Particle production in expanding universes with path integrals

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A method is developed to calculate path integrations for a scalar particle moving in the spatially flat Robertson-Walker spacetime with scale factor $a(t)$. For three cases, which are specified by $a(t)=t$, $a(t)=\sqrt{t}$, and $a(t)=e^{Ht}$, exact path-integral calculations are presented and particle pair productions are studied.

I. INTRODUCTION

One of the methods for studying the particle-production rate in the early Universe is to calculate the overlap amplitude between the initial and the final states of a homogeneous, isotropic expanding universe.[1,2] The determination of the final particle states is not problematic, because at late times the curvature of the Universe varies very slowly on the scales on which the measurements are made. This stage of the Universe, called the adiabatic region, has properties very similar to those of Minkowski space. On the other hand, because of the cosmological singularity one cannot define the initial particle state. To overcome this difficulty one may assume that Einstein equations are not valid before an early time t_0 and invent an adiabatic regime for the initial period. The time t_0 is chosen according to a new physical principle.

Another way of handling the cosmological pair-production problem is the Feynman path-integral method. This method was first used by Chitre and Hartle[3] for calculating particle production in the early Universe; but, it had previously been applied to various other gravitational fields.[4] In this approach the probability of detecting a particle and its antiparticle at very late times at the space-time points x_a and x_b is expressed in terms of the amplitude $K(x_b,x_a)$ for the particle to propagate from x_b to x_a. To calculate $K(x_b,x_a)$ one sums over all the paths from x_b to x_a which are restricted to lie to the future of the initial singularity. This restriction enables one to avoid the difficulties associated with the initial singularity.

Chitre and Hartle studied pair production in the spatially flat Robertson-Walker space-time described by the metric

$$ds^2=-dt^2+a^2(t)[(dx_1)^2+(dx_2)^2+(dx_3)^2] \tag{1}$$

with $a(t)=t$. They made this choice of the scale factor, because in this example they could carry out the calculations exactly. They did not calculate $K(x_b,x_a)$ by summation over the paths, but they solved the covariant Schrödinger equation with boundary conditions implied by the path-integral expression.

In this paper we shall first formulate $K(x_b,x_a)$ as the path integral for the motion of a scalar particle moving in the space-time of Eq. (1) with general scale factor $a(t)$. We shall then carry out exactly the integrations over all paths for three special cases: $a(t)=t$ is studied to demonstrate the agreement between our procedure and the calculations presented in Ref. 3, $a(t)=\sqrt{t}$ and $a(t)=e^{Ht}$, which correspond to the radiation-dominated and the inflationary universes, are considered for their obvious cosmological importance. In developing the path integrals for these cases, we factorize the motion in the space and time coordinates; then, we observe that the amplitude for each motion can be expressed in terms of a flat-space quantum-mechanical Green's function.

II. PATH INTEGRATION FOR PARTICLE MOTION IN THE ROBERTSON-WALKER GEOMETRY

The propagator for a scalar particle of mass μ to go from x_b to x_a in the geometry given by the metric of Eq. (1) is[3]

$$K(x_b,x_a)=\int_0^\infty dW\, e^{-i\mu^2 W}F(W,x_b,x_a)\,. \tag{2}$$

Here $F(W,x_b,x_a)$ is the amplitude for the particle to move from x_b to x_a in a total parameter time W and, it is expressed as the path integral

$$F(W,x_b,x_a)=\int \mathscr{D}(t,\mathbf{x})\exp\left[\frac{i}{4}\int_0^W dw[-\dot{t}^2+a^2(t)\dot{\mathbf{x}}^2]\right] \tag{3}$$

with the overdot standing for derivative with respect to the parameter time w. Since W is not an observable we integrate it out in Eq. (2); and, the weight factor $e^{-i\mu^2 W}$ is introduced for having the usual flat-space-time limit for the propagator $K(x_b,x_a)$. Equation (3) is understood as the limit of graded formulation:

$$F(W,x_b,x_a)=\lim_{\substack{\epsilon\to 0\\ n\to\infty}}\frac{a(t_a)}{a(t_b)}(4\pi i\epsilon)^{-2(n+1)}\times\int\prod_{j=1}^{n}\{dt_j d^3x_j[g(x_j)]^{1/2}\}\prod_{j=1}^{n+1}\exp\left[\frac{\epsilon}{4}\left[-\frac{(t_j-t_{j-1})^2}{\epsilon^2}+a^2(t_j)\frac{(\mathbf{x}_j-\mathbf{x}_{j-1})^2}{\epsilon^2}\right]\right] \tag{4}$$

with $(n+1)\epsilon=W$. In this formula the points a and b correspond to $(n+1)$th and 0th points in the division of the parameter time, respectively; $g(x_j)$ is the determinant of the metric at point x_j:

$$g(x_j)=a^6(t_j)\ . \tag{5}$$

Following Chitre and Hartle we require the amplitude $F(W,x_b,x_a)$ to satisfy the covariant Schrödinger equation which is suggested by the conformal invariance:

$$i\frac{\partial\psi}{\partial W}=\left[-\Delta_2-\frac{R}{6}\right]\psi\ . \tag{6}$$

In fact, the choice of the functional measure in Eq. (4) is suggested by this requirement. As we shall see in studying the specific examples that the symmetrization of that measure with respect to t_a and t_b gives just the necessary contributions to the action functional which brings $F(W,x_b,x_a)$ in agreement with the solutions of Eq. (6). For the metric of Eq. (1) the Laplace-Beltrami operator

$$\Delta_2=-g^{-1/2}\partial_\mu(g^{\mu\nu}g^{1/2}\partial_\nu)$$

takes the form of

$$\Delta_2=\partial_t{}^2+\frac{3(da/dt)}{a}\partial_t-\frac{1}{a^2}\nabla^2 \tag{7}$$

and the curvature scalar R is given by

$$R=6\left[\left[\frac{da/dt}{a}\right]^2+\frac{d^2a/dt^2}{a}\right]\ . \tag{8}$$

Substituting $\psi=e^{-i\mu^2W}f(\mathbf{k},x)$ into Eq. (6) and using the factorized form

$$f(\mathbf{k},x)=\frac{1}{(2\pi)^{3/2}}e^{i\mathbf{k}\cdot\mathbf{x}}g(\mathbf{k},t)\ , \tag{9}$$

the equation satisfied by the time part of the state function becomes

$$\left[\partial_t{}^2+\frac{3(da/dt)}{a}\partial_t+\frac{k^2}{a^2}+\frac{(da/dt)^2}{a^2}+\frac{d^2a/dt^2}{a}+\mu^2\right]g(\mathbf{k},t)=0\ . \tag{10}$$

The path integral of Eqs. (3) or (4) can be obtained from a sort of "Hamiltonian path integral":

$$F(W,x_b,x_a)=\int\mathscr{D}(t,\mathbf{x})\mathscr{D}(k_t,\mathbf{k})\times\exp\left[i\int_0^W dw[k_t\dot{t}+\mathbf{k}\cdot\dot{\mathbf{x}}+k_t{}^2-a^{-2}(t)\mathbf{k}^2]\right] \tag{11}$$

which, in the graded formulation, is given by

$$F(W,x_b,x_a)=\lim_{\substack{\epsilon\to 0\\ n\to\infty}}[a(t_a)a(t_b)]^{-1}a^{-1}(t_a)\times\int\prod_{j=1}^{n}(dt_jd^3x_j)\prod_{j=1}^{n+1}\frac{dk_{tj}d^3k_j}{(2\pi)^4}\prod_{j=1}^{n+1}\exp\left[i\left[k_{tj}(t_j-t_{j-1})+\mathbf{k}_j\cdot(\mathbf{x}_j-\mathbf{x}_{j-1})+\epsilon k_{t_j}{}^2-\frac{\epsilon}{a^2(t_j)}\mathbf{k}_j{}^2\right]\right]\ . \tag{12}$$

One can easily verify that after the integrations over $dk_{tj}d^3k_j$ are carried out the above equation yields Eq. (4).

Integrations over $\prod_{j=1}^{n}d^3x_j$ gives δ functions; then after performing the integrations over $\prod_{j=1}^{n}d^3k_j$ we arrive at

$$F(W,x_b,x_a)=\int\frac{d^3k_{n+1}}{(2\pi)^3}e^{i\mathbf{k}_{n+1}\cdot(\mathbf{x}_a-\mathbf{x}_b)}\times[a(t_a)a(t_b)]^{-1}a^{-1}(t_a)\int\prod_{j=1}^{n}dt_j\prod_{j=1}^{n+1}\frac{dk_{tj}}{2\pi}\prod_{j=1}^{n+1}\exp\left[i\left[k_{tj}(t_j-t_{j-1})+\epsilon k_{tj}{}^2-\frac{\epsilon}{a^2(t_j)}k_{n+1}{}^2\right]\right] \tag{13}$$

which (by dropping the subscript $n+1$ of $\mathbf{k}_{n+1}$) we express in compact notation as

$$F(W,x_b,x_a)=\int\frac{d^3k}{(2\pi)^3}e^{i\mathbf{k}\cdot(\mathbf{x}_a-\mathbf{x}_b)}F(W,t_b,t_a)\ , \tag{14}$$

where $F(W,t_b,t_a)$ is the one-dimensional path integral in the time coordinate:

$$F(W,t_b,t_a)=(a_aa_b)^{-1}a_a{}^{-1}\int\mathscr{D}t\,\mathscr{D}k_t\exp\left[i\int_0^W dw[k_t\dot{t}+k_t{}^2-k^2/a^2(t)]\right] \tag{15}$$

with $k^2\equiv\mathbf{k}^2$. Inserting Eq. (14) into Eq. (2) we get

$$K(x_b,x_a)=\int\frac{d^3k}{(2\pi)^3}e^{i\mathbf{k}\cdot(\mathbf{x}_b-\mathbf{x}_a)}K_{\mathbf{k}}(t_b,t_a)\ . \tag{16}$$

Here K_k is the propagator for the wave number $\mathbf{k}$:

$$K_{\mathbf{k}}(t_b,t_a)=\int_0^\infty dW\,e^{-i\mu^2 W}F(W,t_b,t_a)\ . \tag{17}$$

Now we are ready to develop these propagators for specific cases.

(i) $a(t)=a_0 t$, $a_0=\text{const}$, $0\le t<\infty$. In this example amplitude of Eq. (15) is

$$F(W,t_b,t_a)=a_0^{-3}(t_a t_b)^{-1}t_a^{-1}\int \mathscr{D}t\,\mathscr{D}k_t \exp\left[i\int_0^W dw\left[k_t\dot{t}+k_t^2-\frac{k^2/a_0^2}{t^2}\right]\right]\ . \tag{18}$$

To symmetrize the factor t_a^{-1}, we write it as[5]

$$t_a^{-1}=(t_a t_b)^{-1/2}\exp\left[-\frac{1}{2}\ln\frac{t_a}{t_b}\right]=(t_a t_b)^{-1/2}\exp\left[i\int_0^W dw\frac{i\dot{t}}{2t}\right]\ . \tag{19}$$

After inserting this formula into Eq. (18) and shifting k_t with $-i/2t$ we obtain

$$F(W,t_b,t_a)=a_0^{-3}(t_a t_b)^{-3/2}\int \mathscr{D}t\,\mathscr{D}k_t \exp\left[i\int_0^W dw\left[k_t\dot{t}+k_t^2-\frac{k^2/a_0^2+\frac{1}{4}}{t^2}-\frac{2ik_t}{2t}\right]\right]\ . \tag{20}$$

Note that if we had $(t_a t_b)^{-1}t_b^{-1}$ in place of $(t_a t_b)^{-1}t_a^{-1}$ in Eq. (18) we would have to symmetrize t_b^{-1}, then, we would have the same expression as Eq. (20) with the sign of the imaginary term in the action reversed. The existence of the imaginary term $\pm ik_t/t$ of the action amounts an ϵ order shift in $\dot{t}_j$:

$$\dot{t}_j\to\dot{t}_j\pm i/t_j\ ,$$

that is

$$\frac{t_j-t_{j-1}}{\epsilon}\to\frac{t_j-t_{j-1}}{\epsilon}\pm\frac{i}{t_j} \tag{21}$$

which gives a vanishing contribution as $\epsilon\to 0$. At this point we would like to emphasize that to perform this symmetrization in a rigorous way we could first make an analytical continuation by $w\to -iw$, $t\to it$ in Eq. (18). This would save us from having an imaginary term in the action.

Dropping the term k_t/t in Eq. (20) we arrive at the expression

$$F(W,t_b,t_a)=a_0^{-3}(t_a t_b)^{-3/2}\int \mathscr{D}t\,\mathscr{D}k_t \exp\left[i\int_0^W dw\left[k_t\dot{t}+k_t^2-\frac{k^2/a_0^2+\frac{1}{4}}{t^2}\right]\right]\ . \tag{22}$$

The path integral of this equation is the Green's function for a particle of mass $m=\frac{1}{2}$, moving in the flat "space-time" $(-w,t)$ under the influence of potential:

$$V(t)=\frac{-k^2/a_0^2-\frac{1}{4}}{t^2},\quad t\ge 0\ . \tag{23}$$

Since the exact solution of this path integral is known,[6] we can directly write the final form of Eq. (22):

$$F(W,t_b,t_a)=a_0^{-3}(t_a t_b)^{-3/2}\frac{i(t_a t_b)^{1/2}}{2W}I_{ik/a_0}\left[\frac{it_a t_b}{2W}\right]\exp\left[\frac{1}{4iW}(t_a^2+t_b^2)\right]\ , \tag{24}$$

where I_{ik/a_0} is the modified Bessel function. Inserting this result into Eq. (17) we get

$$K_{\mathbf{k}}(t_b,t_a)=\frac{i}{2a_0^3(t_a t_b)}\int_0^\infty\frac{dW}{W}\exp\left[-i\mu^2 W+\frac{1}{4iW}(t_a^2+t_b^2)\right]I_{ik/a_0}\left[\frac{it_a t_b}{2W}\right] \tag{25}$$

or after dW integrations, for $t_a>t_b$ we have

$$K_{\mathbf{k}}(t_b,t_a)=\frac{\pi}{2a_0^3(t_a t_b)}H^{(2)}_{ik/a_0}(\mu t_a)J_{ik/a_0}(\mu t_b)\ . \tag{26}$$

This result is in agreement with the one obtained in Ref. 3. For massless particles the full propagator $K(x_b,x_a)$ of Eq. (16) can be calculated exactly. To do this we substitute the $\mu\to 0$ limit of Eq. (25) into Eq. (16) and integrate over d^3k:

$$K_0(x_b,x_a)=\frac{-i}{(2\pi)^2}\frac{1}{t_a t_b}\frac{1}{(\ln t_a-\ln t_b)^2+(\mathbf{x}_a-\mathbf{x}_b)^2}\ .$$

For $\mu\ne 0$, using the asymptotic limits of the Bessel functions for $t\to\infty$ (Ref. 7)

$$H^{(2)}_{ik/a_0}(\mu t)\simeq\left[\frac{2}{\pi\mu t}\right]^{1/2}\exp\left[i\frac{\pi}{2}\left[i\frac{k}{a_0}+\frac{1}{2}\right]\right]e^{-i\mu t} \tag{27a}$$

and

$$J_{ik/a_0}(\mu t)\simeq\left[\frac{2}{\pi\mu t}\right]^{1/2}\cos\left[\mu t-\frac{\pi}{2}\left[i\frac{k}{a_0}+\frac{1}{2}\right]\right] \tag{27b}$$

we obtain the late time limit of $K_{\mathbf{k}}$,

$$K_{\mathbf{k}}(t_b,t_a)\simeq\frac{1}{2\mu a_0{}^3(t_a t_b)^{3/2}}(e^{-i\mu(t_a-t_b)}+e^{i\pi/4}e^{-\pi k/a_0}e^{-i\mu(t_a+t_b)}) , \tag{28}$$

which is also the limit for large mass.

(ii) $a(t)=a_0\sqrt{t}$, $a_0=\text{const}$, $0\le t<\infty$. This is the metric of the radiation-dominated universe. Introducing this metric into Eq. (15), we obtain

$$F(W,t_b,t_a)=a_0{}^{-3}(t_a t_b)^{-1/2}t_a{}^{-1/2}\int \mathscr{D}t\,\mathscr{D}k_t\exp\left[i\int_0^W dw\left[k_t\dot t+k_t{}^2-\frac{k^2/a_0{}^2}{t}\right]\right]. \tag{29}$$

This amplitude is very similar to the path integral for the nonrelativistic one-dimensional Kepler motion. Equivalently, if we symmetrize the $t_a{}^{-1/2}$ factor in the same fashion as we did in the previous case, we can reduce the problem in hand to the three-dimensional Kepler problem for s waves. Since the path integral for the H atom is solvable,[8] applying the same techniques we can evaluate Eq. (29) as well. We first make a point transformation:

$$t=u^2,\quad k_t=\frac{1}{2u}k_u,\quad -\infty<u<\infty , \tag{30}$$

and obtain

$$F(W,t_b,t_a)=a_0{}^{-3}(u_a u_b)^{-1}u_a{}^{-1}(2u_a)^{-1}\int \mathscr{D}u\,\mathscr{D}k_u\exp\left[i\int_0^W dw\left[k_u\dot u+\frac{1}{4u^2}k_u{}^2-\frac{k^2/a_0{}^2}{u^2}\right]\right]. \tag{31}$$

In order to get rid of the $(1/u^2)$ factor of the kinetic term, we transform to a new parameter time ξ given by

$$dw=2u^2d\xi,\quad w=\int^{\xi}d\xi\,2u^2 . \tag{32}$$

Introducing this new parameter time together with the identity

$$1=\int_0^\infty d\chi(2u_a{}^2)\delta\left[W-\int_0^\chi d\xi\,2u^2\right]=\int_0^\infty d\chi\int_{-\infty}^{\infty}\frac{dE}{2\pi}e^{iEW}(2u_a{}^2)\exp\left[-iE\int_0^\chi d\xi\,2u^2\right], \tag{33}$$

Eq. (31) becomes

$$\begin{aligned}F(W,t_b,t_a)=&\int_0^\infty d\chi\int_{-\infty}^{\infty}\frac{dE}{2\pi}e^{iEW}\exp\left[-2i\frac{k^2}{a_0{}^2}\chi\right](u_a u_b)^{-1}\\&\times\int \mathscr{D}u\,\mathscr{D}k_u\exp\left[i\int_0^\chi d\xi\left[k_u\frac{du}{d\xi}+\frac{k_u{}^2}{2}-\frac{1}{2}(2\sqrt{E})^2u^2\right]\right].\end{aligned} \tag{34}$$

The path integral in the above expression is the Green's function for the nonrelativistic harmonic-oscillator motion with unit mass and with imaginary frequency $i(2\sqrt{E})$ taking place in the time interval $(0,-\chi)$. We can write down the result of this path integration from the known formulas[9] and get

$$\begin{aligned}F(W,t_b,t_a)=&\int_0^\infty d\chi\int_{-\infty}^{\infty}\frac{dE}{2\pi}e^{iEW}\exp\left[-2i\frac{k^2}{a_0{}^2}\chi\right](u_a u_b)^{-1}\\&\times\left[\frac{i\sqrt{E}}{\pi\sinh(2\sqrt{E}\,\chi)}\right]^{1/2}\exp\left[-i\frac{\sqrt{E}}{\sinh(2\sqrt{E}\,\chi)}[(u_a{}^2+u_b{}^2)\cosh(2\sqrt{E}\,\chi)-2u_a u_b]\right].\end{aligned} \tag{35}$$

To evaluate the propagator $K_{\mathbf{k}}(t_b,t_a)$ for the wave number $\mathbf{k}$ we insert this expression into Eq. (17) and integrate over dW and dE:

$$K_{\mathbf{k}}(t_b,t_a)=-\frac{2\pi}{u_a u_b}\int_0^\infty d\chi\left[\frac{i\mu}{\pi\sinh 2\mu\chi}\right]^{1/2}\exp\left[-2i\frac{k^2}{a_0{}^2}\chi\right]\exp\left[-\frac{i\mu}{\sinh 2\mu\chi}[(u_a{}^2+u_b{}^2)\cosh 2\mu\chi-2u_a u_b]\right]. \tag{36}$$

For the massless particles we take the $\mu\to 0$ limit of the above equation:

$$K_{\mathbf{k}}^{0}(t_b,t_a)=-\frac{\pi_0}{k}\frac{a_0}{u_a u_b}\exp\left[-2i\frac{k}{a_0}(u_a-u_b)\right] . \tag{37}$$

To obtain formulas for the full propagator $K(x_b,x_a)$ for the massive and massless particles, we introduce Eqs. (36) and (37) into Eq. (16). After integrating over d^3k we arrive at

$$K(x_b,x_a)=\frac{ia_0^{\,3}}{2u_a u_b}\int_0^\infty d\chi\left[\frac{\mu\pi}{\chi^3\sinh 2\mu\chi}\right]^{1/2}\exp\left[i\frac{a_0^{\,2}}{8\chi}(\mathbf{x}_a-\mathbf{x}_b)^2\right]\exp\left[\frac{i\mu}{\sinh 2\mu\chi}[(u_a^{\,2}+u_b^{\,2})\cosh 2\mu\chi-2u_a u_b]\right] \tag{38}$$

and

$$K^0(x_b,x_a)=-2\sqrt{2\pi}\,a_0^{\,3}\frac{1}{u_a u_b}\frac{1}{a_0^{\,2}\,|\,\mathbf{x}_a-\mathbf{x}_b\,|\,-4(u_a-u_b)} . \tag{39}$$

Now to be able to study the late time limit of $K_{\mathbf{k}}(t_b,t_a)$ we will decompose it into the parabolic cylinder functions which are the wave functions for the repulsive oscillator. By employing the following bilinear generating function formula of the functions[10]

$$\frac{(\pi/2)^{1/2}}{2\pi i}\int_{c-i\infty}^{c+i\infty}d\nu\frac{t^{-\nu-1}}{\sin(-\pi\nu)}[D_\nu(x)D_{-\nu-1}(iy)+D_\nu(-x)D_{-\nu-1}(-iy)]=(1+t^2)^{-1/2}\exp\left[\frac{1}{4}\frac{1-t^2}{1+t^2}(x^2+y^2)+i\frac{txy}{1+t^2}\right]$$

for $-1<c<0$, $|\arg t\,|<\pi/2$, with the identifications

$$t=e^{-2\mu\chi},\; x=2\sqrt{-i\mu}\,u_a,\; y=2\sqrt{-i\mu}\,u_b ,$$

we can rewrite Eq. (36) as

$$K_{\mathbf{k}}(t_b,t_a)=\frac{\sqrt{\mu}}{2\pi i}e^{-i\pi/4}\int_0^\infty d\chi\exp\left[-2i\frac{k^2}{a_0^{\,2}}\chi\right]\int_{c-i\infty}^{c+i\infty}\frac{d\nu}{\sin(-\pi\nu)}e^{-i\pi\nu/2}e^{2\mu(\nu+1/2)\chi}$$
$$\times[D_\nu((1-i)\sqrt{2\mu}\,u_a)D_{-\nu-1}((1+i)\sqrt{2\mu}\,u_b)$$
$$+D_\nu((-1+i)\sqrt{2\mu}\,u_a)D_{-\nu-1}((-1-i)\sqrt{2\mu}\,u_b)] .$$

After integrating over $d\chi$ we obtain

$$K_{\mathbf{k}}(t_b,t_a)=-2\pi\frac{\sqrt{\mu}}{u_a u_b}\frac{e^{-\pi\lambda/4}}{1+e^{-\pi\lambda}}[D_\gamma(e^{-i\pi/4}2\sqrt{\mu}\,u_a)D_{-\gamma-1}(e^{-i\pi/4}2\sqrt{\mu}\,u_b)$$
$$+D_\gamma(e^{3i\pi/4}2\sqrt{\mu}\,u_a)D_{-\gamma-1}(e^{-3i\pi/4}2\sqrt{\mu}\,u_b)] , \tag{40}$$

where λ and γ are defined as

$$\lambda=2k^2/\mu a_0^{\,2},\;\; \gamma=-\tfrac{1}{2}(1-i\lambda) .$$

By using the well-known asymptotic expansions of the parabolic cylinder functions,[10] we have the $u=\sqrt{t}\to\infty$ limit of Eq. (40):

$$K_{\mathbf{k}}(t_b,t_a)\simeq-2\pi\frac{\sqrt{\mu}}{u_a u_b}\frac{e^{-\pi\lambda/4}}{1+e^{-\pi\lambda}}\left[\exp\left[-i\frac{\pi}{2}(\gamma+1/2)\right](1+e^{2i\pi(\gamma+1/2)})(2\sqrt{\mu})^{-1}u_a^{\,\gamma}u_b^{\,-\gamma-1}e^{i\mu(u_a-u_b)}\right.$$
$$-\frac{\sqrt{2\pi}}{\Gamma(1+\gamma)}e^{i\pi\gamma}(4\mu)^{\gamma}(u_a u_b)^{\gamma}e^{i\mu(u_a+u_b)}$$
$$-\frac{\sqrt{2\pi}}{\Gamma(-\gamma)}e^{i\pi\gamma}(4\mu)^{-\gamma-1}(u_a u_b)^{-\gamma-1}e^{-i\mu(u_a+u_b)}$$
$$\left.+\frac{2\pi}{\Gamma(-\gamma)\Gamma(1+\gamma)}\exp\left[\frac{5i\pi}{4}(\gamma+1/2)\right](2\sqrt{\mu})^{-1}u_a^{\,-\gamma-1}u_b^{\,\gamma}e^{-i\mu(u_a-u_b)}\right] . \tag{41}$$

(iii) $a(t)=e^{Ht}$, $H=$const, $-\infty<t<\infty$. This metric describes the inflationary universe. The radius of the Universe varies from zero to infinity as the parameter t takes values from $-\infty$ to $+\infty$; $H\equiv\dot{a}/a$ is the Hubble parameter which is a constant through the inflationary era. Inserting this expansion factor $a(t)=e^{Ht}$ into the amplitude of Eq. (15) we have

$$F(W,t_b,t_a)=e^{-H(t_a+t_b)}e^{-Ht_a}\int \mathscr{D}t\,\mathscr{D}k_t \exp\left[i\int_0^W dw(k_t\dot{t}+k_t{}^2-e^{-2Ht}k^2)\right] . \tag{42}$$

We observe that this expression is very similar to the path integral of the Morse potential. Thus one can follow the same procedure employed previously in solving that problem.[5] We first make a point transformation

$$t=-\frac{1}{H}\ln u,\quad k_t=-Huk_u,\quad 0\le u<\infty , \tag{43}$$

and arrive at

$$F(W,t_b,t_a)=(u_a u_b)u_a(-Hu_a)\int \mathscr{D}u\,\mathscr{D}k_u \exp\left[i\int_0^W dw(k_u\dot{u}+H^2u^2k_u{}^2-u^2k^2)\right] . \tag{44}$$

As we did in Sec. II (ii), to get rid of the H^2u^2 factor of the kinetic term, we transform to a new parameter time ξ defined by

$$dw=d\xi/2H^2u^2 . \tag{45}$$

Introduction of this parameter time together with the constraint identity into Eq. (44)

$$1=\int_0^\infty d\chi\frac{1}{2H^2u_a{}^2}\delta\left[W-\int_0^\chi d\xi/2H^2u^2\right]=\int_0^\infty d\chi\int_{-\infty}^\infty \frac{dE}{2\pi}e^{iEW}\frac{1}{2H^2u_a{}^2}\exp\left[-i\int_0^\chi d\xi\frac{E}{2H^2u^2}\right] \tag{46}$$

gives

$$F(W,t_b,t_a)=\int_0^\infty d\chi\int_{-\infty}^\infty\frac{dE}{2\pi}e^{iEW}\left[-\frac{u_au_b}{2H}\right]\exp\left[-i\frac{k^2}{2H^2}\chi\right]\int\mathscr{D}u\,\mathscr{D}k_u\exp\left[i\int_0^\chi d\xi\left[k_u\frac{du}{d\xi}+\frac{k_u{}^2}{2}-\frac{E/2H^2}{u^2}\right]\right] . \tag{47}$$

The path integral of the above equation is the same as the path integral of Eq. (22); thus, we can copy the result of that expression by replacing $k^2/a_0{}^2+\frac{1}{4}$ with E/H^2, and $m=\frac{1}{2}$ with $m=1$, and obtain

$$F(W,t_b,t_a)=\int_0^\infty d\chi\int_{-\infty}^\infty\frac{dE}{2\pi}e^{iEW}\left[-\frac{u_au_b}{2H}\right]\exp\left[-i\frac{k^2}{2H^2}\chi\right]\frac{i(u_au_b)^{1/2}}{\chi}I_{\sqrt{1-4E/H^2}/2}\left[\frac{iu_au_b}{\chi}\right]\exp\left[\frac{1}{2i\chi}(u_a{}^2+u_b{}^2)\right] . \tag{48}$$

To get the propagator $K_{\mathbf{k}}(t_b,t_a)$, we repeat the corresponding steps of the previous cases. That is we insert Eq. (48) into (17) and take the integrals over dW and dE:

$$K_{\mathbf{k}}(t_b,t_a)=\frac{i}{2H}\int_0^\infty\frac{d\chi}{\chi}\exp\left[-i\frac{k^2}{2H^2}\chi+\frac{1}{2i\chi}(u_a{}^2+u_b{}^2)\right]I_\nu\left[\frac{iu_au_b}{\chi}\right] , \tag{49}$$

where the index ν is given by

$$\nu=\tfrac{1}{2}(1-4\mu^2/H^2)^{1/2} . \tag{50}$$

After performing $d\chi$ integration we have (for $u_a>u_b$)

$$K_{\mathbf{k}}(t_b,t_a)=\frac{\pi}{2H}(u_au_b)^{3/2}H_\nu^{(2)}\left[\frac{k}{H}u_a\right]J_\nu\left[\frac{k}{H}u_b\right] . \tag{51}$$

For massless particles we put $\nu=\frac{1}{2}$ in the above formula, and arrive at

$$K_{\mathbf{k}}^0(t_b,t_a)=\frac{i}{k}(u_au_b)e^{-i(k/H)u_a}\sin\frac{k}{H}u_b \tag{52}$$

or by using Eq. (16) we obtain a final expression for the full propagator:

$$K^0(x_b,x_a)=-\left[\frac{2H}{\pi}\right]^2\frac{u_au_b}{[2H^2(\mathbf{x}_a-\mathbf{x}_b)^2-(u_a+u_b)^2][2H^2(\mathbf{x}_a-\mathbf{x}_b)^2-(u_a-u_b)^2]} . \tag{53}$$

Since we are always interested in finding the amplitude for observing the pairs at very late times, we need the large t forms of the propagators. For that we use the following limiting values of the Bessel functions as $u\to 0$ (Ref. 7):

$$H_\nu^{(2)}\left[\frac{k}{H}u_a\right]\simeq\frac{e^{i\pi/2}}{\sin\pi\nu}\left[\frac{(k/2H)^{-\nu}}{\Gamma(1-\nu)}u_a{}^{-\nu}-\frac{e^{i\pi\nu}(k/2H)^{\nu}}{\Gamma(1+\nu)}u_a^{\nu}\right] \tag{54a}$$

and

$$J_\nu\left[\frac{k}{H}u_b\right]\simeq\frac{(k/2H)^{\nu}}{\Gamma(1+\nu)}u_b{}^{\nu}\ . \tag{54b}$$

Employing these formulas in Eq. (51) we have

$$K_{\mathbf{k}}(t_b,t_a)\simeq\frac{i}{2H}(u_a u_b)^{3/2}\left[\frac{1}{\nu}u_a{}^{-\nu}u_b{}^{\nu}-\pi e^{i\pi\nu}\frac{(k/2H)^{2\nu}}{\Gamma^2(1+\nu)\sin\pi\nu}(u_a u_b)^{\nu}\right]. \tag{55}$$

For the massless particles this formula gives

$$K_{\mathbf{k}}^0(t_b,t_a)\simeq\frac{i}{H}u_a u_b{}^2\ . \tag{56}$$

For $\mu=H/2$, ν vanishes, and Eq. (55) becomes

$$K_{\mathbf{k}}(t_b,t_a)\,|_{\mu=H/2}\simeq\frac{\pi}{2H}(u_a u_b)^{3/2}\ . \tag{57}$$

III. PAIR-PRODUCTION AMPLITUDES

The amplitude A_{ij} for a pair of particles created by the gravitational field and detected at time t in states $f_i(x)$ and $f_j(x)$ is given by

$$A_{ij}=-A_0\int d^3k_a d^3k_b\int d\sigma^{\mu}(x_a)d\sigma^{\nu}(x_b)f_i^*(\mathbf{k}_a,x_a)\overleftrightarrow{\partial}_{\mu}{}^{a}f_j^*(\mathbf{k}_b,x_b)\overleftrightarrow{\partial}_{\nu}{}^{b}K(x_b,x_a)\ . \tag{58}$$

Here A_0 is the amplitude for no particle production and $d\sigma^{\mu}$ is the element of constant t hypersurface. Because of the integrations over d^3k_a and d^3k_b the above amplitude does not distinguish the wave numbers of the particles. For the metric of Eq. (1), A_{ij} becomes

$$A_{ij}=-A_0\int d^3k_a d^3k_b d^3x_a d^3x_b[a(t_a)a(t_b)]^3 f_i^*(\mathbf{k}_a,x_a)\overleftrightarrow{\partial}_{t_a}f_j^*(\mathbf{k}_b,x_b)\overleftrightarrow{\partial}_{t_b}K(x_b,x_a)\ . \tag{59}$$

In this amplitude we use for $f(\mathbf{k},x)$ and $K(x_b,x_a)$ the type of expressions given by Eqs. (9) and (16). After integrating over $d^3x_{a,b}$ and $d^3k_{a,b}$ we obtain an expression for the creation of a pair with wave numbers $\mathbf{k}$ and $-\mathbf{k}$:

$$A_{ij}=\int d^3k\, A_{ij}(\mathbf{k})\ , \tag{60}$$

where

$$A_{ij}(\mathbf{k})=-A_0[a(t_a)a(t_b)]^3 g_i^*(\mathbf{k},t_a)\overleftrightarrow{\partial}_{t_a}g_j^*(-\mathbf{k},t_b)\overleftrightarrow{\partial}_{t_b}K_{\mathbf{k}}(t_b,t_a)\ . \tag{61}$$

We want to evaluate $A_{ij}(\mathbf{k})$ at a late time for positive-energy particles with large values of physical momenta $\mathbf{p}=\mathbf{k}/a(t)$. Under these conditions, for some forms of $a(t)$, and for massive particles, the solutions of Eq. (10) are reduced to the WKB-type functions constructed by Parker:[1]

$$g(\mathbf{k},t)=\frac{1}{[a(t)]^{3/2}[2w_k(t)]^{1/2}}\exp\left[-i\int^{t}dt' w_k(t')\right] \tag{62}$$

with

$$w_k(t)=\{k^2/[a(t)]^2+\mu^2\}^{1/2}\ . \tag{63}$$

For large t and k, $g(\mathbf{k},t)$ satisfies the positive-energy condition

$$\partial_t g=-iw_k(t)g\ . \tag{64}$$

Validity of these "adiabatic" approximations provide an "almost meaningful" concept of particle. When Eq. (64) is introduced into Eq. (61) we have

$$A_{ij}(\mathbf{k})=-A_0[a(t_a)a(t_b)]^3 g_i^*(t_a)g_j^*(t_b)\{-w_k(t_a)w_k(t_b)-i[w_k(t_a)\partial_{t_b}+w_k(t_b)\partial_{t_a}]+\partial_{t_a}\partial_{t_b}\}K_{\mathbf{k}}(t_a,t_b)\ . \tag{65}$$

This is the amplitude we are going to evaluate for the specific cases for the massive particles.

(i) $a(t)=a_0 t$. Since this is the example studied extensively in Ref. 3, we will not go into any detail of the derivations, but simply state the results for completeness. The amplitude $A_{ij}(\mathbf{k})$ is

$$A_{ij}(\mathbf{k}) \simeq -A_0 e^{i\pi/2} e^{-\pi k/a_0} . \tag{66}$$

For massless particles one gets a vanishing result for the amplitude. Note that the first term of the propagator of Eq. (28) contributes an oscillating term to $A_{ij}(\mathbf{k})$ which drops out at late times. The probability for creation of a pair with wave number $\mathbf{k}$ is

$$P_1(\mathbf{k}) = |A_0|^2 e^{-2\pi k/a_0} . \tag{67}$$

The probability of $n(\mathbf{k})$ pair creation is

$$P_n(\mathbf{k}) = |A_0|^2 e^{-2\pi n(\mathbf{k})k/a_0} . \tag{68}$$

From the conservation of probability condition

$$\sum_{n=0}^{\infty} P_n(\mathbf{k}) = 1 ,$$

one also obtains value of $|A_0|^2$ and gets

$$P_n(\mathbf{k}) = e^{-2\pi n(\mathbf{k})k/a_0}(1 - e^{-2\pi k/a_0}) . \tag{69}$$

The average number of pairs having the wave number $\mathbf{k}$ is

$$N(\mathbf{k}) = \sum_{n=0}^{\infty} n P_n(\mathbf{k}) = \frac{1}{e^{2\pi k/a_0} - 1} . \tag{70}$$

(ii) $a(t)=a_0\sqrt{t}$. In this case time part of the covariant Schrödinger equation given by Eq. (10) is

$$\left[\partial_t^2 + \frac{3}{2t}\partial_t + \frac{k^2/a_0^2}{t} + \mu^2 \right] g(\mathbf{k},t) = 0 . \tag{71}$$

Transforming to the variable $u=\sqrt{t}$ and substituting

$$g = \frac{1}{u} h ,$$

Eq. (71) becomes

$$\left[-\frac{d^2}{du^2} - 4\mu^2 u^2 \right] h = \frac{4k^2}{a_0^2} h . \tag{72}$$

This is similar to the Schrödinger equation of a repulsive harmonic oscillator; and it is solved in terms of the parabolic cylinder functions.[7] Thus, the state function g can be written as

$$g = \mathcal{N} \frac{1}{u} D_{-(1+i\lambda)/2}(e^{i\pi/4} 2\sqrt{\mu}\, u) , \tag{73}$$

with $\lambda = 2k^2/\mu a_0^2$. On the other hand, for large values of u, the WKB solution of Eq. (62) is given by

$$g(\mathbf{k},t) \simeq \frac{1}{a_0\sqrt{2\mu}} \frac{e^{-i\mu u^2}}{u^{3/2}} e^{-i\lambda/4} . \tag{74}$$

We see that if the normalization constant is chosen to satisfy

$$|\mathcal{N}| = a_0^{-3} \mu^{-1/4} e^{-\pi\lambda/8}$$

the asymptotic limit of Eq. (73) coincides with Eq. (74).

To develop the amplitude $A_{ij}(\mathbf{k})$ we substitute the expressions of Eqs. (41) and (74) into the late time limit of Eq. (65) [i.e., we take $w_k(t) \simeq \mu$]; and, put $t_a = t_b = t$. When inserted in $A_{ij}(\mathbf{k})$ only the third term of Eq. (41) gives contribution; the others, because of their rapidly oscillating character, vanish at late times. The final form of the amplitude, apart from a constant phase, is

$$A_{ij}(\mathbf{k}) \simeq A_0 \frac{1}{4} (2\pi)^{3/2} \frac{e^{-3\pi\lambda/4}}{\Gamma(\frac{1}{2} - i\lambda/2)} . \tag{75}$$

Note that, in evaluating the above equation we have also employed the large k (i.e., large λ) limit. The probability for one pair production with wave number $\mathbf{k}$ is

$$P_1(\mathbf{k}) = |A_{ij}(\mathbf{k})|^2 \simeq |A_0|^2 \left[\frac{\pi}{2} \right]^2 e^{-\pi\lambda} . \tag{76}$$

We observe that this formula, which is in agreement with the result of Birrell and Davies,[1] is of a similar nature with the probability of Eq. (67). Therefore all the results obtained in the previous example for the massive pair production are also valid for the radiation-dominated universe, provided that $|A_0|^2$ and k/a_0 are replaced by $|A_0|^2(\pi/2)^2$ and $\lambda/2$, respectively.

For the massless particle the state function $g(\mathbf{k},t)$ of Eq. (62) is

$$g_0(\mathbf{k},t) = \frac{1}{a_0\sqrt{2k}} \frac{e^{-(2ik/a_0)u}}{u} . \tag{77}$$

Inserting this expression and the propagator $K^0_{\mathbf{k}}$ of Eq. (37) in Eq. (61) we see that $A_{ij}(\mathbf{k})$ is precisely equal to zero. This is not surprising since according to the general conclusion of Parker,[1] "In an expanding universe in which a particular type of particle is predominant, the expansion achieved after a long time will be such as to minimize the average creation rate of that particle."

(iii) $a(t)=e^{Ht}$. Let us first consider the massive particles. In the late time limit, i.e., for $\mu \gg ku = k\,e^{-Ht}$ the WKB solution of Eq. (62) becomes

$$g(\mathbf{k},t) \simeq \frac{u^{3/2}}{\sqrt{2\mu}} u^{i\mu/H} \left[\frac{2\mu}{k} \right]^{-i\mu/H} . \tag{78}$$

The exact solution of Eq. (10) for $a(t)=e^{Ht}$, which can be taken as

$$g(\mathbf{k},t) = \left[\frac{H}{\mu} \right]^{\nu} \frac{1}{\sqrt{2\mu}} \Gamma(1+\nu) u^{3/2} J_\nu \left[\frac{k}{H} u \right] \tag{79}$$

approaches to the form

$$g(\mathbf{k},t) \simeq \left[\frac{H}{\mu} \right]^{\nu} \frac{1}{\sqrt{2\mu}} u^{(\nu+3/2)} \left[\frac{k}{2H} \right]^{\nu} \tag{80}$$

as $ku/H \to 0$ and agrees with Eq. (78) for $\mu \gg H$. Note that, since for massive particles $w_k(t) \simeq \mu$, the condition of Eq. (64) is also satisfied.

For $\mu > H/2$, when we insert Eqs. (80) and (55) into Eq. (61) and evaluate it at $t_a = t_b = t$, we obtain

$$A_{ij} \simeq -iA_0 \frac{\pi H}{2\mu} \frac{e^{i\pi\nu}}{\sin\pi\nu} \frac{(2\mu^2/Hk)^\nu}{[\Gamma(1+\nu)]^2}(1-\mu^2/H^2+2\nu) . \quad (81)$$

In $\mu \gg H$ limit, in which the adiabatic approximations are valid, this amplitude is

$$A_{ij} \simeq -A_0 \pi \left[\frac{2\mu^2}{kH} \right]^{i\mu/H} e^{-2\pi\mu/H} \frac{H/\mu}{[\Gamma(i\mu/H)]^2} . \quad (82)$$

The probability expressions for one and n pair productions are

$$P_1(\mu \gg H) = |A_{ij}|^2 = |A_0|^2 \tfrac{1}{4} e^{-2\pi\mu/H} \quad (83)$$

and

$$P_n(\mu \gg H) = \frac{1}{2^{2n}} e^{-2\pi n\mu/H} \left[1 - \frac{e^{-2\pi\mu/H}}{4} \right] . \quad (84)$$

In writing the last formula we have inserted $|A_0| = 1 - \frac{1}{4} e^{-2\pi\mu/4}$ which is obtained from the probability conservation.

At this point it is of interest to discuss the choice of the initial state implicit in our method. For that, let us consider the WKB solution of Eq. (62) at $u \to \infty$:

$$h(\mathbf{k},t) \simeq \frac{1}{\sqrt{2k}} u\, e^{-i(k/H)u} . \quad (85)$$

The correctly normalized exact solution of Eq. (10) which takes this form in the "early time" limit (i.e., for $ku \gg H$) is

$$h(\mathbf{k},t) = \tfrac{1}{2}(\pi/H)^{1/2} \exp\left[-i\frac{\pi}{2}(\nu+1/2) \right] u^{3/2} H_\nu^{(2)} \left[\frac{ku}{H} \right] . \quad (86)$$

If we take $h(\mathbf{k},t)$ as the initial state, by the help of the relation between the Hankel and Bessel functions, we express it as the superposition of $g(\mathbf{k},t)$ and $g^*(\mathbf{k},t)$ of Eq. (79); thus we can identify the Bogoliubov coefficients. We then observed that, for $\mu \gg H$ this procedure leads to the same results for the production amplitudes as the ones we have already obtained.

Some remarks are in order for the choice we made for the out vacuum state. Let us note that, although Eq. (78) has the form of the zeroth-order adiabatic solution, this correspondence is misleading in higher orders for $d^n a(t)/dt^n = H^n = \text{const}$. We choose Eq. (78) as the out vacuum because, it is the late time limit of the exact solution $g(\mathbf{k},t)$ of Eq. (79) for large μ and is also of $e^{i\mu t}$ type. In fact, one can talk of two alternative choices of the vacuum states in de Sitter spacetimes.[11] One choice may be based on the requirement that the vacuum state produces the standard time exponential form in the flat spacetime limit of $H \to 0$. In this case no particle production occurs. Alternatively one may define the particle modes by demanding to have the time exponential form in $|t| \to \infty$ limit. This choice leads to a nonzero and mode-independent value for the pair-production amplitude. Apparently the out state definition we made corresponds to this choice.

Note that since $P_n(\mu \gg H)$ is independent of $\mathbf{k}$, the absolute probability for noncreation of pairs in any mode, which is given by

$$\prod_{\mathbf{k}} |A_0|^2 = \exp\left[\int d^3k \ln |A_0|^2 \right] = \exp\left[\ln(1-\tfrac{1}{4}e^{-2\pi\mu/H}) \int d^3k \right] \quad (87)$$

is divergent.[2] This divergence is due to the infinite-volume limit. However this does not prevent one getting meaningful information. As it was done in Ref. 2, it can be handled by first cutting the integral in Eq. (87) at a value Λ and then letting this cutoff change. Change in the cutoff can be expressed in terms of the change in four-volume as

$$\Delta \int d^3k = \Gamma \Delta V_4 \quad (88)$$

with

$$\Gamma = \frac{2H}{\pi} \ln \left[1 - \frac{1}{4} e^{-2\pi\mu/H} \right] . \quad (89)$$

From Eqs. (87) and (83) we see that Γ, which is finite, is the probability of no pair creation in unit four-volume:

$$\Gamma = \frac{1}{V_4} \frac{1}{\ln(1-\frac{1}{4}e^{-2\pi\mu/H})} \ln \prod_{\mathbf{k}} |A_0|^2 .$$

For $\mu = 0$ the state function with wave number k is obtained from Eq. (79) by putting $\nu = \frac{1}{2}$:

$$g(\mathbf{k},t) \simeq u \sin\frac{ku}{H} . \quad (90)$$

This form can also be deduced from the zero-mass propagator for mode $\mathbf{k}$ given by Eq. (52). Introducing Eqs. (90) and (52) into Eq. (61) we found that the massless particle-production amplitude is equal to zero.

IV. CONCLUSIONS

In recent years considerable experience has been gained in calculating exactly the path integrals for several quantum-mechanical problems in flat spacetime. In this work we have demonstrated that this experience can also be utilized for solving the curved spacetime path integrals for studying the particle production in gravitational fields. Previously, the path integrals were essentially used for defining the boundary conditions of the covariant Schrödinger equations which are satisfied by the propagators. (Actually, parabolic differential equations which are obtained by analytically continuing spacetime are solved.)

For the spatially flat Robertson-Walker universe with scale factor $a(t)$, we converted the path integration over the time coordinate to the path integrations for the nonrelativistic motion of a particle under the influence of a potential $V \simeq a^{-2}(t)$ in flat spacetime. We studied three specific examples of $a(t)$ which correspond to the universes with initial singularities. There exist, of course, some other cases that can also be worked out exactly. For example, the path integrals for the metrics of Eq. (1) with $a(t) \simeq \cosh t$ and $a(t) \simeq \sin t$ can be handled by expressing their propagators in terms of the kernels for the potentials

$V \simeq \cosh^{-2} t$, and $V \simeq \sin^{-2} t$ whose path integrals are exactly solvable.[12]

The results of the examples we have studied can be summarized as follows.

For all of the cases explicit calculations gave vanishing values for the massless particle productions. This is the natural consequence of the conformal invariance.

For $a(t) \simeq t$ and $a(t) \simeq \sqrt{t}$, at high energies we obtained similar pair-production probabilities for the massive particles which are in the form of e^{-2k}. These are the same as the results of Chitre and Hartle[3] and Birrell and Davies.[1]

For the inflationary universe the probability of very massive particle pair production in unit four-volume was found to be proportional to $e^{-2\pi(\mu/H)}$. This agrees with the conclusion of Mottola[2] for the de Sitter space.

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