. (8) becomes

$$K_a{}^{b\cdot}=-\alpha(\mathscr{S}_a{}^b-\tfrac{1}{2}\gamma_a{}^b\mathscr{S}+\tfrac{1}{2}\gamma_a{}^b\rho) -D_aD^b\alpha+\alpha R_a{}^b+\alpha K K_a{}^b\ . \tag{9}$$

The Bianchi identity in spacetime is

$$\nabla^b G_{ba}=0\ . \tag{10}$$

The components can be projected perpendicular and parallel to n^a to obtain

$$G_{nn}{}^{\cdot}=\alpha^{-1}D^a(\alpha^2\perp G_{an})+\alpha G_{nn}K+\alpha G_{ab}K^{ab} \tag{11}$$

and

$$(\perp G_{na})^{\cdot}=D^b(\alpha\perp G_{ab})+\alpha K\perp G_{na}+G_{nn}D_a\alpha\ . \tag{12}$$

These are identities for any geometry. And when the Einstein equations are imposed the G's are replaced by the appropriate components of the stress-energy tensor to obtain the conservation of energy

$$\rho^{\cdot}=\alpha(-D_aj^a+\mathscr{S}^{ab}K_{ab}+\rho K)-2j^aD_a\alpha\ , \tag{13}$$

and the conservation of momentum

$$j^{a\cdot}=\alpha(-D_b\mathscr{S}^{ba}+2K^a{}_bj^b+j^aK)-\mathscr{S}^{ab}D_b\alpha-\rho D^a\alpha\ . \tag{14}$$

These are essentially identical to York's[12] Eqs. (40) and (41).

III. THE EVOLUTION AWAY FROM GOOD DATA

The Bianchi identity (11) and (12) clearly shows that, for a vacuum geometry, if the constraints are satisfied ($\perp G_{an}=G_{nn}=0$) on one hypersurface and if the evolution equations ($\perp G_{ab}=0$) are satisfied then the Lie derivatives of G_{nn} and $\perp G_{an}$ both vanish, so the evolved data on the next slice also satisfy the constraints. Good data evolve into good data. If a stress energy is allowed, then the constraints are evolved only if the stress energy is conserved [Eqs. (13) and (14)].

In the real world the complexity of the Einstein equations often leads us to consider their numerical solutions. First the constraint equations (4) and (5) are solved as coupled, elliptic, nonlinear finite-difference equations. This determines the metric and the extrinsic curvature of one initial slice of the spacetime. Then the finite-difference versions of the evolution equations determine the data on a neighboring slice. In the course of this procedure small inaccuracies are admitted, and the constraint equations are no longer satisfied exactly but only approximately. How do data evolve which do not exactly satisfy the constraint equation?

The evolution of general, vacuum data can be studied by using only the equations above but with new interpretations of ρ, j^a, and $\mathscr{S}_{ab}$.

Consider an arbitrary three-dimensional metric γ_{ab} and extrinsic curvature K_{ab} as initial data on a spacelike hypersurface of a four-dimensional manifold. On this hypersurface let Eqs. (4) and (5) define quantities ρ and j^a which are measures of how close these data come to satisfying the vacuum constraint equations. Clearly if and only if ρ and j^a both vanish, then the constraints are satisfied. Now if the initial data are evolved according to Eqs. (7) and (9), with $G_{ab}=0$, then we can see how both ρ and j^a change in time; in particular we can see whether they grow or diminish.

Later we wish to consider evolution equations that differ from Eqs. (7) and (9) by substitution of the initial-value equations. So, for the time being, let the evolution equation (7) be replaced by

$$\dot{\gamma}_{ab} = -2\alpha K_{ab} + 2\kappa \mathscr{H}_{ab} , \tag{15}$$

where $\mathscr{H}_{ab}$ is a tensor which we will choose later. And, in keeping with our reinterpretation of ρ and j^a, let the tensor $\mathscr{S}_{ab}$ on the right-hand side of Eq. (9) be similarly unspecified. Both $\mathscr{H}_{ab}$ and $\mathscr{S}_{ab}$ will be chosen in such a manner that they vanish precisely when the constraint equations are satisfied. Hence the exact evolution of an exact solution to the constraint equations will be the same whether evolved according to Eqs. (7) and (9) or Eqs. (15) and (9).

It is simple to evaluate both $\dot{\rho}$ and $j^{a\cdot}$ by Geroch's[13] methods. We find

$$\dot{\rho} = D^a D^b \mathscr{H}_{ab} - D^a D_a \mathscr{H}^b{}_b - R^{ab}\mathscr{H}_{ab} + \alpha K^{ab}\mathscr{S}_{ab} + \alpha K\rho - \alpha D_a j^a - 2 j^a D_a \alpha \tag{16}$$

and

$$j^{a\cdot} = K^{ab} D_b \mathscr{H}^c{}_c - K^{bc} D^a \mathscr{H}_{bc} - D_b(\alpha \mathscr{S}^{ab}) + \alpha j^a K + 2\alpha j^b K_b{}^a - \rho D^a \alpha - 2\kappa j^b \mathscr{H}_b{}^a . \tag{17}$$

When combined, these equations yield a useful and interesting identity:

$$\tfrac{1}{2}(\rho^2 + j_a j^a)^\cdot = \rho D^a D^b \mathscr{H}_{ab} - \rho D^a D_a \mathscr{H} - \rho R^{ab}\mathscr{H}_{ab} + j_a K^{ab} D_b \mathscr{H} - j^a K^{bc} D_a \mathscr{H}_{bc} - \kappa j^a j^b \mathscr{H}_{ab} + \alpha\rho K^{ab}(\mathscr{S}_{ab} - \rho\gamma_{ab}) - j_a D_b(\alpha \mathscr{S}^{ab} - \alpha\rho\gamma^{ab}) + 2\alpha K \rho^2 + \alpha j^a j_a K + \alpha j^a j^b K_{ab} - \alpha^{-3} D_a(\alpha^4 \rho j^a) . \tag{18}$$

This identity can be multiplied by α^3 and integrated over the entire three-dimensional hypersurface. Then, with the assumption that the constraint equations are satisfied on the boundaries of the hypersurface, all divergence terms can be discarded. What remains is

$$\int \tfrac{1}{2}\alpha^3(\rho^2 + j_a j^a)^\cdot dV = \int [-D^a(\alpha^3\rho) D^b(\mathscr{H}_{ab} - \gamma_{ab}\mathscr{H}) - \alpha^3 \rho R^{ab}\mathscr{H}_{ab} + \alpha^3 j_a K^{ab} D_b \mathscr{H} - \alpha^3 j^a K_{bc} D_a \mathscr{H}_{bc} - \kappa\alpha^3 j^a j^b \mathscr{H}_{ab} + \alpha(\mathscr{S}^{ab} - \rho\gamma^{ab})(D_{(a}\alpha^3 j_{b)} + \alpha^3 \rho K_{ab}) + 2\alpha^4 K \rho^2 + \alpha^4 j^a j_a K + \alpha^4 j^a j^b K_{ab}] dV . \tag{19}$$

This equation is the cornerstone of our analysis. If the right-hand side of this equation were negative definite, then for any set of the initial data, whether or not it satisfied the constraint equations, the values of ρ and j^a would decrease in a global sense as the geometry evolved from hypersurface to hypersurface. This would be a nice property for the evolution equations to possess. The evolution would preserve the constraint equations in a stable manner.

But the right-hand side of this equation is of indefinite sign. Just the existence of initial data with a positive right-hand side of Eq. (19) does not alone guarantee difficulty with evolution of the data. A perturbation analysis of ρ and j^a might reveal that all of the modes are stable, but a specific combination of the modes could still give a positive right-hand side.

IV. SOME SIMPLE EXAMPLES

The usual[8,12,14,15] set of vacuum evolution equations has $\mathscr{H}_{ab} = 0$, and

$$\mathscr{S}_{ab} = \rho\gamma_{ab} . \tag{20}$$

With these restrictions Eqs. (16) and (17) simplify to

$$\dot{\rho} = 2\alpha K \rho - \alpha^{-1} D_a(\alpha^2 j^a) \tag{21}$$

and

$$j^{a\cdot} = \alpha j^a K + 2\alpha j^b K_{ab} - 2\kappa j^b \mathscr{H}_b{}^a - \alpha^{-1} D_a(\alpha^2 \rho) , \tag{22}$$

and Eq. (19) becomes

$$\int \tfrac{1}{2}\alpha^3(\rho^2 + j^a j_a)^\cdot dV = \int (2\alpha^4 \rho^2 K + \alpha^4 j^a j_a K + \alpha^4 j^a j^b K_{ab}) dV \tag{23}$$

Unfortunately, under many circumstances the right-hand side of this equation is positive. For example, if a slicing condition is chosen to have $K = 0$, with K_{ab} nonvanishing, then K_{ab} is known to have both positive and negative eigenvalues. A j^a which points in the direction of an eigenvector with a positive eigenvalue will yield a positive right-hand side for Eq. (23). This does not guarantee that evolution of such data will necessarily evolve away from solutions to the constraint equations, but it certainly allows for the possibility.

Consider the following simple situation. Let γ_{ab} be a spherically symmetric, conformally flat metric with

$$\gamma_{ab} = \psi^4 f_{ab} , \tag{24}$$

where ψ is the conformal factor and f_{ab} is a flat metric. For such a geometry the scalar curvature is

$$R = -8\psi^{-5}\nabla^2\psi . \tag{25}$$

For the slicing conditions we let $\alpha = 1$ and the shift vector vanish. For initial data we choose $\psi = \psi_0$, a constant, and $K_{ab} = 0$, so we start with flat, empty space and the evolution via Eqs. (7) and (9) ought to leave us with flat, empty space. But if the initial data are perturbed slightly with

$$K_{ab} = \tfrac{1}{3} K \gamma_{ab} \tag{26}$$

and

$$K = a^{-1} \,, \tag{27}$$

where a is a large constant, then Eq. (7) becomes

$$\psi^{-1}\dot{\psi} = -\tfrac{1}{6}K \,, \tag{28}$$

and Eq. (9) becomes

$$\dot{K} = -8\psi^{-5}\nabla^2\psi + K^2 \,. \tag{29}$$

A solution to these equations, with the initial conditions (26) and (27), is

$$K = (a-t)^{-1} \tag{30}$$

and

$$\psi = \psi_0(1-t/a)^{-1/6} \,. \tag{31}$$

And this instability of the evolution of flat space is also manifest in the solution for the derivation from satisfaction of the constraint equations:

$$\rho = -\kappa^{-1}(a-t)^{-1} \,. \tag{32}$$

So if K is perturbed to a constant greater than zero by numerical error, then the constant a is a large positive number. After a sufficiently long time, when t approaches a, K blows up and the evolution is singular. The geometry on each individual hypersurface in this example is flat, but the nonvanishing of K_{ab} keeps the geometry of the entire four-dimensional spacetime not flat.

It is clear that what might start as a small initial perturbation to K could easily grow into a large deviation from the physical system being modeled. Wilson[3] and others have often circumvented this difficulty by resolving the constraint equations after every time step. This fully constrained evolution gives accurate results but demands a relatively large amount of computing power.

Wilson has also carefully applied a different approach to this numerical problem. This other technique is to use only some components of the evolution equations to evolve only some of the dynamical components of the metric and extrinsic curvature. The remaining components of γ_{ab} and K_{ab} are then found by solving the constraint equations. This modification guarantees that the constraints will be satisfied on every hypersurface. But it must be applied with care as this same, simple example demonstrates.

We use the same assumptions as in the previous example, but we evolve only ψ with Eq. (28) and solve Eq. (4) for K thereby guaranteeing the satisfaction of the constraints. A nontrivial solution valid through first order in the small function ϵ is

$$\psi = 1 + \epsilon + \epsilon^2\lambda^2 r^2/18 \,, \tag{33}$$

$$K = -2\epsilon\lambda \,, \tag{34}$$

where

$$\epsilon = \epsilon_0 e^{\lambda t/3} \tag{35}$$

the quantity ϵ_0 is a small, positive constant, and λ is a positive constant. Both the evolution of ψ and the Hamiltonian constraint are satisfied through the first nontrivial order in ϵ but Eq. (29) for the evolution of K is not satisfied, and, in addition, the perturbed geometry clearly evolves away from flat space in an exponential manner.

We could also try evolving K with Eq. (29) and solving the constraint (4) for ψ. Then

$$K = 3/(a-t) \tag{36}$$

and

$$\psi = 1 + \tfrac{1}{8}r^2(a-t)^{-2} \,, \tag{37}$$

where a is a large positive constant, satisfy the evolution of K and the Hamiltonian constraint through the first order in the small quantity a^{-1}, but Eq. (28) for the evolution of ψ does not hold, and the departure of the geometry from flat space grows as $(a-t)^{-2}$.

This technique, to evolve only some of the dynamical quantities and then to solve the constraints for the remaining ones, has been carefully implemented on problems which do not exhibit the pathologies shown in the previous two examples. But the above examples do point out some of the dangers which might befall a blind application of this approach.

V. A REMEDY TO THE PROBLEM OF THE EVOLUTION OF THE CONSTRAINTS

A different approach to this problem is to modify the evolution equations by substitution of the constraint equations. If a choice for $\mathscr{H}_{ab}$ in Eq. (15) and $\mathscr{S}_{ab}$ in Eq. (9) made the right-hand side of Eq. (19) negative definite, then initial data, not precisely satisfying the constraints, would always evolve toward a solution of the constraints. In that case the evolution would in some sense be stable. Our choices for a modification of the evolution equations are

$$\mathscr{H}_{ab} = -L\alpha^3\rho\gamma_{ab} \tag{38}$$

and

$$\alpha\mathscr{S}_{ab} = \alpha\rho\gamma_{ab} - 2L\alpha^3\rho(K_{ab} - \tfrac{1}{3}K\gamma_{ab}) - LD_{(a}\alpha^3 j_{b)} + \tfrac{1}{3}L\gamma_{ab}D_c(\alpha^3 j^c) \,, \tag{39}$$

where L is a constant with a dimension of length which remains unspecified for the moment. Equation (19) now reduces to

$$\int \tfrac{1}{2}\alpha^3(\rho^2 + j^a j_a)^{\cdot}\, dV = \int \langle \alpha^4(2K\rho^2 + Kj^a j_a + K_{ab}j^a j^b) - L\{2D_a(\alpha^3\rho)D^a(\alpha^3\rho) + [D^{(a}\alpha^3 j^{b)} - \tfrac{1}{3}\gamma^{ab}D_c(\alpha^3 j^c)][D_{(a}\alpha^3 j_{b)} - \tfrac{1}{3}\gamma_{ab}D_c(\alpha^3 j^c)]\} - L\alpha^6(K^{ab}K_{ab}\rho^2 + \tfrac{1}{3}K^2\rho^2 - 2\rho j^a D_a K - 2\kappa\rho^3 - 4\kappa\rho j^a j_a)\rangle\, dV \,. \tag{40}$$

The right-hand side of this expression is, unfortunately, not explicitly negative definite. However, under some rather general circumstances, this formulation of the evolution equations is useful anyway.

A slicing condition often employed in numerical relativity is that K vanishes on each hypersurface. In this case Eq. (40) simplifies considerably to

$$\int \tfrac{1}{2}\alpha^3(\rho^2+j^a j_a)^{\cdot}\, dV = \int \langle \alpha^4 K_{ab} j^a j^b - L\{2D_a(\alpha^3\rho)D^a(\alpha^3\rho)+[D^{(a}\alpha^3\rho^{b)}-\tfrac{1}{3}\gamma^{ab}D_c(\alpha^3 j^c)][D_{(a}\alpha^3 j_{b)}-\tfrac{1}{3}\gamma_{ab}D_c(\alpha^3 j^c)]\} - L\alpha^6(K^{ab}K_{ab}\rho^2-2\kappa\rho^3-4\kappa\rho j^a j_a)\rangle dV \,. \tag{41}$$

Two terms here are of third order in ρ and j^a, and these have κ as a factor. If the initial data are only a perturbation away from satisfying the constraints, then these two terms would be of a higher order in the perturbation than the others. The only remaining term which may be positive is the $j^a j^b K_{ab}$ term, all the others are explicitly negative definite.

In fact we now show that for any given data γ_{ab} and K_{ab} there exists a value of L that makes the right-hand side of Eq. (41) negative definite for nonvanishing values of ρ and j^a.

First consider the possibility that

$$D_{(a}j_{b)}-\tfrac{1}{3}\gamma_{ab}D_c j^c=0 \,. \tag{42}$$

This equation was studied by York,[16] who showed that the operator on the left-hand side is positive definite and Hermitian, and that such a j^a is a conformal Killing vector. Furthermore, if we assume that K is zero, contract Eq. (42) with K^{ab}, integrate over all space, integrate by parts, and discard the surface terms because of boundary conditions, then we are left with

$$-\int j_a D_b K^{ab} dV=0 \,. \tag{43}$$

But, when K is zero, then $D_b K^{ab}=\kappa j^a$; so Eq. (42) implies that j^a vanishes. With both coefficients of L negative definite, L can always be chosen sufficiently large that the right-hand side of Eq. (41) is negative unless the constraint equations are satisfied precisely.

The free parameter L is a mixed blessing. Usually in theoretical physics the fewer free parameters introduced the better. But in this case it is easy to understand the meaning of the dimensionful parameter by analogy with a technique sometimes used to solve Laplace's equation. An artificial time and a diffusion coefficient are added to the elliptic Laplace's equation turning it into the parabolic diffusion equation. The diffusion equation is then evolved forward in the artificial time until a steady state is reached, which then satisfies Laplace's equation. In our case if the terms in the evolution equations (9) and (15) not involving L are just discarded, then the geometry will no longer evolve in time, but instead just evolves to a solution of the constraint equations.

In numerical relativity, if a very large value of L were chosen then the computer would spend most of its time checking to make sure that the constraint equations were well satisfied. And, of course, setting L to zero means that the computer spends no time checking the constraint equations during evolution. Clearly a good choice for L makes the right-hand side of Eq. (41) just barely negative.

VI. SUMMARY

The traditional methods used for the numerical evolution of Einstein's equations have often had difficulty in dealing with the constraint equations. Our modification of the evolution equations ensures a solution of the constraints by continuously nudging both the geometry and the extrinsic curvature. We are currently using this technique both to solve the constraint equations alone and also to evolve the full set of Einstein's equations.

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