Multipole expansions of the general-relativistic gravitational field of the external universe

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The generic, vacuum, dynamical gravitational field in the vicinity of a freely falling observer is expanded in powers of distance away from the observer's spatial origin (i.e., in distance away from his timelike-geodesic world line). The expansion is determined fully, aside from coordinate freedom, by two families of time-dependent multipole moments—"electric-type moments" and "magnetic-type moments"—which characterize the gravitational influence of the external universe. These "external multipole moments" are defined covariantly in terms of the Riemann curvature tensor and its spatial derivatives, evaluated on the observer's world line. The properties of these moments are discussed, and an analysis is given of the structure of the gravitational field's multipole expansion for the special case of de Donder coordinates. In de Donder coordinates the expansion involves only integral powers of distance from the origin; no logarithmic terms occur in this multiparameter expansion.

I. INTRODUCTION AND SUMMARY

In theoretical physics, multipole-moment formalisms are very useful tools for dealing with fields. In a linear field theory, we can decompose the field into its multipole components and study each of them separately; and after the behavior of each multipole component is well understood, we can superimpose the components to get the full field and can use the components as aids to understand it. By contrast, in general relativity, the nonlinearity of the theory prevents us from getting a general solution of the vacuum field equations by a simple superposition of various multipole components. This limits somewhat the use of multipole moments in general relativity; but despite this, relativistic multipole formalisms are still very useful.

One reason is that exact solutions of the Einstein field equations are usually hard to interpret; and their interpretation is aided by the construction of a corresponding Newtonian solution with the same multipole moments as the exact solution. Furthermore, although the exact solution is not a simple linear superposition of multipole components, the coupling among the multipole components at each nonlinear order is definite once the coordinates have been fixed.

Another source of the usefulness of general-relativistic multipoles is the fact that, in the real universe, strongly gravitating bodies are almost always separated from each other by such great distances that their gravitational interactions are weak. This permits those interactions to be characterized by multipolar couplings, with only the lowest few multipoles and only quadratic couplings playing significant roles.

A third source of multipolar usefulness is the fact that almost all sources of gravitational radiation in the real universe are thought to have sufficiently slow internal velocities that the lowest few multipoles dominate the radiation, and the nonlinear coupling between them is of only modest consequence.

There are two types of multipole moments: internal moments and external moments. The internal multipole moments are produced by gravitational sources internal to some region; the external multipole moments by sources external to the region. In the Newtonian theory of gravity, the solutions of the Laplace equation $\nabla^2\phi=0$ can be expressed as

$$\phi=\sum_{l,m} I_{lm}Y^{lm}/r^{l+1}+\sum_{l,m}E_{lm}Y^{lm}r^{l} .$$

With one choice of normalization, the internal multipole moments are the expansion coefficients I_{lm} in front of Y^{lm}/r^{l+1}, and the external multipole moments are the E_{lm} in front of $Y^{lm}r^{l}$.

The gravitational interaction of a bounded system and a complicated external universe can be described in terms of couplings between the internal and external multipole moments. For example, the precession of the Earth's spin axis ("precession of the equinoxes") can be described as due to a coupling between the internal quadrupole moment of the Earth and the external quadrupole moment produced by the Sun, the Moon, and the planets—the external moment being, essentially, the "tidal gravitational field" of these "external" sources (Exercise 16.4 of Ref. 1).

In general relativity, because gravity is produced not only by mass, but also by mass motion ("mass current," "momentum density"), there are two families of multipole moments for both internal and external situations: "mass moments" and "current moments." In the internal case, in addition to the mass moments I_{lm}, which are due largely to the nonuniform distribution of mass, there are also current moments S_{lm} due to rotation, pulsation, and other mass motions. In the external case, in addition to the "electric-type moments" E_{lm}, sometimes also called "mass moments," which are essentially the tidal field and its gradients, there are also the "magnetic-type moments" B_{lm}, also called "current moments," which are produced by motions of external masses and which in turn create velocity-dependent tidal forces on test bodies.

The linearized, stationary solution of the vacuum Einstein field equations can be expressed in terms of these multipole moments. For the sake of illustration we will suppress all normalization constants here. In an appropriate de Donder gauge, the metric has the form[2,3]

$$g_{00}=-1+\frac{2M}{r}+\sum_{l=2}^{\infty}\frac{1}{r^{l+1}}(I_{lm}Y^{lm})-\sum_{l=2}^{\infty}r^{l}(E_{lm}Y^{lm}) , \tag{1.1a}$$

$$g_{0i}=-\sum_{l=1}^{\infty}\frac{1}{r^{l+1}}(S_{lm}Y_i^{l,lm})-\sum_{l=2}^{\infty}r^{l}(B_{lm}Y_i^{l,lm}) , \tag{1.1b}$$

$$g_{ij}=\delta_{ij}\left[1+\frac{2M}{r}+\sum_{l=2}^{\infty}\frac{1}{r^{l+1}}(I_{lm}Y^{lm})-\sum_{l=2}^{\infty}r^{l}(E_{lm}Y^{lm})\right] , \tag{1.1c}$$

and the metric density, defined as $\mathfrak{g}^{\mu\nu}=\sqrt{-g}\,g^{\mu\nu}$, where g is the determinant of $g_{\mu\nu}$, is

$$\mathfrak{g}^{00}=-1-\frac{4M}{r}-\sum_{l=2}^{\infty}\frac{1}{r^{l+1}}(I_{lm}Y^{lm})+\sum_{l=2}^{\infty}r^{l}(E_{lm}Y^{lm}) , \tag{1.2a}$$

$$\mathfrak{g}^{0i}=-\sum_{l=1}^{\infty}\frac{1}{r^{l+1}}(S_{lm}Y_i^{l,lm})-\sum_{l=2}^{\infty}r^{l}(B_{lm}Y_i^{l,lm}) , \tag{1.2b}$$

$$\mathfrak{g}^{ij}=\delta_{ij} . \tag{1.2c}$$

Here and throughout this paper geometric units, in which $G=c=1$, are used; Y^{lm} is the spherical harmonic of order lm, and $Y_i^{l,lm}$ is the vectorial spherical harmonic. Aside from normalization, $\mathbf{Y}^{l,lm}=\mathbf{L}Y^{lm}$, where in three-dimensional notation $\mathbf{L}\equiv(1/i)\mathbf{r}\times\nabla$ (see Sec. III and Ref. 2).

When one studies the gravitational couplings inherent in the Einstein field equations one finds that, as in Newtonian gravity, the laws of motion and precession for an internal body are determined by couplings between the body's internal moments $\mathcal{I}_{A_l}$, $\mathcal{S}_{A_l}$ and the external moments produced by the rest of the universe, $\mathcal{E}_{A_l}$, $\mathcal{B}_{A_l}$.[3–5]

Much research has been done on internal-moment expansions. This research has included (i) the formulation of several different definitions of internal moments;[2,5–8] (ii) proofs that these various definitions are equivalent;[5,8,9] (iii) studies of the relationship of these moments to the properties of spacetime;[5,10,11] and (iv) studies of the roles of these moments in gravitational radiation problems[2,12] and in the laws of motion and precession for internal bodies.[3–5]

There are two main approaches to the definition of internal moments in fully nonlinear general relativity. In the first approach ("physical-spacetime approach") one defines the moments as expansion coefficients in physical spacetime for the metric or some other gravitational quantities.[2,7,8] In the second approach ("conformal-space approach") one performs a conformal transformation on asymptotically flat spacetime, thereby moving "infinity" into a finite location Λ; and one then defines the moments in terms of covariant derivatives of certain quantities in the conformal space at Λ.[6]

Thorne[2] is representative of the physical-spacetime approach. In Thorne's formalism, a special class of coordinates, called "asymptotically Cartesian and mass centered" (ACMC), is introduced; and the multipole moments, defined as certain expansion coefficients of the metric, are the same in all such ACMC coordinate systems. The strength of this approach lies in its usefulness in practical, astrophysically motivated calculations.[13,14]

The conformal-space approach is represented by Geroch and Hansen.[6] Their covariant definition of the multipole moments is very elegant and beautiful, because the multipole moments now become completely geometrical objects living at a specific location Λ in the conformal space. However, because the relationship between conformal space and physical spacetime is somewhat complex, this definition of the moments has not found extensive use in practical calculations.

By contrast with these extensive studies of internal moments, studies of external moments have been limited to two. The first was an appendix in Thorne and Hartle (Ref. 3), which presented the general linearized stationary solution of the vacuum field equations in multipole expansion form (in de Donder gauge), and proposed an iterative algorithm for generating the nonlinear corrections that would convert the linearized solution into a fully nonlinear one. The second was a paper by Suen[5] which studied various aspects of the Thorne-Hartle stationary, external formalism and which thoroughly unified it with Thorne's[2] stationary internal formalism to produce a general multipole analysis of the buffer zone surrounding a stationary body in a stationary external spacetime. In both of these studies (Thorne and Hartle and Suen) the external-moment expansions were combined with internal-moment expansions to produce laws of motion and precession for the internal body; see also Zhang.[4]

The purpose of this paper is to extend the stationary external-moment analyses of Thorne and Hartle and of Suen to fully dynamical situations. More specifically, we shall give a rather thorough treatment of the external-moment problem for fully dynamical, fully nonlinear, vacuum systems; and unlike Thorne and Hartle and Suen, we shall base our treatment on moments that are defined in terms of fully covariant, locally measurable quantities. This has the advantage that we do not have to know the external source distributions.

In Sec. II A we shall introduce into physical spacetime a fiducial timelike-geodesic world line λ about which to do our external multipole expansions; and we shall define the moments locally and covariantly as the symmetric and trace-free (STF) parts (equivalent to the irreducible components of a tensor) of the covariant gradients of the Riemann tensor and itself on λ. These moments will then be functions of the proper time measured by a freely falling observer who moves along λ. More specifically, in a local inertial frame of this observer these moments will take the form (Sec. II B)

$$\mathcal{E}_{A_l}=\frac{1}{(l-2)!}\,(R_{0a_10a_2;a_3\cdots a_l})^{\mathrm{STF}} , \tag{1.3a}$$

$$\mathcal{B}_{A_l}=\frac{3}{2(l+1)(l-2)!}\,(\epsilon_{a_1ij}R_{ija_20;a_3\cdots a_l})^{\mathrm{STF}} . \tag{1.3b}$$

[Here $(\cdots)^{\mathrm{STF}}$ means to take the symmetric, trace-free parts on free indices $a_1a_2\cdots a_l$, and ϵ_{a_1ij} is the flat space Levi-Civita tensor.] At linear order, these moments are just the tensorial representation of E_{lm} and B_{lm} appearing in the metric (1.1) and metric density (1.2) up to normalization constants. At higher nonlinear orders, this is no longer the case. However, relations between these "true" multipole moments and those generated by the metric (density) expansion do exist and will be the subject of our study in Sec. IV.

Since these moments are not coordinate invariant in some class of simply specifiable coordinates, we cannot read them off in a straightforward way from the metric (density). The calculation of these moments in particular, and the analysis of the whole external multipole expansion formalism in general must be done in some tightly specified coordinate. The coordinate we shall use is at the same time de Donder and also "locally inertial and Cartesian" (LIC). The metric and metric density take forms similar to (1.1) and (1.2) with additional time derivative terms [Eq. (3.26)], and at higher nonlinear orders, additional r^l [$(n<l)$-pole] terms.

It is a remarkable feature of our de Donder expansion that, although the spacetime is allowed to be fully dynamical and fully nonlinear, the expansion entails only integrals of power of r. By contrast with internal, dynamical expansion,[2] no $\ln r$ terms appear at any nonlinear order; see Sec. III.

In the remainder of this paper, we shall give a more precise description of the basic ideas and some technical details of the main results. In Sec. II, the concept of the fiducial world line will be introduced, the covariant definition of external multipole moments will be given, and special coordinate systems in which to do multipole expansions will be discussed. In Sec. III we shall study the structure of our de Donder coordinate external multipole expansion of the metric density. Finally, in Sec. IV the relationship between our covariantly defined external moments $\mathscr{E}_{A_l}$, $\mathscr{B}_{A_l}$ and the "moments" obtained from the expansion coefficients of the metric density, which we will call $\mathscr{E}^{\gamma}_{A_l}$, $\mathscr{B}^{\gamma}_{A_l}$, will be derived; and various properties of the moments will be discussed.

II. CONCEPTUAL FOUNDATIONS

A. Fiducial world line and external expansion

Consider some preferred observer who wants to study spacetime in his vicinity by observing the motions of an apparatus that he carries with himself. As he moves through spacetime, his motion defines a world line. This world line will be the fiducial world line of our multipole formalism. Although in some special cases, the preferred observer might be defined by the symmetry of the spacetime, quite generally it will be chosen on the basis of the physics of the problem being studied. Our goal is to characterize, in terms of quantities defined on this fiducial world line, the gravitational field of the external universe.

In analyzing the observer's measurements we shall simplify our analysis by temporarily ignoring the gravitational influence of the apparatus. Its influence can be taken into account, after the formalism for describing the external universe is fully developed, by putting back in the gravity of the apparatus and studying the coupling between the apparatus and the external universe. The Thorne-Hartle[3]-Zhang[4] derivation of laws of motion and precession is an example of such a coupling study.

To further simplify the analysis, we shall assume in this paper that the apparatus (and, with it removed, the fiducial world line) is surrounded by a vacuum region of spacetime. This is the situation one usually encounters in practice. An example which does not have this property is a fiducial world tube in an axion filled, Friedman cosmological model (the axions will pass through the apparatus, which we have thrown away, with near impunity, so the apparatus cannot be regarded as in a local vacuum).[15]

We shall also simplify our analysis by assuming that the fiducial world line is a timelike geodesic. This is a case of common interest, e.g., in the Thorne-Hartle laws of motion and precession; but often one wants to use an accelerated world line. It would not be difficult to generalize our multipole formalism to the accelerated case. The principal effects of such acceleration have been studied already at linear, quadrupole order for a dynamical external universe by Ni and Zimmermann;[16] and at fully nonlinear, all-multipole order for a stationary external universe by Suen.[5] However, for simplicity, this paper's fully nonlinear, all-multipole, fully dynamical analysis will assume zero acceleration.

In the spirit of using locally measurable quantities to characterize the external universe, we introduce here three length scales defined on λ to characterize it in order of magnitude. They are[3]

$$\mathscr{R}=(\text{radius of curvature of spacetime on } \lambda)\ , \tag{2.1a}$$

$$\mathscr{L}=(\text{inhomogeneity scale of curvature})\ , \tag{2.1b}$$

$$\mathscr{T}=(\text{time scale for changes of curvature})\ . \tag{2.1c}$$

These length scales are defined formally, in terms of the multipole moments of Eqs. (1.3), by

$$\mathscr{R}=\operatorname{Min}\left[\frac{1}{|\mathscr{E}_{ij}|^{1/2}}, \frac{1}{|\mathscr{B}_{ij}|^{1/2}}\right], \tag{2.1a$'$}$$

$$\mathscr{L}=\operatorname*{Min}_{l>2}\left[\left[\frac{|\mathscr{E}_{ij}|}{(l-2)!\,|\mathscr{E}_{A_l}|}\right]^{1/(l-2)}, \left[\frac{|\mathscr{B}_{ij}|}{(l+1)(l-2)!\,|\mathscr{B}_{A_l}|}\right]^{1/(l-2)}\right], \tag{2.1b$'$}$$

$$\mathscr{T}=\operatorname*{Min}_{n\geq 1}\left[\left[\frac{|\mathscr{E}_{ij}|}{|d^n\mathscr{E}_{ij}/dt^n|}\right]^{1/n}, \left[\frac{|\mathscr{B}_{ij}|}{|d^n\mathscr{B}_{ij}/dt^n|}\right]^{1/n}\right]. \tag{2.1c$'$}$$

Here time t is measured in a local inertial frame on λ (see Sec. II C). The external expansion will be a power series in the dimensionless variables $r/\mathscr{R}$, $r/\mathscr{L}$, and $r/\mathscr{T}$, where r is the spatial distance away from λ. This multiparameter expansion will be valid and will give good accuracy for the first few terms out to a distance

$r_{\max} \sim \mathrm{Min}(\mathscr{R},\mathscr{L},\mathscr{T})$. (Note that if a gravitational shock wave were to pass through λ, $\mathscr{L}$ and $\mathscr{T}$ would be zero at its point of passage and our formalism would be useless for studying its effects.) Of course when one restores a measuring apparatus to the world line λ, its size must be much less than $r_{\max}$ if our formalism is to be useful for studying its couplings to the external universe.[3]

Throughout the rest of this paper, our attention will focus on external fields with measuring apparatus on λ ignored; the fiducial world line λ will be a timelike geodesic inside a vacuum world tube; and we shall have the mathematical task of characterizing the external field in this world tube, out to a distance $\sim r_{\max}$, in terms of the external multipole moments defined on λ.

B. LIC coordinates and de Donder coordinates

In a covariant theory like general relativity, choosing a well-behaved coordinate system is almost as important as solving the field equations. For example, as is well known, an ill-chosen coordinate system can become singular somewhere even though nothing particularly singular happens to the spacetime there. In this subsection, we will discuss two overlapping families of coordinate systems that are well behaved in the vicinity of the fiducial world line λ and in which one might wish to perform multipole expansions.

In parallel with the ACMC coordinate systems of internal problems, we introduce for external problems the class of "locally inertial and Cartesian" (LIC) coordinate systems. A coordinate system (t, x^1, x^2, x^3) is LIC if and only if (i) its spatial origin $x^i=0$ lies on λ at all times; and (ii) the coordinate components of the metric are expandable about λ in powers of $r=[(x^1)^2+(x^2)^2+(x^3)^2]^{1/2}$, and the expansion takes the form

$$g_{\mu\nu}=\eta_{\mu\nu}+\sum_{n=2}^{\infty} r^n[(n\text{-pole})+\cdots+(0\text{-pole})] , \tag{2.2a}$$

or if expressed in terms of metric density perturbation $\bar{h}^{\mu\nu}\equiv\eta^{\mu\nu}-\mathfrak{g}^{\mu\nu}$

$$\bar{h}^{\mu\nu}=\sum_{n=2}^{\infty} r^n[(n\text{-pole})+\cdots+(0\text{-pole})] , \tag{2.2b}$$

where $\eta_{\mu\nu}$ is the flat metric (in Lorentz coordinates) and "l-pole" means a time-dependent, r-independent spherical harmonic of order l. Note that this definition guarantees that on λ the coordinate basis vectors form an orthonormal frame that is Fermi transported along λ.

It is not obvious from the outset that there exist any coordinate systems that have all these properties. However, we shall demonstrate below (Sec. III) that these properties are, in fact, satisfied by a subclass of de Donder coordinate systems.

In LIC coordinates, many calculations become greatly simplified. One example is the definition of the external moments, which reduces the precise definition (2.10), as we will give in Sec. II C, to (1.3) in LIC coordinates. But by contrast with the internal problem, where the moments read off the expansion coefficients are the same in all ACMC coordinate systems even at nonlinear orders, the external moments read off the metric or metric density are not the same in all LIC coordinate systems.

For example, consider a precisely static spacetime, so $\mathscr{B}_{ab}=0$, and examine it up to quartic order in r. More specifically, let the metric density $\mathfrak{g}^{\mu\nu}_{\mathrm{DD}}=\eta^{\mu\nu}-\bar{h}^{\mu\nu}_{\mathrm{DD}}$ in de Donder coordinates x^{μ}_{DD} have the form

$$\bar{h}^{00}_{\mathrm{DD}}(x^{\mu}_{\mathrm{DD}})=2\mathscr{E}_{ab}x^a_{\mathrm{DD}}x^b_{\mathrm{DD}} + r^4_{\mathrm{DD}}[(2\text{-pole})+(0\text{-pole})] , \tag{2.3a}$$

$$\bar{h}^{0i}_{\mathrm{DD}}(x^{\mu}_{\mathrm{DD}})=0 , \tag{2.3b}$$

$$\bar{h}^{ij}_{\mathrm{DD}}(x^{\mu}_{\mathrm{DD}})=r^4_{\mathrm{DD}}[(2\text{-pole})+(0\text{-pole})] + r^2_{\mathrm{DD}}(0\text{-pole}) , \tag{2.3c}$$

where $\mathscr{E}_{ab}$ is independent of time and the (2-pole) and (0-pole) terms are constructed from products of $\mathscr{E}_{ab}$ with itself in such a manner as to guarantee satisfaction of the vacuum Einstein equations and the de Donder gauge condition. Then to quartic order

$$g^{\mathrm{DD}}_{00}(x^{\mu}_{\mathrm{DD}})=-1+\mathscr{E}_{ab}x^a_{\mathrm{DD}}x^b_{\mathrm{DD}} -\tfrac{3}{2}(\mathscr{E}_{ab}\mathscr{E}_{cd})^{\mathrm{STF}}x^a_{\mathrm{DD}}x^b_{\mathrm{DD}}x^c_{\mathrm{DD}}x^d_{\mathrm{DD}} + r^4_{\mathrm{DD}}[(2\text{-pole})+(0\text{-pole})]+r^2_{\mathrm{DD}}(0\text{-pole}) , \tag{2.4a}$$

$$g^{\mathrm{DD}}_{0i}(x^{\mu}_{\mathrm{DD}})=0 , \tag{2.4b}$$

$$g^{\mathrm{DD}}_{ij}(x^{\mu}_{\mathrm{DD}})=r^4_{\mathrm{DD}}[(4\text{-pole})+(2\text{-pole})+(0\text{-pole})] + r^2_{\mathrm{DD}}[(2\text{-pole})+(0\text{-pole})] . \tag{2.4c}$$

Let us make a coordinate transformation

$$t_{\mathrm{FN}}=t_{\mathrm{DD}} , \tag{2.5a}$$

$$x^i_{\mathrm{FN}}=x^i_{\mathrm{DD}}+\tfrac{1}{6}\mathscr{E}_{ia}x^a_{\mathrm{DD}}r^2_{\mathrm{DD}}-\tfrac{1}{3}\mathscr{E}_{ab}x^a_{\mathrm{DD}}x^b_{\mathrm{DD}}x^i_{\mathrm{DD}} . \tag{2.5b}$$

(Incidentally, this transformation brings us into Fermi normal coordinate.[17]) The metric has the form

$$g^{\mathrm{FN}}_{00}(x^{\mu}_{\mathrm{FN}})=-1+\mathscr{E}_{ab}x^a_{\mathrm{FN}}x^b_{\mathrm{FN}} -\tfrac{5}{6}(\mathscr{E}_{ab}\mathscr{E}_{cd})^{\mathrm{STF}}x^a_{\mathrm{FN}}x^b_{\mathrm{FN}}x^c_{\mathrm{FN}}x^d_{\mathrm{FN}} + r^4_{\mathrm{FN}}[(2\text{-pole})+(0\text{-pole})]+r^2_{\mathrm{FN}}(0\text{-pole}) , \tag{2.6a}$$

$$g^{\mathrm{FN}}_{0i}(x^{\mu}_{\mathrm{FN}})=0 , \tag{2.6b}$$

$$g^{\mathrm{FN}}_{ij}(x^{\mu}_{\mathrm{FN}})=r^4_{\mathrm{FN}}[(4\text{-pole})+(2\text{-pole})+(0\text{-pole})] + r^2_{\mathrm{FN}}[(2\text{-pole})+(0\text{-pole})] \tag{2.6c}$$

corresponding to, at linear order,

$$\bar{h}^{00}_{\mathrm{FN}}(x^{\mu}_{\mathrm{FN}})=\tfrac{2}{3}\mathscr{E}_{ab}x^a_{\mathrm{FN}}x^b_{\mathrm{FN}}+O(r^4_{\mathrm{FN}}) , \tag{2.7a}$$

$$\bar{h}^{0i}_{\mathrm{FN}}(x^{\mu}_{\mathrm{FN}})=0 , \tag{2.7b}$$

$$\bar{h}^{ij}_{\mathrm{FN}}(x^{\mu}_{\mathrm{FN}})=r^2_{\mathrm{FN}}[(2\text{-pole})+(0\text{-pole})]+O(r^4_{\mathrm{FN}}) . \tag{2.7c}$$

Note that the $O(r^2)$ quadrupolar part of $\bar{h}^{00}_{\mathrm{DD}}$ is different from that of $\bar{h}^{00}_{\mathrm{FN}}$ (factor 2 versus $\frac{2}{3}$); this shows that, even at linear order, the quadrupole moments read off of

$\bar{h}^{\mu\nu}$ differ in the two coordinate systems. Note further that the $O(r^2)$ quadrupolar parts of $g_{00}^{\rm DD}$ and $g_{00}^{\rm FN}$ are the same, but the 4-pole parts are different (factor 0 versus factor $-\frac{5}{6}$). Thus, if one were to try to read multipole moments off g_{00} one would find different moments at the quadratically nonlinear order in the two coordinate systems. However, in a precisely fixed de Donder coordinate system this confusion of "multipole moments" obtained from metric (density) expansion will be taken away in stationary as well as in dynamic situations.

A coordinate system is de Donder if and only if its metric density satisfies the gauge condition

$$\mathfrak{g}^{\alpha\beta}{}_{,\beta}=0 \ . \tag{2.8}$$

In any de Donder coordinate system, the Einstein field equations and the Landau-Lifshitz pseudotensor are much simplified. This makes de Donder coordinates especially useful in practical calculations. For example, de Donder coordinates were used in the Thorne-Hartle-Zhang derivations of laws of motion and precession;[3,4] in Suen's[5] thorough treatment of stationary systems with both external and internal moments; in studies of gravitational waves from an isolated source by Blanchet and Damour[12] and others; and the proof by Thorne and Gürsel[14] that the free precession of a slowly rotating, rigid, general relativistic body is governed by the classical nonrelativistic Euler equations.

The four de Donder coordinate conditions $\mathfrak{g}^{\alpha\beta}{}_{,\beta}=0$ reduce the number of independent components of $\mathfrak{g}^{\mu\nu}$ to six (there are ten initially). Any further coordinate transformation

$$x^{\mu'}=x^{\mu}+\xi^{\mu}$$

leaves the coordinate system de Donder if (2.8) is not violated. At the linear order, this is achieved if ξ^μ satisfies[18]

$$\Box\xi^{\mu}=0 \ . \tag{2.9}$$

By an appropriate choice of ξ^μ, the number of dynamically independent metric coefficients is further reduced to two (e.g., the two polarizations of a gravitational wave); correspondingly, as we shall see in Secs. III and IV, the full spacetime metric near λ is rigidly fixed in terms of the time development of only two families of moments $\mathscr{E}_{A_l}$ and $\mathscr{B}_{A_l}$. Our criterion for choosing ξ^μ at linear order will be that in the stationary limit (i.e., at zero order in $r/\mathscr{T}$) $\bar{h}_{ij}$ should vanish. The pure time derivative terms in the linearized $\bar{h}_{ij}$ and $\bar{h}_{0i}$ of Eqs. (3.26) will then be forced to be present by the de Donder gauge condition (2.8). At higher, nonlinear orders we shall specialize our de Donder gauge to keep $\bar{h}^{\mu\nu}$ in LIC form (2.2) with "moments" $\mathscr{E}^{\gamma}_{A_l}$, $\mathscr{B}^{\gamma}_{A_l}$ always corresponding to the coefficients in front of r^lY^{lm} or $r^lY_i^{l,lm}$, no matter what the nonlinear order is.

C. Definitions of the multipole moments

In the external problem we lose the ability of reading the moments directly off the metric (density) as one has been able to do in the internal case. However, the expansion around λ in vacuum permits us to define these moments covariantly without performing any conformal transformation. More specifically, we shall define the multipole moments of the external spacetime as STF parts of gradients of the Riemann tensor and itself, evaluated on the physical spacetime's fiducial world line λ:

$$\mathscr{E}_{\alpha_1\alpha_2\cdots\alpha_l}\equiv\frac{1}{(l-2)!}(P^{\beta_1}_{\alpha_1}P^{\beta_2}_{\alpha_2}\cdots P^{\beta_l}_{\alpha_l})^{\rm STF}\times R_{\mu\beta_1\nu\beta_2;\beta_3\cdots\beta_l}u^{\mu}u^{\nu}|_{\lambda} \ , \tag{2.10a}$$

$$\mathscr{B}_{\alpha_1\alpha_2\cdots\alpha_l}\equiv\frac{3}{2(l+1)(l-2)!}(P^{\beta_1}_{\alpha_1}P^{\beta_2}_{\alpha_2}\cdots P^{\beta_l}_{\alpha_l})^{\rm STF}\times \epsilon_{\mu\beta_1}{}^{\gamma\delta}R_{\gamma\delta\beta_2\nu;\beta_3\cdots\beta_l}u^{\mu}u^{\nu}|_{\lambda} \ . \tag{2.10b}$$

Here $P^{\mu}{}_{\nu}=g^{\mu}{}_{\nu}+u^{\mu}u_{\nu}$ is the projection tensor which projects into the local 3-space orthogonal to the 4-velocity $\mathbf{u}$; and $\mathbf{u}$ is the 4-velocity (i.e., tangent unit vector) of the fiducial world line λ, or equivalently of the preferred observer who moves along that world line.

As defined by Eqs. (2.10) the external multipole moments $\mathscr{E}_{\alpha_1\alpha_2\cdots\alpha_l}$, $\mathscr{B}_{\alpha_1\alpha_2\cdots\alpha_l}$ are symmetric, trace-free 4-tensors defined on λ and totally orthogonal to λ. If one changes from λ to some other fiducial world line, these moments will change. When one performs a 3+1 split of spacetime from the viewpoint of the preferred observer, $\mathscr{E}_{\alpha_1\alpha_2\cdots\alpha_l}$ and $\mathscr{B}_{\alpha_1\alpha_2\cdots\alpha_l}$ become fully spatial, three-dimensional, STF tensor fields $\mathscr{E}_{a_1\cdots a_l}=\mathscr{E}_{A_l}$, $\mathscr{B}_{a_1\cdots a_l}=\mathscr{B}_{A_l}$, defined at the origin of the local 3-space of the preferred observer, and evolving with time as measured by that observer; and the definitions (2.10) reduce to those of Eqs. (1.3).

In external expansions, the expansion coefficients are expressed in terms of these multipole moments. Although these moments are defined in a coordinate-independent way, the details of the expansion still depend on the coordinate system used.

The remainder of this paper (Secs. III and IV) will deal with the expansion of the metric density, metric, and Riemann tensor in rigidly fixed de Donder coordinates— which, of course are also LIC. But in proving some of the properties of these expansions, we shall switch from Cartesian coordinates to the corresponding spherical polar coordinate system from time to time.

III. STRUCTURE OF THE EXTERNAL EXPANSION IN A RIGIDLY FIXED DE DONDER COORDINATE SYSTEM

A. Notations and mathematical formulas

In this subsection, we will explain the notations to be used in our de Donder-coordinate-system multipole expansion; and we will give some mathematical formulas which will be used in developing the expansion formalism.

The notation we shall use is that of Thorne.[2] Symmetric traceless tensors with l spatial indices ("STF-l tensors") will be denoted by capital script letters, as $\mathscr{F}_{A_l}$, which is shorthand for $(\mathscr{F}_{a_1a_2\cdots a_l})^{\rm STF}$. A product of l

spatial coordinates $x_{a_1}x_{a_2}\cdots x_{a_l}$ will be denoted X_{A_l}. The raising and lowering of these indices are effected by the Kronecker delta $\delta^i{}_j$. Thus $\mathcal{F}_{A_l}X_{A_l}$ is the same as $\mathcal{F}_{a_1a_2\cdots a_l}x^{a_1}x^{a_2}\cdots x^{a_l}$. The STF harmonics and the spherical harmonics Y_{lm} (scalar), $Y_i^{l',lm}$ (vector) and $T_{ij}^{\lambda l',lm}$ (tensor) will be used extensively in this paper. The scalar spherical harmonics Y^{lm} are well known. The vector and tensor spherical harmonics are obtained by Clebsch-Gordan coupling $Y^{l'm'}$ to the basis vectors

$$\xi^0\equiv\mathbf{e}_z\ ,\quad \xi^{\pm1}\equiv\mp(\mathbf{e}_x\pm i\mathbf{e}_y)/\sqrt{2}\ , \tag{3.1}$$

and the basis tensors

$$\overleftrightarrow{\mathrm{t}}^{\pm2}=\tfrac{1}{2}\,(\mathbf{e}_x\otimes\mathbf{e}_x-\mathbf{e}_y\otimes\mathbf{e}_y)\pm\frac{i}{2}\,(\mathbf{e}_x\otimes\mathbf{e}_y+\mathbf{e}_y\otimes\mathbf{e}_x)\ , \tag{3.2a}$$

$$\overleftrightarrow{\mathrm{t}}^{\pm1}=\mp\tfrac{1}{2}\,(\mathbf{e}_x\otimes\mathbf{e}_z+\mathbf{e}_z\otimes\mathbf{e}_x)-\frac{i}{2}\,(\mathbf{e}_y\otimes\mathbf{e}_z+\mathbf{e}_z\otimes\mathbf{e}_y)\ , \tag{3.2b}$$

$$\overleftrightarrow{\mathrm{t}}^{0}=6^{-1/2}(-\mathbf{e}_x\otimes\mathbf{e}_x-\mathbf{e}_y\otimes\mathbf{e}_y+2\mathbf{e}_z\otimes\mathbf{e}_z)\ , \tag{3.2c}$$

$$\overleftrightarrow{\delta}=\mathbf{e}_x\otimes\mathbf{e}_x+\mathbf{e}_y\otimes\mathbf{e}_y+\mathbf{e}_z\otimes\mathbf{e}_z\ . \tag{3.2d}$$

For a beautiful review of relations between these Clebsch-Gordan-coupled harmonics and STF harmonics and for reviews of some of their properties, see Thorne.[2] As discussed by Thorne, any STF-l tensor $\mathcal{F}_{K_l}$ can be expanded as

$$\mathcal{F}_{K_l}=\sum_m F^{lm}\mathcal{Y}^{lm}_{K_l}\ , \tag{3.3}$$

where $\mathcal{Y}^{lm}_{K_l}$ is related to Y^{lm} through

$$Y^{lm}=\mathcal{Y}^{lm}_{K_l}N_{K_l}\ . \tag{3.4}$$

Therefore we have the correspondence between $\mathcal{E}_{A_l}$ and E_{lm}, $\mathcal{B}_{A_l}$ and B_{lm}.

For the convenience of discussion below, we shall abbreviate Y^{lm}, $Y_i^{l',lm}$, and $T_{ij}^{\lambda l',lm}$ as $H^{l'lm}$. In $H^{l'lm}$, l' is the eigenvalue of the orbital angular momentum operator $L^2=-(\mathbf{r}\times\nabla)^2$

$$L^2H^{l'lm}=l'(l'+1)H^{l'lm}\ ;$$

l is the order of the irreducible representation of the rotation group generated by $H^{l'lm}$; and m is the azimuthal quantum number.[2] In our analysis, we shall often encounter the Poisson equation

$$\nabla^2 f=H^{l'lm}r^{l'+n}\ , \tag{3.5}$$

which can be inverted as

$$f=\frac{1}{(n+2)(n+2l'+3)}\,H^{l'lm}r^{l'+n+2}\quad \text{for } n\neq-2\ . \tag{3.6}$$

One can easily convince oneself that (3.6) is a solution of (3.5) by writing the Laplacian ∇^2 as $(1/r^2)\partial_r(r^2\partial_r)-L^2/r^2$.

An important lemma, which we shall use extensively, is this: *Any scalar, vector, or second-rank symmetric tensor, constructed from products of quantities with the form* $(H^{l'lm}r^{l'+2k})_{,i_1\cdots i_n}$ *(where* $k\geq0$ *and the comma denotes* $n\geq0$ *spatial derivatives) must have the form* $H^{L'LM}r^{L'+2K}$ *with* $K\geq0$. Proof: $H^{l'lm}r^{l'+2k}$ is a sum of terms[2] like $X_{B_s}\mathcal{Y}^{lm}_{C_pA_q}X_{A_q}r^{2k'}$ with the constraints

$$p+q=l\ ,\quad s+q=l'-2(k'-k)\ ,\quad k'\geq k\ . \tag{3.7}$$

When we take one spatial derivative of such a term we get

$$\partial_i(X_{B_s}\mathcal{Y}^{lm}_{C_pA_q}X_{A_q}r^{2k})=\sum_j\delta_{ib_j}X_{B_{j-1}b_{j+1}\cdots b_s}\mathcal{Y}^{lm}_{C_pA_q}X_{A_q}r^{2k}+qX_{B_s}\mathcal{Y}^{lm}_{C_piA_{q-1}}X_{A_{q-1}}r^{2k}+2kr^{2k-2}x_iX_{B_s}\mathcal{Y}^{lm}_{C_pA_q}X_{A_q}$$
$$\sim X_{B'_s}\mathcal{Y}^{lm}_{C'_pA'_q}X_{A'_q}r^{2k_1}\ , \tag{3.8}$$

where $\sim$ reads "a sum of terms of the form," and $k_1=k-\Delta k\geq0$ with $\Delta k=0$ or 1. If we take more derivatives, the range of Δk increases but k_1 remains positive. When we couple two such terms together to get scalar, vector, or tensor spherical harmonics, we obtain

$$X_{B_{s_1}}X_{B'_{s_2}}\mathcal{Y}^{l_1m_1}_{C_{p_1}A_{q_1}}\mathcal{Y}^{l_2m_2}_{C'_{p_2}A'_{q_2}}X_{A_{q_1}}X_{A'_{q_2}}r^{2k'_1+2k'_2}$$
$$\sim(\mathcal{Y}^{Lm}_{A_L}X_{A_L}r^n,\ x_i\mathcal{Y}^{Lm}_{A_L}X_{A_L}r^n,\ \mathcal{Y}^{Lm}_{iA_{L-1}}X_{A_{L-1}}r^n,\ x_ix_j\mathcal{Y}_{A_L}X_{A_L}r^n,\ x_i\mathcal{Y}^{Lm}_{jA_{L-1}}X_{A_{L-1}}r^n,\ \mathcal{Y}^{Lm}_{ijA_{L-2}}X_{A_{L-2}}r^n)\ . \tag{3.9}$$

In Eq. (3.9) the coupled terms have $s\leq2$, $l-2\leq q\leq l$ because we only want to form from them tensors of rank ≤2. Since the number of x_i's has to decrease by pairs, the following relation holds:

$$s_1+s_2+q_1+q_2=s+q+(\text{even number})\ . \tag{3.10}$$

A simple counting of powers of r on both sides of Eq. (3.9) gives

$$s_1+s_2+q_1+q_2+2k'_1+2k'_2=s+q+n\ . \tag{3.11}$$

Equations (3.10) and (3.11) can be combined to give

$$n=2k'_1+2k'_2+(\text{even number})=2K'\ ,\quad K'=0,1,2,\ldots\ .$$

If we look back at the constraints (3.7), it becomes clear that from these coupled terms we can only obtain $H^{L'LM}r^{L'+2K}$ with the condition

$$
\begin{aligned}
2K &= s+q+2K'-L' \\
&= s_1+s_2+q_1+q_2+2k'_1+2k'_2-L' \\
&= l'_1+l'_2+2k_1+2k_2-L' \geq 0 \ .
\end{aligned}
$$

This conclusion can also be proved more elegantly using Clebsch-Gordan coupling techniques, but we shall not do so here.

Finally, for later use we shall cite a few formulas from Mathews:[19]

$$
(f_l(kr)Y_i^{l'lm})_{,i} = \begin{cases} -k\sqrt{l/(2l+1)}f_l(kr)Y^{lm}\ , & l'=l-1\ , \quad (3.12a) \\ 0, \ l'=l\ , & \quad (3.12b) \\ -k\sqrt{(l+1)/(2l+1)}f_l(kr)Y^{lm}\ , & l'=l+1\ , \quad (3.12c) \end{cases}
$$

$$
(f_l(kr)T_{ij}^{2,l'lm})_{,j} = \begin{cases} -k\sqrt{(l-1)/(2l-1)}f_{l-1}(kr)Y_i^{l-1,lm}, \ l'=l-2\ , & (3.13a) \\ -k\sqrt{(l-1)/2(2l+1)}f_l(kr)Y_i^{l,lm}, \ l'=l-1\ , & (3.13b) \\ -k[\sqrt{(l+1)(2l+3)/6(2l+1)(2l-1)}f_{l-1}(kr)Y_i^{l-1,lm} & \\ \quad +\sqrt{l(2l-1)/6(2l+1)(2l+3)}f_{l+1}(kr)Y_i^{l+1,lm}], \ l'=l\ , & (3.13c) \\ -k\sqrt{(l+2)/2(2l+1)}f_l(kr)Y_i^{l,lm}, \ l'=l+1\ , & (3.13d) \\ -k\sqrt{(l+2)/(2l+3)}f_{l+1}(kr)Y_i^{l+1,lm}, \ l'=l+2\ . & (3.13e) \end{cases}
$$

Here $f_l(kr)$ is any spherical Bessel function.

B. Algorithm for constructing the (nonlinear) metric density in a rigidly fixed de Donder coordinate system

Thorne and Hartle[3] have proposed, in their Appendix, an iterative algorithm for generating the multipolar expansion of the fully nonlinear, dynamical metric density in the vacuum region surrounding λ; but they did not analyze the mathematical details or consequences of the algorithm. In this section we shall review briefly the proposed algorithm.

In terms of the metric density $\mathfrak{g}^{\mu\nu}$ or $\bar{h}^{\mu\nu}=\eta^{\mu\nu}-\mathfrak{g}^{\mu\nu}$, the vacuum Einstein field equations in de Donder gauge read

$$\Box\bar{h}^{\mu\nu}=W^{\mu\nu}\ , \tag{3.14a}$$

$$\bar{h}^{\mu\nu}{}_{,\nu}=0\ , \tag{3.14b}$$

where

$$W^{\mu\nu}=-16\pi(-g)t^{\mu\nu}+\bar{h}^{\mu\nu}{}_{,\alpha\beta}\bar{h}^{\alpha\beta}-\bar{h}^{\mu\alpha}{}_{,\beta}\bar{h}^{\nu\beta}{}_{,\alpha}\ ; \tag{3.15}$$

$t^{\mu\nu}$ is the Landau-Lifshitz pseudotensor [Eq. (20.20) of Ref. 1]

$$
\begin{aligned}
16\pi(-g)t^{\mu\nu} &= \bar{h}^{\mu\nu}{}_{,\lambda}\bar{h}^{\lambda}{}_{,\alpha}-\bar{h}^{\mu\lambda}{}_{,\lambda}\bar{h}^{\nu\alpha}{}_{,\alpha}+\tfrac{1}{2}\,g^{\mu\nu}g_{\lambda\alpha}\bar{h}^{\lambda\beta}{}_{,\rho}\bar{h}^{\rho\alpha}{}_{,\beta} \\
&\quad -(g^{\mu\lambda}g_{\alpha\beta}\bar{h}^{\nu\beta}{}_{,\rho}\bar{h}^{\alpha\rho}{}_{,\lambda}+g^{\nu\lambda}g_{\alpha\beta}\bar{h}^{\mu\beta}{}_{,\rho}\bar{h}^{\alpha\rho}{}_{,\lambda})+g_{\lambda\mu}g^{\rho\beta}\bar{h}^{\mu\lambda}{}_{,\beta}\bar{h}^{\nu\alpha}{}_{,\rho} \\
&\quad +\tfrac{1}{8}\,(2g^{\mu\lambda}g^{\nu\alpha}-g^{\mu\nu}g^{\lambda\alpha})(2g_{\beta\rho}g_{\sigma\tau}-g_{\rho\sigma}g_{\beta\tau})\bar{h}^{\beta\tau}{}_{,\lambda}\bar{h}^{\rho\sigma}{}_{,\alpha}\ .
\end{aligned} \tag{3.16}
$$

The box $\Box$ which acts on each component of $\bar{h}^{\mu\nu}$ in (3.14a) is the flat-space d'Alembertian, $\Box=\eta_{\mu\nu}(\partial/\partial x^{\mu})(\partial/\partial x^{\nu})$; and the compatibility of (3.14a) and (3.14b), i.e., the existence of a solution, is guaranteed by the identity $W^{\mu\nu}{}_{,\nu}\equiv 0$.

To solve Eqs. (3.14) one can proceed in either or both of two directions: First, one can expand $\bar{h}^{\mu\nu}$ in nonlinear orders p

$$\bar{h}^{\mu\nu}=\sum G^p\gamma_p^{\mu\nu}\ . \tag{3.17}$$

Here G is the gravitational constant which is actually equal to unity in our units, but is kept to serve the purpose of "bookkeeping." The source terms $W^{\mu\nu}$ are also expanded in this way:

$$W^{\mu\nu}=\sum_p G^pW_p^{\mu\nu}\ , \tag{3.18}$$

where $W_p^{\mu\nu}$ is constructable via Eqs. (3.15), (3.16), from products of $\gamma_{p'}^{\alpha\beta}$ with $p'_1+p'_2+\cdots=p$ and thus with each $p'\leq p-1$. One can then solve the field equations and gauge conditions (3.14) iteratively,

$$\Box\gamma_p^{\mu\nu}=W_p^{\mu\nu}\ , \tag{3.19a}$$

$$\gamma_p^{\mu\nu}{}_{,\nu}=0\ , \tag{3.19b}$$

first for $p=1$ (where $W_p^{\mu\nu}=0$), then for $p=2,3,\ldots$. Unfortunately, Eqs. (3.19) are hard to solve in most cases.

One can also proceed in a second direction. First one makes a multiparameter expansion of the metric density

$\bar{h}^{\mu\nu}$ and the source terms $W^{\mu\nu}$ in terms of the small dimensionless variables $r/\mathscr{R}$, $r/\mathscr{L}$, $r/\mathscr{T}$; and then one decomposes each term into spherical harmonic components of order l:

$$\bar{h}^{\mu\nu}=\sum_{p,n,u,l}(\gamma^{\mu\nu})_{pnul}\,, \tag{3.20a}$$

$$(\gamma^{\mu\nu})_{pnul}\sim\left[\left[\frac{r}{\mathscr{R}}\right]^{2p}\left[\frac{r}{\mathscr{L}}\right]^{n}\left[\frac{r}{\mathscr{T}}\right]^{u}\right]_{\text{order } l}\,, \tag{3.20b}$$

$$W^{\mu\nu}=\sum W^{\mu\nu}_{pnul}\,. \tag{3.21}$$

One can verify from the definition of $\mathscr{R}$ that the p which appears here is the same as the p of the nonlinearity expansion (3.17). The field equations and gauge conditions (3.14) then take the form

$$\nabla^2\gamma^{\mu\nu}_{pnul}=W^{\mu\nu}_{pnul}+\partial_t^2\gamma^{\mu\nu}_{pn(u-2)l}\,, \tag{3.22a}$$

$$\gamma^{\mu 0}_{pn(u-1)l,0}+\gamma^{\mu i}_{pnul,i}=0\,. \tag{3.22b}$$

These equations, like (3.19), can then be solved iteratively. From Eqs. (3.5) and (3.6) one can find a particular integral of (3.22a); this is far easier than solving (3.19). One must then add to this particular integral a homogeneous solution of (3.22a), chosen so as to enforce the gauge condition (3.22b). In Secs. III D and III E, we will elucidate some of the mathematical details and consequences of this algorithm.

C. The linear part of the algorithm

The linear order ($p=1$) of the above algorithm is simplified by the fact that $W^{\mu\nu}_1=0$. The field equations and gauge conditions thus read

$$\Box\gamma^{\mu\nu}_1=0\,, \tag{3.23}$$

$$\gamma^{\mu\nu}_{1\ ,\nu}=0\,. \tag{3.24}$$

These can be solved directly without a multiparameter expansion. The desired solution, which must go as r^n with $n\geq 2$ at the origin because of our LIC coordinates, can be constructed as superpositions of "normal modes":

$$\gamma^{\mu\nu}_1\sim\sum_{l'lm}\int C_{l'lm\omega}j_{l'}(\omega r)e^{i\omega t}H^{l'lm}d\omega \quad \text{for } l\geq 2\,, \tag{3.25}$$

where $j_{l'}$ are spherical Bessel functions and $\omega\sim 1/\mathscr{T}$. The superposition coefficients $C_{l'lm\omega}$ are fixed by the demands that (i) at stationary order [i.e., at leading order $(\omega r)^{l'}$ in a power-series expansion of $j_{l'}$] $\gamma^{\mu\nu}_1=\eta^{\mu\nu}-\mathfrak{g}^{\mu\nu}$ must take on the linearized, stationary, Thorne-Hartle form (1.2), and (ii) the gauge condition (3.21) must be satisfied. A straightforward but rather tedious calculation, including a power-series expansion of $j_{l'}$, then gives (restoring all normalization constants)

$$\bar{h}^{00}=-\sum_{l=2}^{\infty}\frac{4(2l+1)!!}{l(l-1)}\sum_{k=0}^{\infty}\frac{r^{2k}}{2^k k!(2l+1+2k)!!}\,{}^{(2k)}\mathscr{E}^{\gamma}_{A_l}X_{A_l}\,, \tag{3.26a}$$

$$\begin{aligned}\bar{h}^{0i}=&-\sum_{l=2}^{\infty}\frac{2(2l+1)!!}{3(l-1)}\sum_{k=0}^{\infty}\frac{r^{2k}}{2^k k!(2l+1+2k)!!}\,\epsilon_{ipq}x_p\,{}^{(2k)}\mathscr{B}^{\gamma}_{qA_{l-1}}X_{A_{l-1}}\\&+\sum_{l=2}^{\infty}\frac{4(2l+1)(2l+1)!!}{(l+1)l(l-1)}\sum_{k=0}^{\infty}\frac{r^{2k}}{2^k k!(2l+3+2k)!!}\left[{}^{(2k+1)}\mathscr{E}^{\gamma}_{A_l}X_{A_l}x_i-\frac{l}{2l+1}\,{}^{(2k+1)}\mathscr{E}^{\gamma}_{iA_{l-1}}X_{A_{l-1}}r^2\right]\,,\end{aligned} \tag{3.26b}$$

$$\begin{aligned}\bar{h}^{ij}=&\sum_{l=2}^{\infty}\frac{4(2l+1)(2l+1)!!}{3(l+2)(l-1)}\sum_{k=0}^{\infty}\frac{r^{2k}}{2^k k!(2l+3+2k)!!}\left[x_{(i}\epsilon_{j)pq}\,{}^{(2k+1)}\mathscr{B}^{\gamma}_{qA_{l-1}}x_pX_{A_{l-1}}-\frac{l-1}{2l+1}\,\epsilon_{pq(i}\,{}^{(2k+1)}\mathscr{B}^{\gamma}_{j)qA_{l-2}}x_pX_{A_{l-2}}r^2\right]\\&-\sum_{l=2}^{\infty}\frac{4(2l+3)(2l+1)(2l+1)!!}{(l+2)(l+1)l(l-1)}\\&\times\sum_{k=0}^{\infty}\frac{r^{2k}}{2^k k!(2l+5+2k)!!}\\&\times\left[{}^{(2k+2)}\mathscr{E}^{\gamma}_{A_l}X_{A_l}x_ix_j-\frac{2l}{2l+3}\,x_{(i}\,{}^{(2k+2)}\mathscr{E}^{\gamma}_{j)A_{l-1}}X_{A_{l-1}}r^2+\frac{l(l-1)}{(2l+1)(2l+3)}\,{}^{(2k+2)}\mathscr{E}^{\gamma}_{ijA_{l-2}}X_{A_{l-2}}r^4\right.\\&\left.+\frac{l(l-1)}{(2l+1)(2l+3)}\,{}^{(2k+2)}\mathscr{E}^{\gamma}_{ijA_{l-2}}X_{A_{l-2}}r^4-\frac{1}{2l+3}\,{}^{(2k+2)}\mathscr{E}^{\gamma}_{A_l}X_{A_l}r^2\delta_{ij}\right]\,.\end{aligned} \tag{3.26c}$$

Here ${}^{(2k)}\mathscr{E}^{\gamma}_{A_l}$ means $(d^{2k}/dt^{2k})\mathscr{E}^{\gamma}_{A_l}(t)$, and $x_{(i}\epsilon_{j)pq}$ means to take the symmetric part on indices i,j.

It is evident that Eqs. (3.23) and (3.24) are the homogeneous part of Eqs. (3.14). To ensure that the moments $\mathscr{E}^{\gamma}_{A_l}$, $\mathscr{B}^{\gamma}_{A_l}$ read off of the final, fully nonlinear metric density $\bar{h}^{\mu\nu}$ [Eq. (3.17)] are the same as those in $\gamma^{\mu\nu}_1$ with which we start the iteration, all homogeneous parts of $\bar{h}^{\mu\nu}$ satisfying (3.23), (3.24) must be collected in $\gamma^{\mu\nu}_1$. This, then, determines $\gamma^{\mu\nu}_1$ uniquely, and becomes the starting point for the iterative solution of Eqs. (3.22) for $\gamma^{\mu\nu}_2$, $\gamma^{\mu\nu}_3,\ldots$.

D. Multipole expansion structure of $\mathfrak{g}^{\mu\nu}$ at order $p=2$

In this subsection we shall study the structure of $\gamma^{\mu\nu}_2$, i.e., of the metric density at the lowest nonlinear order

$p=2$. This study will lay a foundation for the next subsection where we shall study all higher orders, $p\geq 3$. The equations we must solve are (3.22a) and (3.22b) with $p\geq 2$. First we shall explore the structure of the source term $W_2^{\mu\nu}$ and from it shall infer the structure of a particular integral of (3.22a). Then we shall show how to construct the homogeneous solution of (3.22a) which, when added to the particular integral, produces a solution of the gauge condition (3.22b) without including terms of the form of $\gamma_1^{\mu\nu}$ [Eq. (3.26)].

In terms of the notation introduced in Sec. III A, at linear order, $\gamma_1^{\mu\nu}$ has the structure

$$\gamma_1^{\mu\nu}\sim X_{B_s}{}^{(u)}\mathcal{F}_{C_pA_q}X_{A_q}r^n\ . \tag{3.27}$$

Here $\mathcal{F}_{A_l}$ is either $\mathcal{E}_{A_l}$ or $\mathcal{B}_{A_l}$. Now $W_2^{\mu\nu}$ has the form

$$W_2^{\mu\nu}\sim\gamma^1_{\alpha\beta,\rho}\gamma^1_{\sigma\delta,\lambda}\ ,\quad \gamma^1_{\alpha\beta,\rho\lambda}\gamma^1_{\sigma\delta}\ . \tag{3.28}$$

It is clear that $\gamma^1_{\alpha\beta,0}$ has the same structure ($H^{l'lm}r^{l'+2k}$) as $\gamma^1_{\alpha\beta}$. We also know (Sec. III A) that taking spatial derivatives will not change the structure of $\gamma^1_{\alpha\beta}$. Therefore, in accord with the lemma in Sec. III A

$$W_2^{\mu\nu}\sim H^{l'lm}r^{l'+2k}\ ,\quad k=0,1,2,\ldots\ . \tag{3.29}$$

Using Eq. (3.6) we see that a particular integral of Eq. (3.22a) has the structure

$$p_2^{\mu\nu}\sim H^{l'lm}r^{l'+2k+2}\ , \tag{3.30}$$

aside from the possible exceptions caused by the $\partial_t^2\gamma^{\mu\nu}_{pn(u-2)l}$ term. But if we start from $\gamma^{\mu\nu}_{pn0l}$ we see that $\partial_t^2\gamma^{\mu\nu}_{pn(u-2)l}$ has the same structure as $W^{\mu\nu}_{pnul}$, i.e., no $H^{l'lm}r^{l'+2k\pm1}$ terms in it. Therefore all parts of $p_2^{\mu\nu}$ have the structure $H^{l'lm}r^{l'+2k+2}$.

The general solution of Eq. (3.22a) is a sum of this particular integral $p_2^{\mu\nu}$ and a homogeneous solution $f_2^{\mu\nu}$ of Eq. (3.22a):

$$\gamma_2^{\mu\nu}=p_2^{\mu\nu}+f_2^{\mu\nu}\ ; \tag{3.31}$$

and we must choose $f_2^{\mu\nu}$ so as to guarantee that the gauge condition (3.22b) is satisfied, and that $\gamma_2^{\mu\nu}$ has no terms of the form of $\gamma_1^{\mu\nu}$. As is clear from Eqs. (3.19), (3.31), any such $f_2^{\mu\nu}$ must satisfy two relations:

$$f_2^{\mu\nu}{}_{,\nu}=-p_2^{\mu\nu}{}_{,\nu}\ , \tag{3.32}$$

$$\Box f_2^{\mu\nu}=0\ . \tag{3.33}$$

The first of these is the desired gauge condition $\gamma_2^{\mu\nu}{}_{,\nu}=0$; the second is required by the field equation $\Box\gamma_2^{\mu\nu}=\Box(p_2^{\mu\nu}+f_2^{\mu\nu})=W_2^{\mu\nu}$ where $\Box p_2^{\mu\nu}=W_2^{\mu\nu}$. That Eqs. (3.32), (3.33) are compatible follows from $\Box(p_2^{\mu\nu}{}_{,\nu})=W_2^{\mu\nu}{}_{,\nu}=0$—where $W_2^{\mu\nu}{}_{,}\nu=0$ is the second-order part of the well-known conservation law $W^{\mu\nu}{}_{,\nu}=0$.[20] That there actually does exist a simultaneous solution to Eqs. (3.32), (3.33) we shall show below by explicit construction. That $f_2^{\mu\nu}$ can be chosen so as to keep out of $\gamma_2^{\mu\nu}$ terms of the form of $\gamma_1^{\mu\nu}$ follows from the fact that such terms satisfy the homogeneous forms of (3.32), (3.33) [cf. (3.23), (3.24)] and thus can be added to or subtracted from $f_2^{\mu\nu}$ at will.

We shall go into some detail to further illustrate the idea of adding gauge terms $f_2^{\mu\nu}$ to satisfy (3.22). For simplicity, we will use the nonlinearity expansion formalism, which easily can be converted to the multiparameter expansion formalism by expanding the spherical Bessel function $j_{l'}(\omega r)$.

We assume from the outset that $p_2^{\mu\nu}$ has been chosen so as to not include terms of the form $\gamma_1^{\mu\nu}$; this is easily done by simply removing such terms if they are present. Because $p_2^{\mu\nu}{}_{,\nu}$ satisfies $\Box p_2^{\mu\nu}{}_{,\nu}=0$, we can write it as

$$p_2^{0\nu}{}_{,\nu}=\sum_{l,m,u}{}^0P^{(u)}_{lm}(t)Y^{lm}r^{l+u}=\sum_{l,m}\int{}^0P_{lm\omega}j_l(\omega r)e^{i\omega t}d\omega\,Y^{lm}\ , \tag{3.34a}$$

$$\begin{aligned}p_2^{i\nu}{}_{,\nu}=&\sum_{l',l,m,u}{}^1P^{(u)}_{l'lm}(t)Y_i^{l',lm}r^{l'+u}\\&=\sum_{l,m}\int{}^1P_{l-1,lm\omega}j_{l-1}(\omega r)e^{i\omega t}d\omega\,Y_i^{l-1,lm}+\sum_{l,m}\int{}^1P_{l,lm\omega}j_l(\omega r)e^{i\omega t}d\omega\,Y_i^{l,lm}\\&\quad+\sum_{l,m}\int{}^1P_{l+1,lm\omega}j_{l+1}(\omega r)e^{i\omega t}d\omega\,Y_i^{l+1,lm}\ ,\end{aligned} \tag{3.34b}$$

where ${}^0P_{lm}, {}^0P_{lm\omega}, {}^1P_{l'lm}, {}^1P_{l'lm\omega}$ are expansion coefficients and the superscript (u) denotes u time derivatives. It is straightforward to verify that the d'Alembertian-free functions

$$f_2^{00}=0\ , \tag{3.35a}$$

$$f_2^{0i}=\sum_{l,m}\int{}^1F_{l+1,lm\omega}j_{l+1}(\omega r)e^{i\omega t}d\omega\,Y_i^{l+1,lm}\ , \tag{3.35b}$$

$$\begin{aligned}f_2^{ij}=&\sum_{l,m}\int{}^2F_{l+1,lm\omega}j_{l+1}(\omega r)e^{i\omega t}d\omega\,T_{ij}^{2,l+1,lm}+\sum_{l,m}\int{}^2F_{l+2,lm\omega}j_{l+2}(\omega r)e^{i\omega t}d\omega\,T_{ij}^{2,l+2,lm}\\&+\delta_{ij}\sum_{l,m}\int{}^2F_{l,lm\omega}j_{l(\omega r)}e^{i\omega t}d\omega\,Y^{lm}\end{aligned} \tag{3.35c}$$

satisfy (3.32) as desired, with $p_2^{\mu\nu}{}_{,\nu}$ in the form (3.34), if

$$^1F_{l+1,lm\omega}=\frac{1}{\omega}\left[\frac{2l+1}{l+1}\right]^{1/2}{}^0P_{lm\omega}\,, \tag{3.36a}$$

$$^2F_{llm\omega}=-\frac{1}{\omega}\left[\frac{2l+1}{l}\right]^{1/2}{}^1P_{l-1,lm\omega}\,, \tag{3.36b}$$

$$^2F_{l+1,lm\omega}=-\frac{1}{\omega}\left[\frac{2(2l+1)}{l+2}\right]^{1/2}{}^1P_{llm\omega}\,, \tag{3.36c}$$

$$^2F_{l+2,lm\omega}=\frac{1}{\omega}\left[\frac{2l+3}{l+2}\right]^{1/2}{}^1P_{l+1,lm\omega}+\frac{i}{\omega}\left[\frac{(2l+3)(2l+1)}{(l+2)(l+1)}\right]^{1/2}{}^0P_{lm\omega}-\frac{1}{\omega}\left[\frac{(l+1)(2l+3)}{l(l+2)}\right]^{1/2}{}^1P_{l-1,lm\omega}\,. \tag{3.36d}$$

Moreover, by expanding the spherical Bessel functions in (3.34) and (3.35) and by combining with (3.36), we can rewrite this d'Alembertian-free solution to (3.32) as

$$f_2^{00}=0\,, \tag{3.37a}$$

$$f_2^{0i}=\sum_{l,m,k}\left[\frac{2l+1}{l+1}\right]^{1/2}a_{l+1}^k\,{}^0P_{2m}^{(2k)}(t)Y_i^{l+1,lm}r^{l+1+2k}\,, \tag{3.37b}$$

$$\begin{aligned}f_2^{ij}=&-\sum_{l,m,k}\left[\frac{2(2l+1)}{l+2}\right]^{1/2}a_{l+1}^k\,{}^1P_{llm}^{(2k)}(t)T_{ij}^{2,l+1,lm}r^{l+1+2k}\\&+\sum_{l,m,k}a_{l+2}^k\left[\left[\frac{2l+3}{l+2}\right]^{1/2}{}^1P_{l+1,lm}^{(2k)}(t)+\left[\frac{(2l+1)(2l+3)}{(l+1)(l+2)}\right]^{1/2}{}^0P_{lm}^{(2k+1)}(t)\right.\\&\left.+\left[\frac{(l+1)(2l+3)}{l(l+2)}\right]^{1/2}{}^1P_{l-1,lm}^{(2k+2)}(t)\right]T_{ij}^{2,l+2,lm}r^{l+2+2k}-\delta_{ij}\sum_{l,m,k}\left[\frac{2l+1}{l}\right]^{1/2}a_l^k\,{}^1P_{l-1,lm}^{(2k)}(t)Y^{lm}r^{l+2k}\,,\end{aligned} \tag{3.37c}$$

where a_l^k are the expansion coefficients of the spherical Bessel function:

$$j_l(x)=\sum_k a_l^k x^{l+2k}\,.$$

Notice that all $l'\le l$ components in $f_2^{\mu\nu}$ have been deliberately chosen to be zero except in f_2^{ij} where we have an extra trace term ($l'=l$). This guarantees, as desired, that $\gamma_2^{\mu\nu}$ will not bring our coordinate system outside the LIC class; that $f_2^{\mu\nu}$ will not interfere with the moments $\mathscr{E}_{A_l}^{\gamma}$, $\mathscr{B}_{A_l}^{\gamma}$ read off $\bar{h}^{00}$, $\bar{h}^{0i}$ from the r^l(l-pole) terms, and (as we shall see) that $f_2^{\mu\nu}$ is unique once $p_2^{\mu\nu}$ has been chosen. It should be emphasized that $f_2^{\mu\nu}$ has the same structure as $\gamma_1^{\mu\nu}$ and $p_2^{\mu\nu}$ ($\sim H^{l'lm}r^{l'+2k}$). Therefore $\gamma_2^{\mu\nu}$, as a whole, also has this structure.

E. Multipole expansion structure of $\mathfrak{g}^{\mu\nu}$ in general

In this subsection, we shall examine the structure of $\gamma_p^{\mu\nu}$, i.e., of the metric density $\mathfrak{g}^{\mu\nu}$, at all nonlinear orders $p\ge 3$. As we saw in the previous subsection, $\gamma_2^{\mu\nu}$ has the same structure ($\sim H^{l'lm}r^{l'+2k}$, $k\ge 0$) as $\gamma_1^{\mu\nu}$. Now we shall show using induction that *for any p, $\gamma_p^{\mu\nu}$ has the same structure as $\gamma_1^{\mu\nu}$ ($\sim H^{l'lm}r^{l'+2k}$, $k\ge 0$).*

We have already shown explicitly that this statement is true at $p=2$. Suppose it is also true up to order p. Then at nonlinearity order $p+1$,

$$W_{p+1}^{\mu\nu}\sim\bar{h}_{\alpha\beta,\rho\sigma}\bar{h}_{\delta\lambda},\ \bar{h}_{\alpha\beta,\rho}\bar{h}_{\sigma\delta,\alpha},\ g_{\alpha\beta}g_{\rho\sigma}\bar{h}_{\delta\lambda,\kappa}\bar{h}_{\pi\tau,\xi},\ldots\,.$$

Since for $p'\le p$, $\gamma_{p'}^{\mu\nu}$ has the same structure as $\gamma_1^{\mu\nu}$, $g_{\alpha\beta}$ will also have this structure. The same argument leading to the conclusion about the structure of $W_2^{\mu\nu}$ (previous subsection) then reveals that

$$W_{p+1}^{\mu\nu}\sim H^{l'lm}r^{l'+2k}\,.$$

This in turn leads to

$$p_{p+1}^{\mu\nu}\sim H^{l'lm}r^{l'+2k+2}\,.$$

We can determine $f_{p+1}^{\mu\nu}$ from $p_{p+1}^{\mu\nu}$ in the same way as we did for the $p=2$ case; and here as there it will lead to $f_{p+1}^{\mu\nu}\sim H^{l'lm}r^{l'+2k}$. Therefore we conclude that $\gamma_{p+1}^{\mu\nu}$ has the same structure

$$\gamma_{p+1}^{\mu\nu}\sim H^{l'lm}r^{l'+2k} \tag{3.38}$$

as was to be proved.

In terms of the external moments, each term in $\gamma_p^{\mu\nu}$ can be written in the form

$$\gamma_{pnul}^{\mu\nu}\sim{}^{(u_1)}\mathscr{E}_{A_{l_1}}{}^{(u_2)}\mathscr{E}_{B_{l_2}}\cdots{}^{(u_i)}\mathscr{E}_{C_{l_i}}{}^{(u_1')}\mathscr{B}_{A'_{l'_1}}{}^{(u_2')}\mathscr{B}_{B'_{l'_2}}\cdots{}^{(u_j')}\mathscr{B}_{C'_{l'_j}}X_{D_{l'}}r^{2k}\,, \tag{3.39a}$$

where the superscript (u_i) means d^{u_i}/dt^{u_i}, and where

$$p=i+j\ ,\quad n=\sum_a^i l_a+\sum_b^j l_b'-2p\ ,\quad u=\sum_a^i u_a+\sum_b^j u_b'\ ,$$
$$l=\sum_a^i l_a+\sum_b^j l_b'-2k' \text{ with } k'=0,1,2,\ldots,\ l'=\sum_a^i l_a+\sum_b^j l_b'+u-2k\ . \tag{3.39b}$$

This general structure tells us two things. One is that the iterative algorithm does not suffer any breakdown at any step if we keep r small compared to $\mathscr{R}$, $\mathscr{L}$, $\mathscr{T}$. Therefore, a fully nonlinear vacuum solution can be generated from the corresponding linearized stationary solution once the gauge is fixed. The second is quite suggestive. That there is no logarithmic term in the expansion suggests that in an appropriate gauge the external expansion might be analytic. This also justifies the use of linearized theory even in a dynamic situation.

IV. EXTERNAL MULTIPOLE MOMENTS

In the Introduction, we mentioned that the curvature-defined external multipole moments $\mathscr{E}_{A_l}$, $\mathscr{B}_{A_l}$, and the moments $\mathscr{E}^{\gamma}_{A_l}$, $\mathscr{B}^{\gamma}_{A_l}$ one straightforwardly reads off the expansion are different at nonlinear orders, but are expressible in terms of each other. We shall return to this issue here and infer from these relations some general properties of the external multipole moments; and we shall give the fully explicit form of the relations between the moments at quadratic order.

A. General form of the relation between the true multipole moments and those of the iterative algorithm and some properties of the moments

Let us denote by $\mathscr{E}^{\gamma}_{A_l}$ and $\mathscr{B}^{\gamma}_{A_l}$ the moments one reads off the fully nonlinear metric density $\mathfrak{g}^{\alpha\beta}$ by the standard procedure (Sec. I) of linearized theory, or equivalently the moments that one puts into $\gamma_1^{\mu\nu}$ to start our iterative algorithm (Sec. III C); and let us denote by $\mathscr{E}_{A_l}$ and $\mathscr{B}_{A_l}$ the true multipole moments as defined covariantly by Eqs. (1.3). Our iterative algorithm will produce a $\mathfrak{g}^{\mu\nu}$ and thence a $g_{\mu\nu}$ which is a sum of terms of the form (3.39), and the Riemann tensor computed from this $g_{\mu\nu}$, when covariantly differentiated and evaluated at $x^i=0$, will produce, via definition (1.3)

$$\mathscr{E}_{A_l}=\mathscr{E}^{\gamma}_{A_l}+\sum_{\{l_i\},\{l_j'\}} e_{K,N}(\mathscr{E}^{\gamma}_{B_{l_1}}\mathscr{E}^{\gamma}_{C_{l_2}}\cdots\mathscr{E}^{\gamma}_{D_{l_k}}\mathscr{B}^{\gamma}_{E_{l_1'}}\cdots\mathscr{B}^{\gamma}_{F_{l_n'}})^{\rm STF}\ , \tag{4.1a}$$

$$\mathscr{B}_{A_l}=\mathscr{B}^{\gamma}_{A_l}+\sum_{\{l_i\},\{l_j'\}} b_{K,N}(\mathscr{E}^{\gamma}_{B_{l_1}}\mathscr{E}^{\gamma}_{C_{l_2}}\cdots\mathscr{E}^{\gamma}_{D_{l_k}}\mathscr{B}^{\gamma}_{E_{l_1'}}\cdots\mathscr{B}^{\gamma}_{F_{l_n'}})^{\rm STF}\ . \tag{4.1b}$$

Here

$$e_{K,N}=e(\{l_i\mid i=1,\ldots,k\}\ ,\ \{l_j'\mid j=1,\ldots,2n\})\ ,$$
$$b_{K,N}=b(\{l_i\mid i=1,\ldots,k\}\ ,\ \{l_j'\mid j=1,\ldots,2n+1\})\ ,$$

are numerical coefficients whose values we have not computed except at quadratic order (see Sec. IV C below); the sum is over all combinations of l_i, l_j'; and due to the fact that we evaluated (1.3) at $x^i=0$, the indices $B_{l_1}, C_{l_2},\ldots,D_{l_k},E_{l_1'},\ldots,F_{l_n'}$ must be such that there are precisely l free indices $a_1a_2\cdots a_l\equiv A_l$ and no contractions between indices before we take the STF part; i.e. $\sum l_i+\sum l_i'=l$ and $B_{l_1}C_{l_2}\cdots D_{l_k}E_{l_1'}\cdots F_{l_n'}$ is a permutation of A_l.

Because γ_1^{00} and γ_1^{0i} are ordinary scalar and 3-vector fields, and because the multipole expansion of γ_1^{00} [Eq. (3.26)] is linear in $\mathscr{E}^{\gamma}_{A_l}$, while that of γ_1^{0i} [Eq. (3.26)] is linear in $\epsilon_{jpq}x_p\mathscr{B}^{\gamma}_{qA_{l-1}}$, $\mathscr{E}^{\gamma}_{A_l}$ must have electriclike parity $(-1)^l$, while $\mathscr{B}^{\gamma}_{A_l}$ (like ϵ_{jpq}) must have magneticlike parity $(-1)^{l+1}$. [Our conventions for defining parity are the same as in Thorne and Hartle;[3] see the paragraph containing their Eq. (2.3).] Similarly, because $R_{0a_10a_2;a_3\cdots a_l}$ and $R_{ija_20;a_3\cdots a_l}$ are ordinary tensor fields, the definitions (1.3) of $\mathscr{E}_{A_l}$ and $\mathscr{B}_{A_l}$ with the crucial ϵ_{a_1ij} in (1.3b) guarantee that $\mathscr{E}_{A_l}$ has electriclike parity $(-1)^l$ and $\mathscr{B}_{A_l}$ has magneticlike parity $(-1)^{l+1}$.

This set of parities implies that in each term of expression (4.1a) for $\mathscr{E}_{A_l}$, there must be an even number of $\mathscr{B}^{\gamma}_{C_l}$'s; and in each term of expression (4.1b) for $\mathscr{B}_{A_l}$ there must be an odd number of $\mathscr{B}^{\gamma}_{C_l}$'s. This explains why in $e_{K,N}$, j runs from 1 to $2n$; while in $b_{K,N}$, from 1 to $2n+1$.

The relations (4.1) can also be inverted to find out $\mathscr{E}^{\gamma}_{A_l}$ and $\mathscr{B}^{\gamma}_{A_l}$ once $\mathscr{E}_{A_l}$ and $\mathscr{B}_{A_l}$ are given. This is done step by step starting from $l=2$. For example

$$\mathscr{E}^{\gamma}_{ab}=\mathscr{E}_{ab}\ ,\quad \mathscr{E}^{\gamma}_{abc}=\mathscr{E}_{abc}\ ,$$
$$\mathscr{E}^{\gamma}_{abcd}=\mathscr{E}_{abcd}-e_{2,0}(\mathscr{E}_{ab}\mathscr{E}_{cd})^{\rm STF}-e_{0,2}(\mathscr{B}_{ab}\mathscr{B}_{cd})^{\rm STF},\ldots,$$
$$\mathscr{B}^{\gamma}_{ab}=\mathscr{B}_{ab}\ ,\quad \mathscr{B}^{\gamma}_{abc}=\mathscr{B}_{abc}\ ,$$
$$\mathscr{B}^{\gamma}_{abcd}=\mathscr{B}_{abcd}-b_{1,1}(\mathscr{E}_{ab}\mathscr{B}_{cd})^{\rm STF},\ldots .$$

It is straightforward to verify that this inversion is unique at all orders; i.e., Eqs. (4.1) determine $\mathscr{E}^{\gamma}_{A_l}$ and $\mathscr{B}^{\gamma}_{A_l}$

uniquely in terms of $\mathscr{E}_{C_l'}$ and $\mathscr{B}_{C_l'}$. With the help of this, we can infer an important property of the external multipole moments $\mathscr{E}_{A_l}$ and $\mathscr{B}_{A_l}$.

First, it is not hard to generalize Suen's[5] Theorems 1, 2 to our case: The metric density generated from this algorithm with our specified choice of $f_p^{\mu\nu}$ is unique up to a time-independent rotation of the spatial coordinates. To prove this theorem, let us consider two de Donder coordinates related by

$$x'^{\mu}=x^{\mu}+\xi^{\mu} \ .$$

We make the same nonlinearity expansion

$$\xi^{\mu}=\sum_{p=0}^{\infty} G^p \xi_p^{\mu} \tag{4.2}$$

as in Suen.[5] By expanding Suen's equation relating the metric density in two coordinates

$$\mathfrak{g}'^{\mu\nu}(x'^i)=\frac{1}{L}L^{\mu}{}_{\alpha}L^{\nu}{}_{\beta}\mathfrak{g}^{\alpha\beta}(x^i) \ ,$$

where $L^{\mu}{}_{\alpha}=(\partial x'^{\mu})/(\partial x^{\alpha})$ and $L=|\det(L^{\mu}{}_{\alpha})|$, we obtain to G^0 order

$$\xi_0^{\mu,\nu}+\xi_0^{\nu,\mu}-\eta^{\mu\nu}\xi_{0,\alpha}^{\alpha}=0 \ . \tag{4.3}$$

This can be readily reduced to the flat spacetime Killing equations

$$\xi_0^{\mu,\nu}+\xi_0^{\nu,\mu}=0 \ . \tag{4.4}$$

The solutions of Eq. (4.4) are linear combinations of the ten Killing vectors. Because the spatial origin of our coordinate is tied to λ at all times, the boost and translation generators will not contribute to ξ^{μ}. What is left are just the three spatial rotations. Having understood the effects of ξ_0^{μ}, we set it to zero to facilitate the discussion of the G^1 order part of the gauge change. At G^1 order, with ξ_0^{μ} set to zero,

$$\delta\gamma_1^{\mu\nu}\equiv\gamma_1'^{\mu\nu}-\gamma_1^{\mu\nu}=\xi_1^{\mu,\nu}+\xi_1^{\nu,\mu}-\eta^{\mu\nu}\xi_{1,\alpha}^{\alpha} \ ; \tag{4.5}$$

and to presume de Donder gauge ξ_1^{μ} must be d'Alembertian-free at G^1 order. For $\delta\gamma_1^{\mu\nu}$ to change $\gamma_1^{\mu\nu}$, it must have the same form as $\gamma_1^{\mu\nu}$, i.e., all $l'\le l$ terms of $\delta\gamma_1^{ij}$ and all $l'<l$ terms of $\delta\gamma_1^{0i}$ must vanish. This together with (4.5) and $\Box\xi_1^{\mu}=0$ implies that all $l\ge 2$ terms in $\delta\gamma_1^{\mu\nu}$ vanish. Furthermore, the primed coordinate system must also be LIC, which means that all $l'<2$ terms in $\delta\gamma_1^{\mu\nu}$ must vanish. This reduces ξ_1^{μ}, like ξ_0^{μ}, to only spatial rotations. To facilitate the discussion of the G^2 order part of gauge change we now set $\xi_0^{\mu}=\xi_1^{\mu}=0$; then

$$\delta\gamma_2^{\mu\nu}\equiv\gamma_2'^{\mu\nu}-\gamma_2^{\mu\nu}=\xi_2^{\mu,\nu}+\xi_2^{\nu,\mu}-\eta^{\mu\nu}\xi_{2,\alpha}^{\alpha} \ . \tag{4.6}$$

For $\delta\gamma_2^{\mu\nu}$ to change $\gamma_2^{\mu\nu}$, it must have the same form as $f_2^{\mu\nu}$ since $f_2^{\mu\nu}$ is the homogeneous part of the solution. The same arguments as above then lead to the conclusion that ξ_2^{μ} also only represents a spatial rotation. This can be repeated up to any p. The result is the theorem we set out to prove.

Since the multipole moments $\mathscr{E}_{A_l}^{\gamma}$, $\mathscr{B}_{A_l}^{\gamma}$ of the iterative algorithm are determined uniquely from the true external moments $\mathscr{E}_{A_l}$, $\mathscr{B}_{A_l}$, it must be true that for any set of external multipole moments $\mathscr{E}_{A_l}$ and $\mathscr{B}_{A_l}$, there corresponds a spacetime, determined uniquely inside a world tube of size $r_{\max}\sim\mathrm{Min}(\mathscr{R},\mathscr{L},\mathscr{T})$. In other words, giving a set of multipole moments on a fiducial world line λ is equivalent to specifying a spacetime inside the world tube surrounding λ.

B. Explicit quadratic-order form of the relation between the moments

Now that we have explored the general structure of the external multipole moments $\mathscr{E}_{A_l}$, $\mathscr{B}_{A_l}$, built up from the expansion coefficients $\mathscr{E}_{A_l}^{\gamma}$, $\mathscr{B}_{A_l}^{\gamma}$ of the metric density, we shall sketch a derivation of the explicit form of the expressions for $\mathscr{E}_{A_l}$, $\mathscr{B}_{A_l}$ in terms of $\mathscr{E}_{A_l}^{\gamma}$, $\mathscr{B}_{A_l}^{\gamma}$ accurate to quadratic order.

The starting point is Eqs. (1.3). The covariant derivatives in the definition (1.3) of $\mathscr{E}_{A_l}$, $\mathscr{B}_{A_l}$ can be expressed explicitly in terms of partial derivatives of the Riemann tensor $R_{\alpha\beta\gamma\delta}$ and connection coefficients $\Gamma_{\beta\gamma}^{\alpha}$; accurate to quadratic order they are

$$R_{0i0j;A_l}=R_{0i0j,A_l}+\text{terms of the form }(\Gamma R_{,\ldots})_{,\ldots} \ ; \tag{4.7a}$$

$$\epsilon_{aij}R_{ijb0;A_l}=\epsilon_{aij}R_{ijb0,A_l}+\text{ terms of the form }\epsilon(\Gamma R_{,\ldots})_{,\ldots} \ . \tag{4.7b}$$

The Γ-correction terms are straightforward to evaluate to the desired quadratic order, since they involve $\Gamma_{\beta\gamma}^{\alpha}$ and $R_{\alpha\beta\gamma\delta}$ only at the linear order. Thus, we shall not discuss them in detail. The first terms in Eqs. (4.7a) and (4.7b), by contrast, involve $R_{\alpha\beta\gamma\delta}$ at quadratic order ($p=2$) and thus require some discussion. Fortunately, their evaluation is simplified by the fact that, because the external moments are evaluated at $x^i=0$, pieces of $R_{\alpha\beta\gamma\delta}$ with the form $((n<l)\text{-pole})r^l$ will not contribute to the final answer; we shall ignore such pieces in the derivations below.

The metric accurate to order $p=2$ is[20]

$$\mathfrak{g}^{\mu\nu}=\sqrt{-g}\,g^{\mu\nu}=\eta^{\mu\nu}-(\bar{h}^{\mu\nu}-\tfrac{1}{2}\,\bar{h}\eta^{\mu\nu})-\tfrac{1}{2}\,\bar{h}\bar{h}^{\mu\nu}+\tfrac{1}{8}\,\eta^{\mu\nu}(\bar{h}^2+2\bar{h}^{\alpha\beta}\bar{h}_{\alpha\beta}) \ , \tag{4.8}$$

where

$$\bar{h}^{\mu\nu}=\eta^{\mu\nu}-\mathfrak{g}^{\mu\nu} \ , \quad \bar{h}=\eta_{\mu\nu}\bar{h}^{\mu\nu} \ .$$

This can be inverted to give

$$g_{\mu\nu}=\eta_{\mu\nu}+\bar{h}_{\mu\nu}-\tfrac{1}{2}\,\eta_{\mu\nu}\bar{h}+\bar{h}_{\mu}{}^{\alpha}\bar{h}_{\alpha\nu}-\tfrac{1}{2}\,\bar{h}\bar{h}_{\mu\nu}+\tfrac{1}{8}\,\eta_{\mu\nu}(\bar{h}^2-2\bar{h}^{\alpha\beta}\bar{h}_{\alpha\beta}) \ . \tag{4.9}$$

Here $\bar{h}^{\mu\nu}$ consists of a linear part $\gamma_1^{\mu\nu}$ given by Eqs. (3.26) plus quadratic corrections $\gamma_2^{\mu\nu}=p_2^{\mu\nu}+f_2^{\mu\nu}$ [Eq. (3.31) and associated discussion]:

$$\bar{h}^{\mu\nu}=\gamma_1^{\mu\nu}+p_2^{\mu\nu}+f_2^{\mu\nu} . \tag{4.10}$$

To simplify the calculations, all $r^l((n<l)\text{-pole})$ terms will be dropped. By virtue of the forms of $\gamma_1^{\mu\nu}$ and of the $W_2^{\mu\nu}$ constructed from it, $p_2^{\mu\nu}$ can be chosen to consist solely of such terms and thus can be ignored. From this fact plus the traceless nature of γ_1^{ij} we conclude that

$$\bar{h}=-\bar{h}^{00}+\delta_{ij}f_2^{ij} .$$

Furthermore, accurate to quadratic order in p,

$$\bar{h}_{\alpha\beta}\bar{h}^{\alpha\beta}=(\bar{h}^{00})^2-2\bar{h}^{0a}\bar{h}^{0a} .$$

These relations reduce Eqs. (4.8), (4.9) to

$$g^{\mu\nu}=\eta^{\mu\nu}-\bar{h}^{\mu\nu}-\tfrac{1}{2}\,\bar{h}^{00}\eta^{\mu\nu}+\tfrac{1}{2}\,\delta^{ij}f_{ij}\eta^{\mu\nu}+\tfrac{1}{2}\,\bar{h}^{00}\bar{h}^{\mu\nu}+\tfrac{1}{8}\,\eta^{\mu\nu}[(\bar{h}^{00})^2-2\bar{h}^{0a}\bar{h}^{0a}] , \tag{4.11}$$

$$g_{\mu\nu}=\eta_{\mu\nu}+\bar{h}_{\mu\nu}+\tfrac{1}{2}\,\bar{h}^{00}\eta_{\mu\nu}-\tfrac{1}{2}\,\delta^{ij}f_{ij}\eta_{\mu\nu}+\bar{h}_\mu^{\alpha}\bar{h}_{\alpha\nu}+\tfrac{1}{2}\,\bar{h}^{00}\bar{h}_{\mu\nu}-\tfrac{1}{8}\,\eta_{\mu\nu}[(\bar{h}^{00})^2-2\bar{h}^{0a}\bar{h}^{0a}] . \tag{4.12}$$

The Riemann curvature tensor, as obtained from this expression for the metric, is

$$R_{0i0j}=\psi_{,ij}+3(\psi^2)_{,ij}-\tfrac{1}{4}\,(\mathbf{A}^2)_{,ij}-\tfrac{1}{4}\,H_iH_j+g_ig_j-\tfrac{1}{4}\,(\delta^{ab}f_{ab})_{,ij}+O(G^3) , \tag{4.13a}$$

$$\epsilon_a{}^{ij}R_{ijb0}=-H_{a,b}-2\psi H_{a,b}+6H_ag_b-2\epsilon_{aij}A^ig^j{}_{,b}+O(G^3) , \tag{4.13b}$$

$$R_{iajb}=\delta_{ib}g_{j,a}-\delta_{ij}g_{b,a}+\delta_{aj}g_{b,i}-\delta_{ab}g_{j,i}+O(G^2) , \tag{4.13c}$$

where

$$\psi=-\tfrac{1}{4}\,\gamma_{1(l-2)0l}^{00} , \quad A^j=-\gamma_{1(l-2)0l}^{0j} , \quad g_i=-\psi_{,i} , \quad H_i=\epsilon_i{}^{ab}A_{b,a} . \tag{4.14}$$

We need $f_2\equiv\delta_{ab}f_2^{ab}$ in order to compute $\mathscr{E}_{A_l}$. As may be seen from Eq. (3.37c), only ${}^1P_{l-1,lm}(t)$ contributes to the trace of the gauge-correcting term f_2. A simple dimensional analysis plus parity considerations tells that only p_2^{ij} contributes to ${}^1P_{l-1,lm}(t)$. It is not too hard to find p_2^{ij} from W_2^{ij}. But further simplification is still possible. Since we only want $r^l(l\text{-pole})$ terms in f_2, we may drop the time derivative and trace terms in W_2^{ij}; this results in

$$W_2^{ij}=-4g^ig^i-H^iH^j . \tag{4.15}$$

By substituting Eqs. (3.26), (4.14) into Eq. (4.15), we can easily find a particular integral p_2^{ij} for γ_2^{ij}. The gauge term determined from it [with all $r^l((n<l)\text{-pole})$ terms dropped] is

$$f_2=\delta_{ij}f_2^{ij}$$
$$=\sum_{l'=2}^{l-2}\frac{12}{(2l-1)l(l-l'-1)(l'-1)}\left[\mathscr{E}^{\gamma}_{A_{l-l'}}\mathscr{E}^{\gamma}_{A_{l'}}+\frac{(l'+1)(l-l'+1)}{9}\mathscr{B}^{\gamma}_{A_{l-l'}}\mathscr{B}^{\gamma}_{A_{l'}}\right]X_{A_l} . \tag{4.16}$$

Now we can substitute Eqs. (3.26), (4.7), (4.14), (4.16) into Eqs. (1.3). After an extremely tedious but straightforward calculation, we obtain

$$\mathscr{E}_{A_l}=\mathscr{E}^{\gamma}_{A_l}+\sum_{l'=2}^{l-2}\frac{l(l-1)}{l'(l'-1)(l-l'-1)}\left[\frac{3}{l-l'}-\frac{l'(l-2)}{l(l-1)(2l-1)}+\frac{2(l-l'-1)}{l(l'+1)}\right](\mathscr{E}^{\gamma}_{A_{l-l'}}\mathscr{E}^{\gamma}_{A_{l'}})^{\mathrm{STF}}$$
$$+\sum_{l'=2}^{l-2}\frac{l(l-1)(l'+1)(l-l'+1)}{9(l'-1)(l-l'-1)}\left[\frac{1}{(l-l'+1)(l'+1)}-\frac{2}{ll'}-\frac{l-2}{l(l-1)(2l-1)}\right](\mathscr{B}^{\gamma}_{A_{l-l'}}\mathscr{B}^{\gamma}_{A_{l'}})^{\mathrm{STF}}+O(G^3) , \tag{4.17a}$$

$$\mathscr{B}_{A_l}=\mathscr{B}^{\gamma}_{A_l}+\frac{2}{l+1}\sum_{l'=2}^{l-2}\frac{1}{(l'-1)(l-l'-1)}$$
$$\times\left[\frac{l-1}{l'}+\frac{l-l'+2}{l+1}+\frac{l'-1}{(l-l')(l+1)}+\frac{(l+l')(l-l'+1)(l-l'-1)}{l'(l'+1)(l+1)}\right](\mathscr{B}^{\gamma}_{A_{l-l'}}\mathscr{E}^{\gamma}_{A_{l'}})^{\mathrm{STF}}$$
$$+O(G^3) . \tag{4.17b}$$

These, then, are the relations between the moments $\mathscr{E}^{\gamma}_{A_l}$, $\mathscr{B}^{\gamma}_{A_l}$ of the de Donder coordinate iterative algorithm (Sec. III) and the true, curvature-defined moments $\mathscr{E}_{A_l}$, $\mathscr{B}_{A_l}$.

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