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Microwave anisotropies from cosmic strings

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The anisotropies in the microwave background radiation in the cosmic-string theory of galaxy formation are calculated using a general-relativistic analysis. On small scales ($\theta \simeq 0.5°$) we predict $\langle (T_1 - T_2)^2 \rangle^{1/2}/T \simeq 4\times 10^{-6}$, on large scales ($\theta \simeq 6°$) $\langle (T_1 - T_2)^2 \rangle^{1/2}/T \simeq 10^{-5}$. The dominant effect is the Sachs-Wolfe effect from string loops. The above amplitudes are significantly smaller than the best current observational upper bounds.

I. INTRODUCTION

Cosmic strings[1] have recently generated considerable interest as a mechanism for forming structures in the Universe such as galaxies and clusters of galaxies.[2] In particular, it has recently been shown[3] that the correlation function of string loops agrees both in magnitude and in shape with the measured correlation function of Abell clusters. Cosmic-string loops are seed masses around which galaxies and clusters of galaxies can accrete. The one free parameter in the cosmic-string theory of galaxy formation, the mass μ per unit proper length of string, was recently determined[4] by requiring that the seed mass have the correct magnitude to account for the observed overdensity in Abell clusters. This value, $\mu G \sim 10^{-6}$, also gives the correct amplification factor of the galaxy-galaxy correlation function over the cluster-cluster correlation function. The calculations were performed for cold dark matter and $\Omega = 1$.

Inhomogeneities in the matter distribution which become galaxies also cause fluctuations in the microwave background. However, apart from a dipole anisotropy which is usually attributed to Earth's peculiar velocity, no temperature anisotropies have been detected so far. Present observations give $\langle \delta T^2 \rangle^{1/2} < 3\times 10^{-5}$ on small angular scales and $\langle \delta T^2 \rangle^{1/2} < 7\times 10^{-5}$ on large scales.[5] Indeed, the smoothness of the microwave is strong observational evidence for using Robertson-Walker (RW) models. The observational bounds on δT have proved to be a tough constraint on models which try to explain the existence of galaxies by the growth of linear energy-density perturbations. For example, models with baryonic dark matter and $\Omega = 1$ are ruled out.[6]

In this paper we present a general-relativistic analysis of the temperature fluctuations.[7] We show that the predicted anisotropies are below present observational bounds. We conclude that, on small angular scales $\theta \simeq 0.5°$,

$$\left\langle \left[\frac{\delta T}{T} \right]^2 \right\rangle^{1/2} \simeq 8\times 10^{-7} \tag{1.1}$$

and on large angular scales $\theta \simeq 6°$

$$\left\langle \left[\frac{\delta T}{T} \right]^2 \right\rangle^{1/2} \simeq 3\times 10^{-6} \,. \tag{1.2}$$

Thus, in the cosmic-string theory of galaxy formation, formation of galaxies and clusters of galaxies is compatible with the absence of observed fluctuations in the microwave background radiation (MBR). The basic reason is simple. Matter can accrete around string loops starting at the time t_{eq} of equal matter and radiation.[8] Loops provide seed masses for galaxies and clusters of galaxies. This is in contrast with models with linear adiabatic perturbations and hot dark matter, in which perturbations on these scales get wiped out by dissipation. It can easily be checked that the mean separation of loops which gives rise to galaxies (clusters of galaxies) exceeds the radius at t_{eq} of the shell which eventually collapses to the radius of a galaxy (clusters of galaxies) by a factor of about 6. Hence the density contrast within the above radius exceeds the mean density contrast by a factor of 200. The former is relevant for formation of structures, the latter for rms temperature anisotropies. Thus the low value[8] of the rms energy density fluctuations $\delta\rho/\rho \sim 10^{-6}$ is consistent with the high density contrast $\delta\rho/\rho \sim 10^{-4}$ required in regions which collapse to form galaxies and clusters of galaxies. This is explained in more detail in Appendix C. In models with linear adiabatic perturbations (random phases) and cold dark matter it is necessary to have $\delta\rho/\rho \sim 10^{-4}$ at t_{eq} in order to be able to explain the formation of structures.

II. FORMATION AND EVOLUTION OF COSMIC STRINGS

We begin with a brief review of the formation and evolution of cosmic strings. Cosmic strings[1] are one-dimensional topological defects which arise in a large class of grand unified theories. The requirements for strings to be formed is that a Higgs field Φ acquires a nonvanishing vacuum expectation value during a phase transition in the early Universe and that the manifold M_0 of degenerate vacua is nonsimply connected. This makes it possible that as one traverses a loop in space $\mathbf{x}(\sigma)$, $\Phi(\mathbf{x}(\sigma))$ will traverse a noncontractible loop in M_0. If one then imagines contracting the loop $\mathbf{x}(\sigma)$ to a point, $\Phi(\mathbf{x}(\sigma))$ must leave M_0, i.e., be in a nonvacuum state. This means there is a localized region of enhanced energy density, which is a string. Strings are either infinite or closed loops. Numerical simulations of the formation of strings[9,10] have shown that about 80% of the string density is in the form of infinite open lengths and the rest is in the form of closed loops.

As noted by Kibble,[11] if the phase transition occurs sufficiently rapidly, Φ will take on random vacuum values on scales larger than some coherence length L. L is of the order of the horizon size.

As the Universe expands, the network of "infinite" strings is stretched and straightened out (by "infinite" we mean infinite lengths and loops with a radius bigger than the horizon). Strings frequently intersect. There is a probability p that strings which cross will exchange partners and reconnect the other way. For $p \sim 1$ the number of infinite strings per comoving volume will decrease as the strings chop themselves up into loops. Loops are formed with a radius of order the horizon size. This picture has recently been established in numerical simulations.[10,12] These simulations show that the coherence length $L(t)$ increases linearly in time

$$L(t)=\lambda t \tag{2.1}$$

with $\lambda \simeq 1$, and that the energy density in string decreases as radiation. There are a few infinite strings per horizon volume.

Loops of string smaller than the horizon retain constant physical size. They oscillate and lose energy by gravitational radiation until they disappear.[13] At any given time there are loops with radii R between 0 and t. Loops with an initial radius smaller than $\gamma G\mu t$, where $\gamma \sim 5$ (Refs. 10 and 12) have decayed by gravitational radiation. Loops remaining with radius $R<\gamma\mu G t$ had an initial radius of the order of $\gamma\mu G t$. The distribution of loops is characterized by the number density $n(R,t)$. $n(R,t)dR$ is the number of loops per unit proper volume at time t with radius between R and $R+dR$. For $t \gtrsim t_{eq}$,

$$n(R,t)=\frac{\nu}{(\gamma G\mu)^{5/2}}\left[\frac{t_{eq}}{t}\right]^2\frac{1}{t_{eq}{}^4},\quad R<\gamma\mu G t\ , \tag{2.2}$$

$$n(R,t)=\frac{\nu}{R^{5/2}t_{eq}{}^{3/2}}\left[\frac{t_{eq}}{t}\right]^2,\quad \gamma\mu G t_{eq}<R<t_{eq} \tag{2.3}$$

(loops which entered the horizon before t_{eq}), and

$$n(R,t)\simeq\frac{\nu}{R^2t^2},\quad t_{eq}<R<t\ . \tag{2.4}$$

According to numerical simulations $\nu \simeq 10^{-2}$. The mass of a loop with radius R is

$$M(R)=\beta\mu R \tag{2.5}$$

with $\beta \simeq 9$.

The centers of loops are not randomly distributed. For loops around which clusters of galaxies accrete, the correlations become the cluster-cluster correlation function.

The correlation function $C(l,R,t)$ of loops of radius R at physical separations l at time t has recently been measured in numerical simulations.[3] For scales l less than the mean separation of loops the result is

$$C(l,R,t)\simeq\epsilon\left[\frac{d_R}{l}\right]^2,\quad R<l<d_R\ , \tag{2.6}$$

with $\epsilon \sim 0.2$. d_R is the mean separation of loops of radius R or greater, and can be determined from (2.2)–(2.4). In particular, for $\gamma\mu G t<R<t_{eq}$ we have

$$d_R=\left[\frac{3}{2\nu}\right]^{1/3}R^{1/2}t_{eq}{}^{1/2}\left[\frac{t}{t_{eq}}\right]^{2/3}\ .$$

These correlations are determined by the detailed manner in which initial (parent) loops split up into final (child) loops.

On larger scales loops have the same correlations as the network of strings they were chopped off from. Hence $C(l,R,t)$ can be determined by taking the Fourier transform of the power spectrum from infinite strings. As yet the numerical simulations do not have the range to measure correlations for $l>d_R$, but a model proposed in Ref. 8 was to assume the infinite strings are a randomly distributed network of random walks. For these,

$$\langle\delta\rho(\mathbf{k})\delta\rho(\mathbf{k}')\rangle=12\rho_s(2\pi)^3\delta^3(k+k')L^{-1}\mu k^{-2}\ , \tag{2.7}$$

where ρ_s is the total density of infinite strings, and L is the step length (2.1).

From this,

$$\xi(l)=\int\frac{d^3k}{(2\pi)^3}e^{-i\mathbf{k}\cdot l}\int\frac{d^3k'}{(2\pi)^3}\left\langle\frac{\delta\rho(\mathbf{k})\delta\rho(\mathbf{k}')}{\rho_s{}^2}\right\rangle$$
$$=\frac{3\mu}{\pi\rho_s L l}\ . \tag{2.8}$$

A simple way to understand this is as follows. If one sits on a string and goes out a distance l, the average length of string between l and $l+dl$ will be given by

$$\frac{\rho_s}{\mu}4\pi l^2dl+d\lambda\equiv\frac{\rho_s}{\mu}4\pi l^2dl[1+\xi(l)]\ ,$$

where $d\lambda$ is the excess over average due to the string one is sitting on. Since the string is Brownian, $\lambda\propto l^2/L$, $d\lambda\propto l\,dl/L$, and we find $\xi\propto l^{-1}$.

The correlation (2.8) translated into the correlation function for loops on large scales:

$$C(l,R,t)\simeq\epsilon d_R/l,\quad l>d_R\ , \tag{2.9}$$

with $\epsilon \sim 0.2$ also.

III. DENSITY FLUCTUATIONS FROM STRINGS AND CONSTRAINTS

If cosmic strings exist they will cause stress-energy perturbations. The evolution of galaxies and clusters from strings has been calculated[2,4,8] and gives encouraging agreement with observations. The next step is to calculate the microwave anisotropy due to strings. To do this we will need the two-point mass correlation function. In this section we discuss the relevant properties of the string perturbations.

Assume that the background spacetime is RW with spatially flat sections:

$$ds^2=dt^2-a^2(t)\delta_{ij}dx^i dx^j \ . \tag{3.1}$$

Strings introduce fluctuations in the stress energy $T_{\mu\nu}=T^{(0)}_{\mu\nu}+\delta T_{\mu\nu}$ and the metric $g_{\mu\nu}=g^{(0)}_{\mu\nu}+h_{\mu\nu}$, so that $(h_{\mu\nu},\delta T_{\mu\nu})$ satisfy the linearized Einstein equations. What can we way about the perturbed energy density $\delta\rho\equiv\delta T^{00}$?

Strings form in a phase transition when some internal symmetry is broken. Let

$$\rho=\rho^{\rm cl}(t)+\langle T^H_{00}\rangle \ , \tag{3.2}$$

where $\rho^{\rm cl}$ is the energy density of a classical fluid and T^H_{00} is the contribution to the energy from the Higgs field. Before the phase transition at the temperature T_c, $T^H_{00}=\Psi(t)$ is homogeneous and isotropic and

$$\rho(t)=\rho^{\rm cl}(t)+\Psi(t) \ . \tag{3.3}$$

As the temperature drops below the phase transition temperature T_c, the Higgs field Φ "rolls into" one of the vacuum states in all of space except along the strings. The initial potential energy of the scalar field decays into radiation $\delta\rho_{\rm R}(\mathbf{x},t)$. Some of the Higgs energy density goes into strings, as discussed in the previous section. Therefore, after the phase transition

$$\rho=\rho^{\rm cl}(t)+\delta\rho_{\rm R}(\mathbf{x},t)+\delta\rho_{\rm string}(\mathbf{x},t) \ . \tag{3.4}$$

Then density perturbation is the difference between ρ in the physical spacetime and ρ in the fictitious spacetime [Eqs. (3.4) and (3.3)], and has a string part and a "Higgs radiation" part:

$$\begin{aligned}\delta\rho&=\delta\rho_{\rm string}(\mathbf{x},t)+\delta\rho_{\rm R}(\mathbf{x},t)-\Psi(t)\\ &\equiv\delta\rho_{\rm string}+\delta\rho_{\rm HR} \ .\end{aligned} \tag{3.5}$$

The evolution of the strings has been studied in some detail[10,12,14] but we do not have a detailed model of the complicated dynamics of the radiation perturbations. However, by using general constraints on stress-energy perturbations we will be able to place strong restrictions on the behavior of $\delta\rho$.

It has been shown[15] that in RW spacetimes arbitrary stress-energy perturbations are not possible; rather, $\delta T^{\mu\nu}$ must obey certain integral constraints. Let $\mathscr{G}$ be a volume contained in a spatial hypersurface in coordinate system (3.1), with boundary $\delta\mathscr{G}$. When δT^0_κ can be neglected compared to δT^0_0, the integral constraints reduce to (for flat spatial sections)

$$\begin{aligned}\int_V dv\,\delta\rho&=\int_{\partial V} da_l B^l \ ,\\ \int_V dv\,\delta\rho\mathbf{x}&=\int_{\partial V} da_l \mathbf{p}^l \ .\end{aligned} \tag{3.6}$$

The boundary terms are linear functions of $h_{\mu\nu}$, and vanish if $h_{\mu\nu}$ vanishes on $\partial\mathscr{G}$. Therefore if the perturbations $\delta\rho$ and $h_{\mu\nu}$ are localized in space, the boundary terms are zero if we take $\mathscr{G}$ large enough.

First suppose we have a RW universe in which a single closed loop of string forms at time t_c with radius l_0. At a later time t the geometry will be unperturbed outside the causal light cone of the initial loop, $\delta\rho(l,t)=h_{\mu\nu}(l,t)\equiv 0$ for

$$l>L(t)=2t-2t^{1/2}t_c^{\ 1/2}+l_0 \ . \tag{3.7}$$

Therefore the density perturbation for a single loop plus its radiation partner must satisfy the integral constraints with a zero boundary term:

$$\begin{aligned}\int_{V_{\rm big}}\delta\rho_1 dv&=0 \ ,\\ \int_{V_{\rm big}}\delta\rho_1 l\,dv&=0 \ ,\end{aligned} \tag{3.8}$$

where $V_{\rm big}$ is a volume larger than the horizon, and

$$\delta\rho_1=\delta\rho_{\rm loop}+\delta\rho_{\rm HR} \ . \tag{3.9}$$

Some loops are present in the initial data but most loops actually form by chopping off infinite strings, which are executing random walks. These infinite strings must obey the constraints (3.6), but it is certainly not clear that the boundary terms vanish. However, if we assume that the chopping process is a causal process, governed by the microphysics of the string, then a loop which is formed at t is not affected by pieces of the string which are further than $t-t_c$ from it. That is, $\delta\rho_1$ from a chopped loop is approximately the same as $\delta\rho_1$ for an initial data loop.

Therefore we require that the density perturbation for a single loop plus Higgs radiation $\delta\rho_1$ obey the integral constraints with a zero boundary term [Eq. (3.8)].

At time t, the contribution to the density perturbation at l, $\delta\rho(l,t)$, provided by loops of radius between R and $R+dR$, and located at sites $\{l_a\}$ is

$$\delta\rho(l,t;R)=\sum_a \delta\rho_1(l-l_a,t;R) \ , \tag{3.10a}$$

where

$$\delta\rho_1(l)\equiv 0 \quad \text{for } l>L \tag{3.10b}$$

and $\delta\rho_1$ satisfies (3.8). The loops oscillate periodically, and on scales greater than its radius, a loop can be approximated as a spherical source with radius R and total mass $\beta R\mu$:

$$\frac{\delta\rho_{\rm loop}}{\rho}(l,t;R)=6\pi\beta G\mu\frac{t^2}{R^2}B\left[\frac{l}{R}\right], \quad t>t_{\rm eq} \ , \tag{3.11}$$

where

$$B(x)\equiv 0 \quad \text{for } x\geq 1 \tag{3.12}$$

and

$$\int d^3x\, B = 1 \ . \tag{3.13}$$

A convenient approximation is to use a Gaussian, $B = e^{-l^2/R^2}/\pi^{3/2}$, even though the Gaussian has an infinite tail.

During the radiation-dominated era, there is very little accretion of matter onto the loop seed masses. The constraints (3.8) give us an estimate of $|\delta\rho_{\rm HR}|$, if we approximate $\delta\rho_{\rm HR}$ as evenly spread out over the causal volume

$$\left|\frac{\delta\rho_{\rm HR}(t;R)}{\rho}\right| \sim \frac{R^3}{L^3}\left|\frac{\delta\rho_{\rm loop}}{\rho}\right| \sim \frac{9}{16}\beta G\mu\frac{R}{t}, \quad R \le t < t_{\rm eq} \ . \tag{3.14}$$

Now we can write down the two-point mass correlation function ξ for loops, which is needed to compute temperature fluctuations. ξ depends on the distribution of sites $\{l_a\}$ in (3.10). Loops which break off different strings are randomly distributed, but there are long-range correlations between the centers of loops which break off the same string. Let $P(l)dl$ be the probability of finding a loop within a distance l, given that one loop is at $l=0$. Then $C(l)$, the correlation function for the centers of loops, is defined by

$$P(l)dl = \frac{1}{V}4\pi l^2 dl[1+C(l)] \ .$$

Substituting (3.10) for $\delta\rho$, averaging the sites l_a over all space and using (3.8) gives, for the contribution to ξ from loops with radii between R and $R+dR$,

$$\xi(l,t;R) \equiv \left\langle \frac{\delta\rho}{\rho}(0,t;R)\frac{\delta\rho}{\rho}(l,t;R)\right\rangle = \xi_1 + \xi_{12} \ , \tag{3.15a}$$

where

$$\xi_1(l,t;R) = n(t,R)\int d^3z \frac{\delta\rho_1}{\rho}(\mathbf{z})\frac{\delta\rho_1}{\rho}(\mathbf{z}+\mathbf{l}) \tag{3.15b}$$

and

$$\xi_{12}(l,t;R) = n(t,R)^2 R \int d^3z\, d^3u \frac{\delta\rho_1}{\rho}(\mathbf{u})\frac{\delta\rho_1}{\rho}(\mathbf{z}) \times C(\mathbf{u}+\mathbf{z}-\mathbf{l})\Delta \ . \tag{3.15c}$$

ξ_1 is the contribution from the random part of the loop distribution, and ξ_{12} is from correlations between different loops of the same size. Δ is a constant of order 1 and will be set to 1 in the following (see Appendix A of Ref. 8). In (3.12) we have let $N(R)/V(t) \to n(t,R)dR$.

The mass correlation function at emission time t_E is obtained by integrating over all radii:

$$\xi(l,t_E) = \int_0^{t_E} dR\, \xi(l,t_E,R) \ . \tag{3.16}$$

The constraints (3.8) on $\delta\rho_1$ give us useful information about ξ. First consider ξ_1, the correlation of one loop with itself. The constraints imply[16]

$$\int_{V_{\rm big}} dv\, \xi_1(l) = 0 \tag{3.17a}$$

and

$$\int_{V_{\rm big}} dv\, l^2 \xi_1(l) = 0 \tag{3.17b}$$

for $V_{\rm big}$ a volume larger than the horizon volume.

These conditions on ξ_1 have implications for microwave fluctuations. On small scales photons will see a monopole moment of ξ, due to the loop at the center, but on large scales ξ looks like a quadrupole source (the function ξ must have at least two zeros). We will see that on large scales this implies a depression of the magnitude of the temperature fluctuations, and a change in their angular dependence.[17]

So far we have discussed the density perturbation $\delta\rho = \rho_{\rm loops} + \delta\rho_{\rm HR}$ for $t < t_{\rm eq}$. When the Universe becomes matter dominated, matter will accrete onto loop seeds and subsequently grow according to linear perturbation theory except in the immediate vicinity of the smallest loops. When $p=0$ and the flow is irrotational, one can always choose coordinates which are synchronous and comoving.[18] We include only scalar modes since these dominate at late times and choose coordinates which are synchronous and comoving. The solutions for the perturbed density $\delta\rho$ and metric $\bar{h}_{ij} \equiv h_{ij}/a^2$ in a flat, pressureless RW universe can be conveniently written as[7,17]

$$\begin{aligned} \frac{\delta\rho}{\rho} &= -\frac{1}{2}\frac{\eta^2}{{\eta_E}^2}\nabla^2 A(x), \quad t > t_{\rm eq} \ , \\ \frac{\partial}{\partial\eta}\bar{h}_{ij} &= -2\frac{\eta}{{\eta_E}^2}A_{,ij} \ . \end{aligned} \tag{3.18}$$

(Here x^i is a comoving coordinate and $a\, d\eta = dt$.)

Equation (3.18) will be used to determine the Sachs-Wolfe effect, one contribution to the microwave anisotropies. The model for $\delta\rho$ for $t < t_{\rm eq}$ is needed to determine what $\delta\rho$ is like on last scattering—i.e., the initial conditions for (3.18).

Finally, before turning to the computation of the microwave anisotropies, we must say something about the infinite strings. When the infinite strings are formed in the phase transition there will also be a perturbation Higgs radiation component, just as for loops

$$\delta\rho_\infty = \delta\rho_{\infty\,\rm st} + \delta\rho_{\infty\,\rm HR} \ . \tag{3.19}$$

Does $\delta\rho_\infty$ obey the constraints (3.8) with a zero boundary term? The answer is not straightforward, since certainly an infinite string and its metric perturbation pierce the boundary of any volume. However, consider a loop which is formed over an interval Δt at the phase transition with a radius much greater than the horizon size: $R \gg 2t_c$. (Numerical simulations[10] show that some of these large loops do form.) As already argued this loop satisfies the constraints with a zero boundary term. Now, only parts of the loop which are within a distance Δt of any point x can influence $\delta\rho$ and $h_{\mu\nu}$ at x. For example, parts of the loop "on the other side" are causally disconnected from x. Assuming that the same microphysics determines the structure of $\delta\rho$ for infinite strings as for large loops, we conclude that the constraints (3.8) hold (at least approximately) for $\delta\rho_\infty$.

IV. MICROWAVE ANISOTROPIES

Perturbations in the matter and metric will cause fluctuations in the observed temperature. A photon is emitted at $E=(t_E,l_E)$ and received at $0=(t_0,0)$. We will assume that the observed microwave background photons were emitted from the last scattering surface when hydrogen recombined. We will make the approximation that recombination happened at a single temperature $T_E \simeq 4000$ K, which is a red-shift $1+z_E \simeq 1.5\times 10^3$.

One contribution to δT is from fluctuations in the photons $\delta\rho_\gamma$ on last scattering, and another is a Doppler shift from the peculiar velocities of the source and observer.[19–22] A third contribution is the Sachs-Wolfe effect: the photon frequency is perturbed as it propagates on the perturbed null geodesic.[7,23–25]

Let $k=k_{(0)}+\delta k$ be the four-momentum of the photon and $u=u_{(0)}+v/a$ be the four-momentum of a local observer. v is the (proper) peculiar velocity. Then the gauge invariant expression for the temperature fluctuations is[26,27]

$$\frac{\delta T_0}{T}=\frac{1}{4}\frac{\delta\rho_\gamma(\mathbf{x}_E,t_E)}{\rho}+\frac{\delta T_{\rm SW}}{T}+\frac{\delta T_{\rm Dop}}{T}\;, \tag{4.1}$$

where[7]

$$\frac{\delta T_{\rm Dop}}{T}=v^r(0)-v^r(E)\;, \tag{4.2}$$

$$\frac{\delta T_{\rm SW}}{T}=a^2(t_E)\delta k^0(E)-a^2(t_0)\delta k^0(0)$$
$$=-\tfrac{1}{2}\int_0^{\Delta\eta} a^4(\bar h_{\mu\nu,\eta}k^\mu_{(0)}k^\nu_{(0)}-2\bar h_{0\beta,\alpha}k^\beta_{(0)}k^\alpha_{(0)})d\eta\;. \tag{4.3}$$

We will calculate the Sachs-Wolfe effect in detail, and estimate the other terms. We will see that the Sachs-Wolfe is the dominant effect, and that $\delta T_{\rm SW}$ is below current observational bounds. The calculations here are all for angular scales greater than the size of the largest loops last scattering. On smaller scales the detailed internal dynamics is important. Neglecting reionization our analysis is accurate for $\theta>0.5°$, which subtends a loop of size t_E.

For scalar perturbation in a pressureless universe the solutions are found in (3.18) and the Sachs-Wolfe integral can be evaluated exactly:[7]

$$\frac{\delta T_{\rm SW}}{T}=\frac{1}{\eta_E{}^2}[A(x)+\eta\mathbf{n}\cdot\nabla A(\mathbf{x})]_E^0\;. \tag{4.4}$$

It can be checked from the following solutions that the first term in (4.4) dominates: $\delta T/T\simeq(1/\eta_E{}^2)A(\mathbf{x})$. Consider the expectation value of the temperature difference between two points separated by an angle θ. Let Δl be the (proper) separation between the emission points at t_E. Then

$$\Delta l=6t_E(1+z_E)^{1/2}\sin\frac{\theta}{2}\;. \tag{4.5}$$

Solving Poisson's equation (3.18) for $A(\mathbf{x})$ and taking the expectation value, the temperature fluctuations are[27]

$$\frac{\langle(T_1-T_2)^2\rangle}{T^2}=\frac{4}{81}\frac{1}{t_E{}^4}\left[\int_0^{\Delta l}dl'l'^2\xi(l')\left[\Delta l-l'+\frac{1}{3}\frac{l'^2}{\Delta l}\right]+\tfrac{1}{3}\Delta l^2\int_{\Delta l}^{\infty}dl'l'\xi\right]\;. \tag{4.6}$$

Now we can read off the different behavior of $\langle(T_1-T_2)^2\rangle$ implied by the general-relativistic constraints. Suppose we are looking on large scales $\Delta l>2L(t_E)$ $(\theta\gtrsim 3°)$, and consider the random contribution $\xi_1(l)$. [Note that from definitions (3.10b) and (3.15b), it follows that $\xi(l)=0$ for $l>2L$.] From (3.17) the only nonzero term gives $\langle\Delta T^2\rangle\propto\int dl'l'^3\xi_1$, independent on θ. On the other hand, on small scales $\Delta l<2L(t_E)$ (3.17) does not hold. As long as ξ is peaked at the origin, the leading term gives $\langle\Delta T^2\rangle\propto\Delta l\int dl'l'^2\xi_1$ (see, e.g., Ref. 23).

Equations (3.15) and (3.16) give the correlation function in two parts. We treat these each in turn.

A. Random loops

In this subsection we compute $\langle\Delta T^2\rangle$ from ξ_1, which is the contribution to the correlation function from randomly distributed loops. Recall that ξ_1 has contributions from the loops and from the Higgs radiation. Since the Higgs radiation is spread over a much larger volume than the loop, on small scales, $\Delta l<L(t_E)=2t_E(\theta\lesssim 3°)$ the loops dominate.

Hence, on small scales the constraints do not apply. Substituting (3.15b), and using $\int d^3l\,\delta\rho(z+l)=\beta\mu R$, the mass of the loop, we have, on small scales $(\theta\lesssim 3°)$,

$$\frac{\langle(T_1-T_2)^2\rangle}{T^2}\simeq\frac{4}{81}\frac{\Delta l}{t_E{}^4}\int_0^{\Delta l}dl\,l^2\xi_1(l),\quad \Delta l<3t_E$$
$$\simeq\frac{2}{27\pi}\nu\left[\sin\frac{\theta}{2}\right](1+z_E)^{1/2}(6\pi\beta G\mu)^2\;, \tag{4.7}$$

in agreement with Ref. 8. With $\beta=9$, $\nu=10^{-2}$, $1+z_E=1.5\times10^3$, $G\mu=2\times10^{-6}$,

$$\frac{\langle(T_1-T_2)^2\rangle^{1/2}}{T}\simeq 5\times10^{-6}\left[\frac{\theta}{3°}\right]^{1/2}\;. \tag{4.8}$$

On large scales $\Delta l>3t_E$ the constraints (3.17) hold. The only nonzero term in (4.6) is the variance from a single point:

$$\frac{\langle(T_1-T_2)^2\rangle}{T^2}=\frac{4}{81}\frac{1}{t_E{}^4}\int dl\,l^3\xi_1,\quad \Delta l>3t_E\;. \tag{4.9}$$

This integral cannot be evaluated exactly without knowing how $\xi_1(l)$ behaves on all scales where it is nonzero, for which we would need an understanding of the dynamics of the Higgs radiation. However, an upper bound may be obtained by using a maximal value for the length of ξ, $l \sim 3t_E$:

$$\frac{\langle (T_1-T_2)^2\rangle^{1/2}}{T} \sim \left[\frac{\nu}{27\pi}\right]^{1/2}(6\pi\beta G\mu) \sim 4\times 10^{-6}, \quad \Delta l > 3t_E \ . \tag{4.10}$$

B. Correlated loops

There is a contribution ξ_{12} to the mass correlation function from the correlations between centers of different loops (2.6), (2.9), and (3.15c).

The site correlation function $C(l,R,t)$ breaks up into a piece for $l<d_R$ and one for $l>d_R$. Now for $l>d_R$, all of $\delta\rho_1$ will be included in an integration volume, so $\delta\rho_1$ will satisfy the constraints (3.8). On the other hand, for small volumes the constraints do not hold, and so the estimates of magnitudes are different. Therefore we must estimate the $l>d_R$ and $l<d_R$ parts separately.

(i) $C(l,R,t)=\epsilon(d_R/l)\theta(l-d_R)$. [Note: Here $\theta(x)$ is the Heaviside function.]

On larger scales, the density perturbation $\delta\rho_1$ which is in the integrand must satisfy the constraints (3.8). Exploiting this, ξ_{12} breaks up into a long-range and a short-range piece. For $R<t_E$,

$$\xi_{12}(l,t_E,R)=n^2(t_E,R)R\epsilon d_R(t_E)\times\left[\frac{4\pi}{5}Q\frac{L^3}{l^3}\theta(l-2L)+S(l)\right], \tag{4.11}$$

where Q and S are nonlocal moments of $\delta\rho_1$, given in Appendix A. They are bounded as

$$|S|\sim|Q|\lesssim(\beta G\mu R)^2 \tag{4.12}$$

and $S(l)$ vanishes outside the sound cone; $S(l)=0$ for $l>L_c=L(t_{eq})(t_E/t_{eq})^{2/3}$.

It should be stressed that the bounds (4.12) are conservative, based on general properties. For example, Q is a kind of quadrupole moment, and actually vanishes for a spherically symmetric perturbation.

$S(l)$ vanishes for $l>L_c$ and the monopole term in (4.6) dominates. We find

$$\frac{\langle (T_1-T_2)^2\rangle^{1/2}}{T} \simeq 2\epsilon^{1/2}\nu^{5/6}\beta(G\mu)^{3/4}\left[\frac{t_E}{t_{eq}}\right]^{5/2}\left[\sin\frac{\theta}{2}\right]^{1/2}\times(1+z_E)^{1/4}, \quad \Delta l>2t_E\left[\frac{t_{eq}}{t_E}\right]^{1/3} .$$

On small scales, $\Delta l\lesssim 3t_E$, this is bounded above by

$$\frac{\langle (T_1-T_2)^2\rangle^{1/2}}{T}\simeq 5\times10^{-8}, \quad \Delta l\lesssim 3t_E \ , \tag{4.13}$$

and on large scales by

$$\frac{\langle (T_1-T_2)^2\rangle^{1/2}}{T}\lesssim 3\times10^{-7}\left[\sin\frac{\theta}{2}\right]^{1/2}, \quad \Delta l\gtrsim 3t_E \ . \tag{4.14}$$

Q is a long-range contribution, vanishing for $l<2L$; hence the main contribution is from the last term in (4.6). From Q we obtain

$$\frac{\langle (T_1-T_2)^2\rangle^{1/2}}{T}\lesssim 8\nu^{5/6}\epsilon^{1/2}\beta(1+z_E)^{1/2}(G\mu)^{3/4}\left[\frac{t_{eq}}{t_E}\right]^{1/6}\left[\frac{\theta}{2}\right]\lesssim 2\times10^{-6}, \quad \Delta l<3t_E \tag{4.15}$$

and on large scales

$$\frac{\langle (T_1-T_2)^2\rangle^{1/2}}{T}\lesssim 16\nu^{5/6}\epsilon^{1/2}\beta\left[\sin\frac{\theta}{2}\right]^{1/2}(1+z_E)^{1/4}(G\mu)^{3/4}\left[\frac{t_{eq}}{t_E}\right]^{1/6}\lesssim 6\times10^{-6}\left[\sin\frac{\theta}{2}\right]^{1/2}, \quad \Delta l>3t_E \ . \tag{4.16}$$

All of these are below current observational bounds.

(ii) $C(l,R,t)=\epsilon(d_R{}^2/l^2)\theta(d_R-l)\theta(l-R)$. First consider large scales, $\Delta l>3t_E$. The first term in (4.6) dominates. (This follows from noting that the last term is zero and that $\Delta l>d_R$.) It is useful here to interchange the order of the dl and dR integrations. Then one finds

$$\Delta l\int_0^{d_R}dl\,l^2n^2R\int d^3z\,d^3u\frac{\delta\rho_1}{\rho}(u)\frac{\delta\rho_1}{\rho}(z)\frac{\epsilon d_R{}^2}{|\mathbf{u}+\mathbf{z}-l|^2}\lesssim\frac{M_{\rm loop}{}^2\epsilon}{\rho^2}d_R{}^3\Delta l n^2R \ , \tag{4.17}$$

and the temperature fluctuations are dominated by the largest loops:

$$\frac{\langle (T_1-T_2)^2\rangle^{1/2}}{T}\simeq(\tfrac{8}{27}\epsilon\nu)^{1/2}(6\pi\beta G\mu)(1+z_E)^{1/4}\left[\sin\frac{\theta}{2}\right]^{1/2}\lesssim 3\times10^{-5}\left[\sin\frac{\theta}{2}\right]^{1/2}, \quad \Delta l>3t_E \ . \tag{4.18}$$

This is also a bound on small scales $\Delta l<t_E$, since in this case all functions in the integrand are positive, and the range of integration is simply made smaller. The result (4.17) should be regarded as a conservative upper bound—possible cancellations in the integrals of $\delta\rho$ from the "Higgs radiation" have not been taken into account. On the other hand, it is at least plausible that the largest contribution to the Sachs-Wolfe effect comes from the

clumping of the largest loops at t_E.

It is worth repeating that here we have calculated the acceleration term in the Sachs-Wolfe effect (4.4) from loops. It can be checked from the solution (3.18) that the acceleration term in $\delta T_{\rm SW}$ dominates the velocity term $\langle \delta T_{{\rm SW},a}{}^2\rangle / \langle \delta T_{{\rm SW},v}{}^2\rangle \sim 1/(1+z_E)$.

On angular scales considered here other effects are potentially important: infinite strings, statistical fluctuations in the number of loops, and the peculiar velocity of the fluid on last scattering. These are considered next.

V. SACHS-WOLFE EFFECT FROM INFINITE STRINGS

The random walk network of infinite strings gives rise to a nonvanishing contribution to the two-point mass correlation function and hence by Eq. (4.6) to the MBR anisotropies. In this section we calculate the MBR anisotropies due to infinite strings.

As in the case of loops it is important to take into account the underdensities in radiation which compensate the overdensity. As discussed in Ref. 8, the overdensity in strings $\delta\rho_\infty(\mathbf{x})$ in infinity strings can be written as

$$\delta\rho_\infty(\mathbf{x}) = \sum_i \int ds_i \mu \delta(\mathbf{x}-\mathbf{d}_i - l_i(s_i)) . \tag{5.1}$$

The sum runs over the set of infinite strings described by the curves $l_s(s_i)$. $\mathbf{d}_i$ is the position of the string for $s_i = 0$ (see Fig. 1). The probability distribution of $l_i(s_i)$ is given by the Wiener measure.

As explained in Sec. II there will be compensating underdensities in radiation which beginning at the time of the phase transition will spread with the speed of sound. The total energy density perturbation $\delta\rho(\mathbf{x},t)$ will satisfy the constraints (3.8), at least on large scales $\geq L$ (see Sec. III). Hence we write

$$\delta\rho_\infty(\mathbf{x},t) = \sum_i \int ds_i \mu F(\mathbf{x}-\mathbf{d}_i - l_i(s_i)) , \tag{5.2}$$

where $F(\mathbf{x})$ satisfies the constraints (3.8).

The computation of the mass correlation function of infinite strings is described in Appendix B. The result is

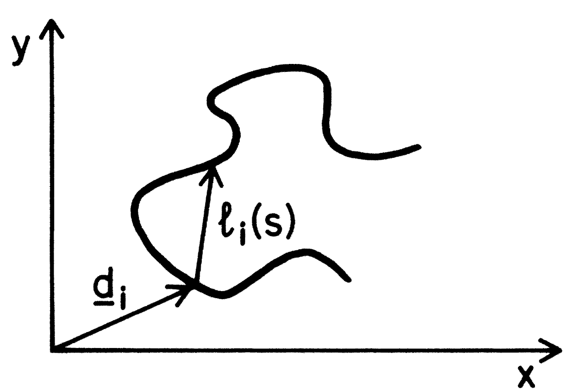

FIG. 1. Parametrization of the random walk.

$$\xi(l,t) = \langle \delta\rho(l,t)\delta\rho(0,t)\rangle \rho_0{}^{-2}(t)$$
$$= \frac{n_H}{8L^3(2\pi)^6}\mu^2 \int d^3z\, f(z)F(z+l)\rho_0{}^{-2}(t) . \tag{5.3}$$

f is constructed from F and n_H is the number of infinite strings per horizon volume. The important point is that both $f(z)$ and $F(z)$ vanish outside the sound cone. Therefore $\xi(l,t)=0$ for $l > 2c_s L(t)$. We obtain the important result that infinite strings do not give rise to long-range correlations, even though their length is infinite. Therefore ξ also satisfies the constraints (3.17).

A simple model gives an estimate of the magnitude of ξ (see Appendix B):

$$\xi(l,t_E) \simeq \left[\frac{\mu G}{4\pi^2}\right]^2 \frac{9n_H}{8L(t_E)^3} t_E{}^4 \frac{1}{4\pi l}, \quad l < c_s L(t_E) ,$$
$$\xi(l,t_E) = 0, \quad l < c_s L(t_E) . \tag{5.4}$$

Finally we use Eq. (4.6) to compute the temperature fluctuations.

On large scales $[\Delta l > L(t_E)]$ the second term in (4.6) vanishes and using the constraints (3.17) we find (with $L=2t$ and $n_H = 1$)

$$\frac{\langle (T_1 - T_2)^{1/2}\rangle}{T} \sim 10^{-1}\frac{G\mu}{4\pi^2} \sim 10^{-8} . \tag{5.5}$$

The contribution from infinite strings is angle independent on large scales and has a smaller amplitude than the Sachs-Wolfe effect from loops.

On small scales $[\Delta l < L(t_E)]$ the second term in Eq. (4.6) dominates. Hence

$$\frac{\langle (T_1 - T_2)^{1/2}\rangle}{T} \sim 10^{-1}\frac{G\mu}{4\pi^2}(1+z_E)^{1/2}\sin\frac{\theta}{2}$$
$$\leq 10^{-8} \text{ for } \theta \leq 3^\circ . \tag{5.6}$$

VI. DOPPLER CONTRIBUTIONS TO THE MICROWAVE BACKGROUND

Peculiar velocities in the fluid which emits radiation on last scattering cause fluctuations in the microwave background radiation:

$$\left[\frac{\delta T}{T}\right]_{\rm Dop} = \mathbf{n}\cdot\delta\mathbf{v}(t_E) , \tag{6.1}$$

where $\mathbf{v}(t_E)$ is the proper velocity on last scattering. The dominant effect is due to intrinsic velocities in the Higgs radiation perturbation; other effects are due to the peculiar velocities of loops and infinite strings, which by momentum conservation induce peculiar velocities in Higgs radiation. Small loops start to accrete matter before decoupling[8] and thus induce velocities gravitationally. However for larger loops, which are relevant to the angular scales considered here, this will be insignificant.

We shall first consider the contribution from intrinsic velocities in the Higgs radiation perturbation from loops. Since $\delta\mathbf{v}(t_E)$ for each loop vanishes outside a sphere of radius $c_s L(t_E) \simeq 2c_s t_E$, the contribution to

$\langle (T_1-T_2)^2\rangle^{1/2}/T$ will be angle independent on large scales. On small scales it varies as $\sin\theta$.

As already pointed out, we do not have a detailed understanding of the dynamics of $\delta\rho_{\rm HR}$. However, we can estimate the amplitude of the effect as follows.

From the continuity equation, the intrinsic velocities in the photon fluid around a single loop are of the order

$$v \sim c_s \frac{\delta\rho_{\rm HR}}{\rho} = c_s \frac{\delta\rho_{\rm HR}}{\rho}\frac{a(t_E)}{a(t_{\rm eq})} \sim c_s{}^{-2}\frac{9\beta}{16}\mu G \frac{R}{t_E}\frac{a(t_E)}{a(t_{\rm eq})}\,, \tag{6.2}$$

where c_s is the speed of sound in the photon fluid and $\rho_{\rm CD}$ is the energy density in cold dark matter, the component which does not couple to the photon fluid. When averaging the contributions from a distribution of loops, the volume factor for a single loop is of the order $4\pi(2c_s t_E)^3/3$. Therefore

$$\int d^3l \langle (\mathbf{v}_1\cdot\mathbf{n}_1 - \mathbf{v}_2\cdot\mathbf{n}_2)^2\rangle \sim \tfrac{1}{3}v^2\frac{4\pi}{3}(2c_s t_E)^3 \tag{6.3}$$

and integrating over all radii R gives

$$\left[\frac{\langle (T_1-T_2)^{1/2}\rangle}{T}\right]_{\rm Dop} \sim (2\pi c_s)^{1/2} v^{1/2}\frac{3\beta}{4}\mu G \frac{a(t_E)}{a(t_{\rm eq})} < 3\times 10^{-6}\,, \tag{6.4}$$

using $v^2=1$, $h=0.5$, $c_s{}^2=\frac{1}{3}$, and $1+z_{\rm eq}=2.5\times 10^4\Omega h^2$. The above gives an upper bound on the effect, since it assumes that the entire Higgs radiation remains in radiative form after $t_{\rm eq}$. The ratio of z factors in Eq. (6.2) is absent if we assume that $\delta\rho_{\rm HR}$ separates into matter and radiation in exactly the same way as the background ρ_0.

Next, we estimate the MBR anisotropies due to the peculiar velocities of loops. A fraction

$$f(t) = \tfrac{9}{16}\beta\mu G \frac{R}{t_E}\frac{a(t_E)}{a(t_{\rm eq})}c_s{}^{-3} \tag{6.5}$$

of the radiation about a loop of radius R is moving with velocity[28,8]

$$v(t) < \left[\frac{R}{t}\right]^{2/3} v_0\,, \tag{6.6}$$

where $v_0 \sim 0.1$ (Ref. 12) is the mean translational velocity of loops at the time of their formation.

For the same reasons as above, the temperature fluctuations induced by this effect will be constant on large scales and vary as $\sin\theta$ on small scales. The amplitude can be estimated by now familiar arguments. There is a volume factor $4\pi 8c_s{}^3 t_E{}^3/3$; averaging over velocities gives a factor $\frac{1}{3}$ and integrating over radii a factor $\frac{3}{7}vt_E$. Thus

$$\frac{\langle (T_1-T_2)^2\rangle^{1/2}}{T} \sim \left[\frac{6\pi}{7}v\right]^{1/2}\frac{3\beta}{4}\mu G v_0 \frac{1+z_{\rm eq}}{1+z_E} < 3\times 10^{-7}\,. \tag{6.7}$$

As expected, this contribution is suppressed by v_0 compared to the contribution from intrinsic sound waves.

The contribution from the peculiar velocities of infinite strings can be estimated in a similar way. A fraction

$$f \le \tfrac{3}{2}\mu G c_s{}^{-2}\frac{a(t_E)}{a(t_{\rm eq})}$$

of the radiation fluid is moving with velocity v_0. The Higgs radiation from any step length of string spreads over a volume $c_s{}^2(2t_E)^3$. There are n_H infinite strings per volume element $(2t_E)^{-3}$. Hence with $n_H \sim 1$

$$\frac{\langle (T_1-T_2)^2\rangle^{1/2}}{T} \sim n_H{}^{1/2}\tfrac{3}{2}v\mu G v_0 \sim 10^{-7}\,.$$

The angular dependence is as for the other two velocity effects.

In Ref. 8 we determined the MBR anisotropies due to gravitational lensing by cosmic strings, an effect which was first discussed by Kaiser and Stebbins.[29] The contribution had a similar magnitude to Eq. (6.4).

We conclude that all Doppler contributions to the microwave background anisotropies are at least 1 order of magnitude below current observational limits on scales ($>0.5°$) for which our analysis is reasonable.

VII. PERTURBATIONS OF THE LAST SCATTERING SURFACE

Energy-density perturbations on the last scattering surface lead to fluctuations in the emission temperature of the microwave background radiation:

$$\left.\frac{\delta T}{T}\right|_{\text{last scattering}} = \frac{1}{4}\frac{\delta\rho_{\rm rad}}{\rho_{\rm rad}}(\mathbf{x}_E, t_E) \tag{7.1}$$

[see Eq. (4.1)].

The mean fluctuation in radiation energy between two points z and $-z$ on the last scattering surface is

$$\begin{aligned}\left\langle \frac{[\delta\rho_{\rm rad}(z)-\delta\rho_{\rm rad}(-z)]^2}{\rho^2}\right\rangle &= \frac{8\pi}{(2\pi)^6}\int_0^\infty dk\,k^2|\delta_k|^2\int_{-1}^{1} d\cos\theta\,\sin^2(kz-\cos\theta)\\ &= \frac{8\pi}{(2\pi)^6}\left[\frac{2}{3}\int_0^{z^{-1}} dk\,k^4|\delta_k|^2 z^2 + \frac{1}{2}\int_{z^{-1}}^\infty dk\,k^2|\delta_k|^2\right]\,,\end{aligned} \tag{7.2}$$

where $|\delta_k|^2$ is defined by

$$\left\langle \frac{\delta\tilde\rho_{\rm rad}(\mathbf{k})\delta\tilde\rho_{\rm rad}(\mathbf{k}')}{\rho^2}\right\rangle = \delta^3(\mathbf{k}+\mathbf{k}')|\delta_k|^2\,. \tag{7.3}$$

On large scales ($k < t_E{}^{-1}$) the correlations $|\delta_k|^2$ in radiation are roughly equal to those in strings. One way to see this is by the constraints (3.7). As shown in Ref. 17, the total two-point momentum-space correlation function decays as k in the limit $k \to 0$. The correlation function from infinite strings however is much larger[8]:

$$|\delta_k|_{\text{infinite strings}}^2 \simeq (6\pi\mu G)^2 k^{-2} t_E \ . \tag{7.4}$$

Hence the compensating fluctuations in radiation energy density must be of equal magnitude. We can see this explicitly by using a simple model for the Higgs radiation energy density perturbation

$$\delta\rho_{\text{HR}} = \sum_i \int ds_i d^2 z_i A e^{-z^2/L^2} \times \delta^3(\mathbf{x} - \mathbf{d}_i - l_i(s_i) - z) \ , \tag{7.5}$$

where z_i are the coordinates in the plane normal to the string at point s_i. The amplitude A is determined by the constraints (3.7):

$$A = \mu(\pi L^2)^{-1} \ . \tag{7.6}$$

Then, following the methods of Appendix B we immediately obtain

$$|\delta_k|^2 \simeq (6\pi\mu G)^2 k^{-2} t_E e^{-(kL)^2/4} \ . \tag{7.7}$$

On small scales ($k > t_E^{-1}$) the correlation function is exponentially suppressed. This conclusion holds also for string loops, as shown in Ref. 8. Thus, by Eq. (7.2)

$$\left\langle \left[\frac{\delta\rho_{\text{rad}}(z) - \delta\rho_{\text{rad}}(-z)}{\rho_{\text{rad}}} \right]^2 \right\rangle^{1/2} \sim \frac{\mu G}{2\pi^2} \frac{1+z_{\text{eq}}}{1+z_E} \frac{z}{t_E} \sim 2 \times 10^{-7} \frac{z}{t_E} \ . \tag{7.8}$$

On large scales

$$\left\langle \left[\frac{\delta\rho_{\text{rad}}(z) - \delta\rho_{\text{rad}}(-z)}{\rho_{\text{rad}}} \right]^2 \right\rangle^{1/2} \sim \frac{3}{4\pi^2} \mu G \frac{1+z_{\text{eq}}}{1+z_E} \sim 3 \times 10^{-7} \ . \tag{7.9}$$

The effect is thus clearly subdominant.

VIII. CONCLUSIONS

We calculated the anisotropies in the cosmic-string theory of galaxy formation using a general-relativistic analysis for all angular scales for which the details of the last scattering surface can be neglected (neglecting reionization this minimal angle is 0.5°). Using the value $\mu G \sim 10^{-6}$ obtained by independent astrophysical considerations[4] we predict small scale anisotropies ($\theta \simeq 0.5°$)

$$\frac{\langle (T_1 - T_2)^{1/2} \rangle}{T} \simeq 2 \times 10^{-6} \tag{8.1}$$

and large scale anisotropies ($\theta \simeq 6°$)

$$\frac{\langle (T_1 - T_2)^{1/2} \rangle}{T} \simeq 4 \times 10^{-6} \ , \tag{8.2}$$

in both cases at least an order of magnitude below the best current observational bounds. We assume cold dark matter, $\Omega = 1$ and $h = 0.5$. These dominant terms come from the Sachs-Wolfe terms (4.8), (4.10), and (4.18).

We considered the Sachs-Wolfe effect due to string loops and infinite strings and estimated the effects due to peculiar velocities of the last scattering surface and of energy density fluctuations on last scattering. On all angular scales considered here, the Sachs-Wolfe effect gives the dominant contribution.

The precise numerical values of our predictions should be considered as upper bounds rather than as exact values. Our ignorance of the dynamics of Higgs radiation is the main reason we were not able to calculate all the effects exactly.

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APPENDIX A: MASS CORRELATIONS OF CASUAL PERTURBATIONS

Let

$$I(l) = \int d^3z \, d^3u \frac{\delta\rho_1}{\rho}(\mathbf{u}) \frac{\delta\rho_1}{\rho}(\mathbf{z}) C(\mathbf{u} + \mathbf{z} - l) \ , \tag{A1}$$

$$C(l) = \frac{k}{l} \theta(l - L) \ . \tag{A2}$$

[See Eqs. (3.15c) and (2.5).] Now since $\delta\rho_1$ satisfies the integral constraints (3.8), we can use a lemma[16] to rewrite $\delta\rho_1$ in a convenient form. This will enable us to separate the correlation function $I(l)$ into a long-range and short-range piece. Specifically, $\delta\rho_1$ can be written as

$$\delta\rho_1(l) = \nabla^2 f + \frac{\partial f}{\partial l} \delta(l - L) \ , \tag{A3}$$

where $f(l) = 0$ for $l > L$ and

$$\int d\Omega \left. \frac{\partial f}{\partial l} \right|_{l=L} = 0 \ , \qquad \int d\Omega \left. \frac{\partial f}{\partial l} \right|_{l=L} Y_{1m}(\Omega) = 0 \ , \tag{A4}$$

for some function $f(l)$.

Substitute (A2) and (A3) into (A1), and integrate by parts. One finds

$$I(l) = kL^3 \left[\frac{4\pi}{5} Q \frac{L^3}{l^3} \theta(l - 2L) + S(l) \right] \ , \tag{A5}$$

where

$$Q=\int d\Omega_v \int d\Omega_z \frac{\partial f}{\partial z}\bigg|_{z=L} \frac{\partial f}{\partial v}\bigg|_{v=L} Y^*_{2m}(\Omega')Y_{2m}(\Omega_z)\left[1+O\left[\frac{L}{l}\right]\right], \quad \hat{\Omega}'=(l+L\hat{\Omega}_v)\,|\,l+L\hat{\Omega}_v\,|^{-1}, \tag{A6}$$

and

$$S(l)=\frac{1}{L^2}\int d^3z\,\nabla^2 f(z)\int d\Omega_v \mathbf{n}\cdot\left[\nabla f-\frac{f\mathbf{n}}{L}\right]_{|z-l+v|=L}$$
$$+\frac{1}{L^2}\int d\Omega_v \frac{\partial f}{\partial v}\bigg|_{v=L}\int d\Omega_z \mathbf{n}\cdot\left[\nabla f-\frac{f\mathbf{n}}{L}\right]_{|z-l+\hat{\Omega}_v L|=L}. \tag{A7}$$

Note that $S(l)=0$ for $l>L$.

If $\delta\rho_1$ is spherical then (A4) implies that Q is zero. However, we expect that $\delta\rho_{\rm HR}$ will not be spherical. To bound the magnitude of Q, we compute Q for a very nonspherical example. Let $\delta\rho_{\rm loop}$ be approximated by a Gaussian equation (3.11) and $\delta\rho_{\rm HR}$ by a box with sides $D(t)$ and height $h(t) \ll D$:

$$\frac{\delta\rho_1}{\rho}=\frac{\delta\rho}{\rho}\bigg|_{\rm loop} - 2\beta\pi G\mu\frac{R}{3h}\theta(h-z)\theta(D-x)$$
$$\times\theta(D-y)\theta(x)\theta(y)\theta(z)\,. \tag{A8}$$

With this model one can calculate $f(l)\simeq\nabla^{-2}\delta\rho_1/\rho$ and hence estimate

$$\frac{\partial f}{\partial l}\simeq-\beta G\mu R\frac{D^4}{l^4}\sin^2\theta\sin\phi\cos\phi\ .$$

[(A4) implies that the monopole and dipole terms in $\partial f/\partial l$ vanish.] (A6) then implies $|Q|\lesssim(\beta G\mu R)^2$.

The bound we use for S is more subtle: we find the contribution to $\delta T/T$ from S is dominated by the smallest loops. It is unrealistic to spread $\delta\rho_{\rm HR}$ from these as far as $L(t_E)$. Instead we assume it spreads as far as the sound cone $D<L_S\sim L(t_{\rm eq})(t_E/t_{\rm eq})^{2/3}$ (the velocity of sound dropping rapidly after $t_{\rm eq}$). This gives the same magnitude for S but says S should vanish outside the sound cone; $S(l)=0$, $l>L_S$.

APPENDIX B: MASS CORRELATION FUNCTION FROM INFINITE STRINGS

In this appendix we derive Eq. (5.3) for the mass correlation function from infinite strings and estimate its magnitude using a simple model for the Higgs radiation underdensity.

The starting point is Eq. (5.2). Since F satisfies the constraints (3.8) and vanishes outside the sound cone we can as in Appendix A use the theorem of Ref. 16 and write F as

$$F(x)=\nabla^2 f(\mathbf{x})\,. \tag{B1}$$

$f(\mathbf{x})$ vanishes outside for forward sound cone of $x=0$. In dropping the boundary term [see Eq. (A3)] we made the simplifying assumption that the underdensity from each point on the string expands in a radially symmetric way. Asymmetries can be included following the methods of Sec. IV.

We compute the energy density correlation function $\xi(\mathbf{x}-\mathbf{y})$ using the methods of Ref. 8. In order to easily take the expectation value with respect to the Wiener measure we write $f(\mathbf{x})$ in terms of its Fourier transform $\tilde{f}(k)$. Then

$$\langle\delta\rho(\mathbf{x},t)\delta\rho(\mathbf{y},t)\rangle=\sum_{i,j}\int ds_i ds_j \mu^2\nabla_x{}^2\nabla_y{}^2$$
$$\times\int d^3\mathbf{k}\,d^3\mathbf{k}' e^{i\mathbf{k}\cdot\mathbf{x}+i\mathbf{k}'\cdot\mathbf{y}}\langle e^{-i\mathbf{k}\cdot\mathbf{d}_i-i\mathbf{k}'\cdot\mathbf{d}_j-i\mathbf{k}\cdot\mathbf{l}_i(s_i)-i\mathbf{k}'\cdot\mathbf{l}_j(s_j)}\rangle\tilde{f}(\mathbf{k})\tilde{f}(\mathbf{k}')$$
$$=\frac{N}{V}\int ds\,ds'\mu^2\nabla_x{}^2\nabla_y{}^2\int d^3\mathbf{k}\,e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}|\tilde{f}(\mathbf{k})|^2\langle e^{-i\mathbf{k}\cdot[l(s)-l(s')]}\rangle_W\,, \tag{B2}$$

where N is the number of infinite strings in the cutoff volume V. The subscript W indicates that the remaining expectation value is with respect to Wiener measure.

If L is the correlation length of the Brownian walk, then

$$\langle e^{-i\mathbf{k}\cdot[l(s)-l(s')]}\rangle_W=e^{-Lk^2|s-s'|}\ . \tag{B3}$$

By radial symmetry the angular part of the $\mathbf{k}$ integration is trivial. We obtain

$$\langle\delta\rho(\mathbf{x},t)\delta\rho(\mathbf{y},t)\rangle=\frac{Nl_s}{V}\frac{\mu^2}{L}\nabla_x{}^2\nabla_y{}^2\frac{1}{\tau}$$
$$\times\int_0^\infty dk\,k^{-1}|\tilde{f}(\mathbf{k})|^2\sin kl\ , \tag{B4}$$

where $l=\mathbf{x}-\mathbf{y}$ and l_s is the total length of one string. There are n_H infinite strings which cross any horizon volume (n_H is of the order 1). Hence the prefactor Nl_s/V, the total length in string per volume, is $n_H L^{-2}$. Using

$$\frac{1}{l}\int_0^\infty dk\, k^{-1}\sin kl\, |\tilde{f}(\mathbf{k})|^2 = (2\pi)^{-6}\frac{\pi}{2}\int d^3z\, d^3z' f(z) f(z')\frac{1}{|z-z'+l|} \tag{B5}$$

in (B4) and noting that ∇^2 acting on $|z-z'+l|^{-1}$ yields a δ function we obtain Eq. (5.3).

In order to evaluate the magnitude of $\xi(\mathbf{r},t)$ we shall assume a simple model for $F(\mathbf{x},t)$. We shall assume that the radiation underdensity is uniform inside the sound cone with radius $c_s L(t)$:

$$F(x)=\delta(x)-a\theta(c_s L(t)-|x|) \tag{B6}$$

with

$$a(t)=\frac{3}{4\pi c_s{}^3 L^3(t)} . \tag{B7}$$

Taking the inverse Laplacian with the boundary condition that F vanish at infinity we obtain

$$f(l)=-\frac{1}{4\pi l}\left[1-\frac{3}{2}\frac{l}{c_s L}+\frac{1}{2}\left[\frac{l}{c_s L}\right]^3\right],\quad l<c_s L ,$$
$$f(l)=0,\quad l>c_s L . \tag{B8}$$

Thus, using the expression for ρ_0 in the matter-dominated era, we get

$$\xi(l,t_E)=\left[\frac{\mu G}{4\pi^2}\right]^2\frac{9n_H}{8L^3(t)}t_E{}^4 \times\left[f(l)-a\int d^3z\, f(z)\theta(c_s L-|z+l|)\right] . \tag{B9}$$

The first term dominates for small l; for $l\sim c_s L$ both are of the same order of magnitude. To a good approximation we can thus work with Eq. (5.4).

APPENDIX C: FORMATION OF STRUCTURES

In this appendix we outline why the small rms value $|\delta_k|^2\sim 10^{-6}$, which leads to the small values for the rms microwave anisotropies computed in this paper, is consistent with formation of galaxies and clusters of galaxies.

In models with linear adiabatic perturbations the rms density fluctuations at $t_{\rm eq}$ must be of the order 10^{-4}, in order to be able to grow to order unity at $z\sim 1$. In models in which structures form by accretion about seed masses the situation is different. The requirement is that the density contrast in the region which collapses to a galaxy (cluster of galaxies) is of the order 10^{-4} at $t_{\rm eq}$. If the radius $r_i(t_{\rm eq})$ of this region is smaller than the mean separation $d(t_{\rm eq})$ of the structures, then the rms density contrast can be substantially smaller.

With $r_i(t_{\rm eq})=at_{\rm eq}$ and $d(t_0)=b$ Mpc we obtain

$$\frac{r_i(t_{\rm eq})}{d(t_{\rm eq})}=12.5\frac{a}{b}h^{-2} . \tag{C1}$$

For clusters of galaxies $a=7\times10^{-1}h$ (from Ref. 4) and $b=55h^{-1}$,

$$\frac{r_i(t_{\rm eq})}{d(t_{\rm eq})}\simeq 0.16 ; \tag{C2}$$

for galaxies we find

$$\frac{r_i(t_{\rm eq})}{d(t_{\rm eq})}\simeq 0.15 . \tag{C3}$$

These values yields $\delta\rho/\rho\sim 10^{-4}$ within the regions which eventually collapse.

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