Studies of the hydrodynamic evolution of matter produced in fluctuations in $\bar{p}p$ collisions and in ultrarelativistic nuclear collisions

H. von Gersdorff and Larry McLerran
Fermi National Accelerator Laboratory, Batavia, Illinois 60510

M. Kataja and P. V. Ruuskanen
Physics Department, University of Jyväskylä, Seminaarinkatu 15, SF-40100 Jyväskylä, Finland
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In this first paper of a series of two, we present a comprehensive study of the hydrodynamic evolution of matter produced in the central region of ultrarelativistic heavy-ion collisions and in high-multiplicity fluctuations of $\bar{p}p$ collisions. We shall begin with a discussion of the limits of the applicability of a perfect-fluid hydrodynamic description of high-energy collisions. A simple bag-model equation of state is argued to have qualitative and semiquantitative features expected from lattice gauge theory and present theoretical understanding. We also discuss the boundary conditions for the perfect-fluid hydrodynamic equations, and what classes of simple events would correspond to simple initial conditions. The decoupling of matter at low energy density and methods for computing transverse-momentum distributions of hadrons are analyzed. We finally present the details of the computer code which we use to numerically solve the hydrodynamic equations.

I. INTRODUCTION

Many theoretical studies have shown that energy densities are achieved in ultrarelativistic nuclear collisions and in fluctuations in $\bar{p}p$ collisions which may allow for the production of a quark-gluon plasma.[1-6] If the time scale which characterizes the expansion of such matter is long enough, thermodynamic parameters may be used to meaningfully characterize the system and the matter may expand reversibly with little entropy production according to the equations of perfect-fluid hydrodynamics. In such circumstances, the equations which describe this evolution depend only on the condition of the matter at some fixed time, and upon the equation of state which relates energy density and pressure. The boundary conditions may be chosen either as initial conditions, if there is a reliable theoretical description of the initial conditions, or as the final configuration at very late times when the matter freezes out and subsequently evolves free streaming into particle detectors. In the latter case, experimental data provides much of the information needed to solve the hydrodynamic equations, since parameters in the hydrodynamic simulation must be adjusted to produce the observed particle multiplicities.

The equation of state of hadronic matter may be computed in principle in Monte Carlo numerical simulations.[7-16] At present, such computations provide little more than qualitative and semiquantitative insight. It is fairly well established that hadronic matter makes a rapid transition between matter with the few degrees of freedom associated with a hadronic gas to matter associated with the large number of degrees of freedom of a quark-gluon plasma.[12-16] The exact nature of this transition is uncertain, in particular whether it is a first-order phase transition, but the rapid change of physical quantities such as the entropy density, by an order of magnitude in a narrow temperature interval of order tens of MeV's, is almost uncontested.[17-19] The value of the temperature at which this transition occurs is not well established, but theoretical speculation centers on 200 MeV, although the temperature may be as high as 400 MeV or as low as 100 MeV and still be within the intrinsic uncertainties of present numerical computations.[20] An equation of state with all of the properties needed to adequately describe a rapid transition between an ideal pion gas, a description valid at very low temperatures, and that of an ideal quark-gluon plasma, valid for very high temperatures, is provided by the MIT bag model. In such a model, the transition between these ideal gases is discontinuous as a function of temperature. The transition temperature may be tuned by varying the bag constant, which for our purposes will be considered to be an adjustable parameter. For the types of computations which we wish to perform, treating the transition as discontinuous or as smooth will provide only small corrections, since we shall only be concerned with gross semiquantitative and qualitative features of the matter as it evolves after production. With better knowledge of the equation of state, and estimates of viscous coefficients, a precise quantitative comparison between theory and experiment should eventually be possible.

What we are doing more precisely is approximating the transition between a quark-gluon plasma, which is an ideal gas at high energy density, and a hadronic gas, which is an ideal gas of pions at low density, as a discontinuous sudden change between these two ideal gasses. For example, the entropy scaled by T^3 goes to a constant at high density which is the number of degrees of freedom of a quark-gluon plasma. At temperatures low compared to the deconfinement temperature, but large enough so

that the massless pion gas approximation is valid, S/T^3 goes to another constant. The ratio of these constants is the ratio of degrees of freedom of a quark-gluon plasma to those of a pion gas, which is large. In general we expect a gradual transition between these two limiting cases with a possible discontinuity at the deconfinement temperature. Monte Carlo studies indicate that the change in degrees of freedom between the pion gas and the quark-gluon plasma is rapid, and happens in a fairly narrow temperature range. Our approximation makes the change in S/T^3 discontinuous, with the entire change occurring at the deconfinement temperature. This approximation is shown in Fig. 1. Without better Monte Carlo data than exists at present, it is difficult to have a precise quantitative assessment of the reliability of our approximation.

We have set many goals in this study of the evolution of matter produced in high-energy hadronic collisions. We would first like to understand the qualitative and semiquantitative features of the matter as it evolves after production. For example, how much time does the system spend as a quark-gluon plasma? How much time as a mixed phase? How much time as hadronic matter before the matter decouples? How does this depend upon the initial conditions and the baryon number of the colliding nuclei? Do shock waves form in the matter, and if they do, how much entropy is produced? Is much collective transverse flow generated by the expansion of the matter, and how is this reflected in the transverse momentum of hadrons? These qualitative and quantitative features of the matter once understood may generate enough insight into the nature of these collision processes to suggest new signals and more refined computations of physical observables.

In the preliminary and modest study which we present here, such qualitative and semiquantitative features of fluctuation in $\bar{p}p$ collisions and head-on nucleus-nucleus collisions are studied. We present full three-space-dimensional simulations of such collisions allowing for a realistic equation of state with a mixed phase and phase transition. The principle difference between the results here and those of previous workers, with the exception of Pratt who considered in detail spherical expansion is that we allow for a mixed phase.[21–26] (In Pratt's analysis, some attempt was made to treat the Lorentz-invariant cylindrical expansion characteristic of the central region of ultrarelativistic nuclear collisions. We differ from Pratt in that we allow shock discontinuities to propagate through and rarefy the mixed phase with arbitrary entropy change. We in fact find that the favored situation is maximal entropy change across the shock discontinuity.) If there is not a first-order phase transition, this mixed phase is simply the region where the energy density varies quickly but the pressure does not. We find that it is essential to include such a mixed phase since the system spends much of its time in this phase. The existence of such a mixed phase, without extreme supercooling assumes that the nucleation rate of the hadronic matter from the quark-gluon plasma is fast compared to the expansion rate.

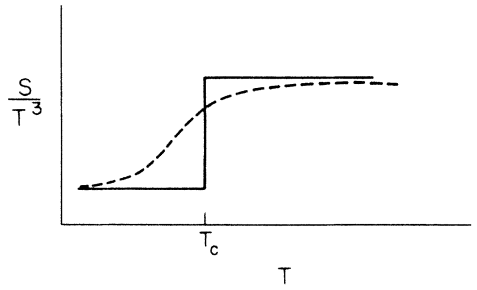

FIG. 1. The entropy per T^3 vs T. The unbroken line represents our approximation, the dashed curve is a guess for a realistic relationship.

Our results are encouraging. With a modest amount of computing time, such collisions may be studied for a variety of assumed initial conditions and parameters which characterize the equation of state. It is easy to imagine that more detailed computations which treat nonzero impact-parameter collisions of nuclei with various baryon numbers A may be carried out without too much increased effort. Entropy generation by viscosity may be included. The fragmentation region might be studied. Various physical quantities such as flavor ratios, photon and dilepton distributions, particle transverse-momentum distributions, Hanbury Brown–Twiss correlations, and collective variables such as flow and thrust might be determined. Once this ambitious program has been carried out, an event generator with a few adjustable parameters may be used for fluctuations in $\bar{p}p$ collisions and ultrarelativistic nuclear collisions with hopefully the same reliability as Monte Carlo simulations of jet processes which are used in jet experiments at the $\bar{p}p$ collider.

The first step in this ambitious program is the hydrodynamic simulation which we present here. Our results should be adequate to describe the production of pions and nucleons in the central region for impact-parameter zero collisions of equal-A nuclei at ultrarelativistic energies (boost invariant cylindrical geometry),[4] and high-multiplicity spherical or uniform rapidity fluctuations in $\bar{p}p$ collisions. Details of our computations are sensitive to unknown features of these collisions such as the dependence of the central region multiplicity upon the baryon number A, and the time at which the matter first begins to flow as an almost perfect fluid, τ_i. Many qualitative and semiquantitative features such as the expansion time and the average transverse momentum of hadrons are not so sensitive to these uncertainties, and may provide signals for the production of a quark-gluon plasma.

The outline of this series of two papers is the following. In the first paper we shall discuss in detail the hydrodynamic description of hadronic collision processes as well as its limitations. In the first section we shall review the hydrodynamics of ultrarelativistic nuclear collisions and fluctuations in $\bar{p}p$ collisions. We shall discuss the limits to the validity of a perfect-fluid hydrodynamic description. In the second section we discuss in detail the boundary conditions for the hydrodynamic collisions, and also how experimental data may be used to infer some features of these conditions. In the third section we dis-

cuss the general features of simple events which may be analyzed by the techniques which we present. The decoupling of hadronic matter is discussed in the fourth section, and methods for extracting the transverse-momentum distributions of hadrons are reviewed. We finally discuss the detailed features of our computer code which simulates these collisions in the fifth section.

In the second paper we present the results of our computations. We discuss the qualitative features of our results such as the time the matter spends in various phases, and the dependence of transverse momentum upon the equation of state. We further explore A dependences of the transverse momentum for nuclear collisions, and the dependence of transverse-momentum distributions upon the mass of the emitted particle.

II. THE HYDRODYNAMIC EQUATIONS AND THEIR APPROXIMATE VALIDITY

In this section we shall discuss the perfect-fluid hydrodynamic equations. We shall begin with a general discussion which does not make reference to the specific initial conditions peculiar to high-energy collision processes. We begin with the equation for conservation of energy-momentum:

$$\partial_\mu T^{\mu\nu}=0 \tag{1}$$

an equation which is always true. There may also be equations for conservation of various currents such as baryon number:

$$\partial_\mu J^\mu=0 . \tag{2}$$

This latter equation will be irrelevant for our later studies which revolve around processes in the central region. In this kinematic domain, the total baryon number (baryons minus antibaryons) is small at very high energies. The smallness is demonstrated by showing that the energy per unit baryon number is large compared to the temperature, so that thermal excitations dominate the contributions to the stress-energy tensor. In the remainder of this paper we shall study processes at zero baryon number density, and may therefore ignore the added complications arising from Eq. (2). Such currents must of course be taken into account when the fragmentation region is studied.

If in addition to conservation of energy-momentum, we require that the expansion of the matter takes place slowly compared to natural collision times, that is slow enough that the expansion be irreversible, then the entropy current is also conserved[27,28]

$$\partial_\mu s^\mu=0 . \tag{3}$$

In this circumstance, it may be shown that the stress-energy tensor must have the form

$$T^{\mu\nu}=(\epsilon+P)u^\mu u^\nu+Pg^{\mu\nu} , \tag{4}$$

where ϵ is the energy density and P is the pressure measure in a frame comoving with the fluid. The fluid four-velocity vector is u which satisfies the constraint

$$u^2=-1 . \tag{5}$$

The conservation of energy-momentum plus an equation of state which relates ϵ to P

$$P=P(\epsilon) \tag{6}$$

are sufficient to determine P, ϵ, and u from condition specified at some arbitrary time. Conservation of entropy follows from conservation of $T^{\mu\nu}$, together with standard thermodynamic relation for the entropy. Note that

$$s^\mu=\sigma u^\mu , \tag{7}$$

where σ is the entropy measured in a comoving frame.

If we can argue that the systems we consider are described by perfect-fluid hydrodynamics, then the computation of properties of the matter produced in ultrarelativistic nuclear collisions or in high-multiplicity hadron collisions are determined only by conditions measured at some fixed time and on the equation of state for matter. In the next section we shall discuss the boundary conditions. As discussed above, the general features of the equation of state which relates ϵ and P are also known.

We now turn to the question of the validity of the adiabatic, or isentropic fluid flow assumption. The criteria that the fluid flow be isentropic is simply that the collision times be fast compared to the expansion time. If the expansion is not adiabatic, a description of the fluid flow becomes considerably more difficult. New parameters enter the hydrodynamic equations, the coefficients of shear and bulk viscosity, and the form of the equations are more involved. The viscous coefficients are difficult to estimate in QCD, but we shall soon review what is known of them. If these viscous corrections to the hydrodynamic equations are sufficiently large, then the approximation which reduces the kinetic equations to local equations with the standard form of the viscous corrections to the perfect-fluid hydrodynamic equations may itself break down, and the correct hydrodynamic equations may involve many more parameters. The point is that for our purposes, the viscous hydrodynamic equations are only reliable if the corrections arising from nonzero viscosity are small.

Another reason besides mathematical simplicity for wishing to apply hydrodynamics only for perfect fluids is that for a perfect fluid, the entropy is conserved. Entropy conservation relates particle multiplicities at early times to that at later times. If the expansion is isentropic, a window penetrates through the haze of hadronic interactions which allows us to reconstruct primeval particle distributions from those observed in the final state of the collision.

The stress-energy tensor, allowing for the effects of viscous flow is

$$T^{\mu\nu}=T_0^{\mu\nu}+\Delta T^{\mu\nu} , \tag{8}$$

where T_0 is the stress-energy tensor for a perfect fluid, as given by Eq. (1), and ΔT is the correction which allows for entropy production, that is, for viscous flow. By allowing the energy-momentum tensor to be piecewise continuous, these perfect-fluid hydrodynamic equations do allow for some entropy production through the medium of shock discontinuities. The most general form for ΔT may

be extracted in an expansion in powers of gradients of the energy density and fluid velocity vector times a characteristic scattering length. This characteristic scattering length is the mean free path for dilute systems such as gases. This procedure for evaluating ΔT is discussed in Refs. 29 and 30, and we shall not repeat the derivation here. The result is

$$\Delta T^{\mu\nu}=\zeta(g^{\mu\nu}-u^{\mu}u^{\nu})\nabla\cdot u + \eta[\nabla^{\mu}u^{\nu}+\nabla^{\nu}u^{\mu}-\tfrac{2}{3}(g^{\mu\nu}-u^{\mu}u^{\nu})\nabla\cdot u] . \quad (9)$$

The derivative operator ∇ is a derivative orthogonal to the direction of fluid flow

$$\nabla^{\mu}=\partial^{\mu}-u^{\mu}u\cdot\partial . \quad (10)$$

The coefficients of shear and bulk viscosity are η and ζ. For the zero baryon number density fluids which we consider, the heat conductivity is zero. This expression is only valid to first order in an expansion where spatial gradients are weak, and if they are not, Eq. (9) is simply incorrect. For systems with sharp discontinuities, ΔT is more complicated and for practical purposes may not be computable, except in certain approximations where the sharp variations in the energy density and fluid velocity are approximated as shock discontinuities, that is, the fluid is treated as piecewise slowly varying. Put another way, when viscous corrections to the hydrodynamic equations become of the same order as the contribution associated with a perfect fluid, the framework of conventional viscous fluid hydrodynamics falls apart, and for practical purposes, we may say that hydrodynamics is no longer applicable for a description of the dynamics. This means only that perfect-fluid dynamics is inapplicable even when supplemented with viscous corrections. The full stress-energy tensor is of course conserved, but the form of this equation expressed in terms of ϵ, P, and u is extremely complicated and in general nonlocal.

The question which we shall attempt to address in the remainder of this section is to what degree a perfect-fluid hydrodynamical description provides a valid approximate description of the matter evolving after a hadronic collision. To begin this discussion it is useful to introduce a mean free path for quarks and gluons in hadronic matter. This length scale characterizes the surface thickness of the matter, and the length scale which must be compared to the length scale of gradients in the matter distribution. If the surface thickness is small compared to the spatial size of the system, and if the mean free path is short compared to the scale sizes over which the matter distribution varies appreciably, then it is plausible that the perfect-fluid hydrodynamic description is correct. Of course, it is possible that the naive considerations of mean free paths, which are rigorously valid for weakly interacting fluids, may be misleading when applied to hadronic matter where nonperturbative effects may be important. Later we shall therefore more carefully formulate the issue of the applicability of perfect-fluid hydrodynamics in terms of magnitudes of viscous coefficients. These coefficients may in principle be computed using the fluctuation-dissipation theorem and are defined outside the domain of weak-coupling expansions.

The simplest estimate of the mean free paths uses the quark-parton additive cross-section model of hadronic interactions. The basic assumption of this extremely naive picture is that the quark-hadron cross section is $\frac{1}{3}$ that of hadron-hadron:

$$\sigma_{qh}\sim\tfrac{1}{3}\sigma_{hh}\sim 13 \text{ mb} . \quad (11)$$

This cross section will be treated as a constant and independent of the energy density of the matter through which the quark propagates. This assumption is in contradiction with the properties of quark interactions at very high energy densities when perturbative QCD may be used. We are assuming that the energy densities are sufficiently low that the effects of the matter do not significantly alter the basic two-body quark interactions. We shall soon present perturbative QCD estimates.

The mean free path is

$$\lambda_{\text{mfp}}\sim 1/\sigma n , \quad (12)$$

where n is the number density of hadrons. At ordinary nuclear-matter energy densities, $\lambda_{\text{mfp}}\sim 5$ fm. Assuming the energy density scales with T^4 as it would be either an ideal gas of pions or a quark-gluon plasma, then $n\sim\epsilon^{3/4}$. The mean free path is therefore

$$\lambda\sim 0.5 \text{ fm}, \quad \epsilon\sim 1\text{–}2 \text{ GeV/fm}^3 , \quad (13)$$

$$\lambda\sim 0.01 \text{ fm}, \quad \epsilon\sim 200 \text{ GeV/fm}^3 . \quad (14)$$

For either of these two energy densities, the mean free path is extremely small compared to a typical nuclear radius, and effects of transverse surface area are quite small for nuclei of reasonable size. In the last case, even for protons, the surface effects would be small. Also as we shall soon see, in the expansion of matter produced in high-energy hadronic collisions, the densities typically scale as a power of time as measured in the local comoving frame. The expansion rate, $(1/n)dn/d\tau$, where n is some typical density such as energy density or entropy density is therefore of the order of the inverse proper time τ after the collision took place. For times which are therefore larger than λ, viscous corrections are small. Depending upon the initial energy density, such times may be quite short.

The additive quark-parton model can be improved for thermal systems by requiring that particle interactions be screened for momentum transfers less than the temperature. If we approximate the differential cross section as $d\sigma/dt\sim\sigma_0 e^{-t/\kappa}$ where $\kappa\sim 400$ MeV, the mean free path is still 0.5 fm at $\epsilon\sim 1\text{–}2$ GeV/fm^3 but is lengthened to about 0.05 fm at $\epsilon\sim 200$ GeV/fm^3.

At very late times, the matter density eventually becomes so low that the mean free path becomes very large. When the mean free path becomes so large that in the entire future history of the system, a particle may be expected to interact on the average less than once, we shall assume that this particle freezes out. At this time the local densities of particles follow the trajectories of free particles and the hydrodynamic description ends. This freezeout may not occur at a sharply defined time, and the matter may therefore propagate with mean free paths comparable to expansion times for some time. If this is

the case, at this stage of the expansion, viscous effects must play a major role. We shall use in explicit simulations, a freezeout temperature $T \sim 50-150$ MeV. Such low temperatures appear to be consistent with the slow rate of expansion which we observe at late times in the explicit hydrodynamic simulations. The main limitation seems to be that the system freezes out when the mean free path becomes of the order of the transverse size of the system which we consider. In a massless pion gas, the mean free path is 10 fm when the temperature is about 100 MeV, and we would probably expect freezeout by this temperature. To be more precise requires a cascade computation which we have not performed. At such late times and low temperatures, the matter is not doing much work and a sloppy treatment of freezeout should not alter the predicted final-state distribution of the matter. The number of pions is also expected to be conserved at temperatures much higher than this, and the multiplicity distributions are not much altered. Put another way, at such low temperatures, interactions are so weak that multiplicity and momentum distributions are not expected to be altered much in the future of the system. We shall check this assumption in our later computations presented in the second paper of this series.

These additive quark-model estimates must surely be modified for high energy densities where perturbative QCD adequately describes the dynamics. At these high energy densities, the quark and gluon cross sections become small, and approach zero as $\sigma \sim \alpha_s^2/q^2$, where α_s is the QCD interaction strength and q is some typical energy scale, $q \sim T$. The mean free path is

$$\lambda \sim 1/(\alpha_s^2 T) . \tag{15}$$

At large temperatures, $\alpha_s \sim 1/\ln(T)$ and $\lambda \sim \ln^2 T/T$.

Two groups have independently computed the mean free paths of quarks and gluons in a quark-gluon plasma, along the lines previously advocated by Shuryak.[31] These different computations differ in the way that small angle scatterings are treated, where high-order perturbative, and possibly nonperturbative, corrections are required. Also, the value of the strong-interaction coupling constant which is used in this evaluation is somewhat ambiguous since it is not precisely clear at what momentum scale the coupling constant is to be evaluated, that is, should the momentum scale be T or $10T$. Finally, at the temperatures for which we shall apply their results, the effects of higher-order perturbative corrections due to inelastic scattering should be important. The lowest-order computations only evaluate the effect of elastic scattering, and these higher-order corrections should reduce the mean free path and increase the total cross section. Given these intrinsic ambiguities, it is impossible to draw any precise conclusion. What we shall do is give a range of values which span the results of Hosoya and Kajantie and of Danielewicz and Gyullasy, and allow for some uncertainty in uncomputed contributions.[29,30] We find for all values of energy density in the range of $\epsilon \sim 1-1000$ GeV/fm^3

$$\lambda_g \sim \tfrac{1}{20}-\tfrac{1}{2} \text{ fm} , \tag{16}$$

$$\lambda_q \sim \tfrac{1}{5}-2 \text{ fm} . \tag{17}$$

The gluon mean free path is λ_g and that of the quark is λ_q in this equation. The variation in mean free path as the energy density varies over this wide interval is at most a factor of 2 in our estimates. The mean free path may therefore be effectively regarded as a constant as the energy density varies over this range. The gluon mean free path is about a factor of 4 smaller than that of the quark as a consequence of the larger color charge of the gluon, which forces it to interact more strongly than the quarks.

For the mean free paths of Eqs. (16) and (17), one would expect the surface effects for quarks to be large for large nuclei only under the most pessimistic scenario. Under optimistic scenarios, the corrections even for hadrons might be small. For gluons, one would not expect large effects for large nuclei, but might find large effects in hadrons. For large nuclei, the effects of finite nuclear size should be manageably small, but for hadron interactions the situation is entirely unclear. For very-high-multiplicity $\bar{p}p$ collisions, the ratio of mean free path to spatial size may however be favorable, and similar to the case for a nucleus-nucleus collision. In such a situation, even if the mean free path at the time of matter formation is not small compared to the spatial size of the system, if the initial energy density is sufficiently large, then after some expansion, the system may have large energy density and be in a large spatial volume. At such a time, however, it is difficult to abstract boundary conditions for hydrodynamic equations, and also the usefulness of the hydrodynamic description of the evolution of the produced matter as a basis for understanding the characteristic features of final distributions and providing a framework to calculate the rates of specific signals becomes marginal.

To what extent perfect-fluid expansion is modified by viscous corrections is resolved by using the condition that the rate of entropy production due to viscous terms be small compared to the change in the entropy density due to expansion. This criterion may be formulated precisely in terms of viscous coefficients, but we shall here formulate the problem semiquantitatively and qualitatively in terms of mean free paths. The change in the entropy density due to expansion is given by the perfect-fluid hydrodynamic equations, for power-law expansion typical of solutions to these equations, as

$$ds/d\tau \sim -s/\tau . \tag{18}$$

The change in the entropy density due to entropy production is

$$ds/d\tau \sim (\tau_c/\tau)s/\tau , \tag{19}$$

where τ_c is the collision time. The criterion that perfect-fluid hydrodynamics be valid is therefore simply

$$\tau_c/\tau \ll 1 . \tag{20}$$

Since the collision time is roughly independent of energy density, and therefore of τ, after some time τ, the system always is capable of expanding to a good approximation as a perfect fluid. This is because as a consequence of the similarity solutions of the hydrodynamic equations, at later times the system is expanding more slowly.

The collision times given by Eqs. (16) and (17) show that for times $\tau > \frac{1}{5}-2$ fm, the quarks may expand isentropically, and the gluons for times $\tau > \frac{1}{20}-\frac{1}{2}$ fm. These numbers are not inconsistent with the assumption that after matter forms at a time $\tau_i \sim \frac{1}{10}-1$ fm, the matter quickly thermalizes and expands to a fair approximation as a perfect fluid. At the earliest times, there is the greatest entropy production, and as time evolves, the system behaves more and more as a perfect fluid. To resolve this problem more precisely, it would be nice to have a nonperturbative estimate of the viscous coefficients.

Perturbative estimates of collision times have been used to estimate the coefficients of shear and bulk viscosity. Hosoya and Kajantie find[31]

$$\zeta = 0 , \tag{21}$$

$$\eta = \frac{0.2}{\alpha_s{}^2 \ln \alpha_s} T^3 . \tag{22}$$

The evaluation of Danielewicz and Gyulassy gives a result which is a factor of 3 larger.[30] The hydrodynamic equations may be used to estimate the total amount of entropy production

$$s_{\text{final}} \sim s_{\text{initial}} (1 + \tau_c / \tau_i) \tag{23}$$

a result which is exact to first order in viscous corrections for $(1+1)$-dimensional hydrodynamic expansion. This equation illustrates the increasing effects of entropy production at increasingly early proper times.

At very early times there is entropy production due to a variety of effects, and it would be extremely valuable to have a controlled theoretical analysis of the preequilibrium quark-gluon plasma.[32] It would seem that such an analysis is tractable since at early times the energy density is high and the effects of interactions are weak. Such an analysis would be required to rigorously derive the inside-outside cascade within QCD. The initial conditions for the hydrodynamic equations would follow from knowledge of the initial-state nuclear wave functions, about which little is presently known. A spectrum of fluctuations could be derived, and the parameter τ_i could be computed. The magnitude and importance of coherent phenomenon could be deduced.

Another possible place where perfect-fluid hydrodynamics might break down is when the quark-gluon plasma expands through a first-order phase transition, or if the quark-gluon plasma must be produced from hadronic matter by undergoing a first-order phase transition. In either of these possible scenarios, large-scale density fluctuations might be produced, and a global hydrodynamic description might break down.[33,34] The system might break apart into droplets of matter which might slowly burn, or explosively detonate the plasma. The possibility that the system might break up into slowly burning droplets has been proposed by Van Hove, and would occur if the plasma spinoidally decomposed, that is, the system falls apart rapidly with a large volume change and consequent large density fluctuations.[33] If the plasma could supercool, then explosive detonation droplets might form. If these large-scale density fluctuations were not too strong, the matter might recombine in the hadron phase, and a viscous expansion would smooth out the density fluctuations. There would be some entropy production, but the final matter distribution might be considerably smoothed out. If the density fluctuations were too severe, the plasma might break apart into isolated droplets each of which might be treated hydrodynamically.

Although in exceptional circumstances, large-scale density fluctuation might be expected to occur, we argue that for average collisions, such fluctuations should be smoothed out. In a typical collision, large-scale density fluctuations are seeded when the pion gas begins to dominate the volume of our expanding system. At this time, we can describe the system as a pion gas with droplets of plasma embedded in it. If the droplets of plasma are equally spaced, as should be approximately the case for average collisions, the first density for which plasma becomes embedded in a pion gas, rather than pion embedded in plasma, is given by computing the fractional volume occupied by closest packed spheres. The ratio of plasma volume to total volume is $f \sim (\frac{4}{3}\pi R_d{}^3)/(8 R_d{}^3) \sim \frac{1}{2}$. At the time that this occurs the separation between droplets is twice the droplet size, $d \sim 2R_d$. As the system expands, the separation between the droplets increases. Assuming that the expansion is $1+1$ dimensional, the separation between droplets is determined by requiring that the droplets of plasma uniformly fill all of the volume. If this is the case, as it should be if the motion of the plasma droplets is random, the total volume of the system increases by $\tau/\tau_{1/2}$ where $\tau_{1/2}$ is the time at which the closest packing of plasma droplets occurs. Since $\tau_{1/2} \sim 2\tau_q$, where τ_q is the time it took for the plasma to lower its density to a small enough value so as to enter the mixed phase, and since the last time the droplets appear in the system is when the system completes the mixed phase is τ_h, the separation between droplets when they disappear from the system is $d \sim 2R_d(\tau_h/2\tau_q)^{1/3}$. As we will argue in later sections, the ratio between τ_h/τ_q is given by the ratio of degrees of freedom of a hadron gas and that of a quark-gluon plasma, ~ 15. The separation is therefore $d \sim 4R_d$.

At this late time, the expansion time for the system is τ_h. The typical diffusion length during one expansion time is therefore $l_{\text{dif}} \sim (\tau_h/\tau_s)^{1/2}\lambda$ where $\tau_s \sim \lambda$ are the scattering time and mean free path, respectively. For a pion gas at a temperature of 200 MeV with a pion cross section of 20 mb, this mean free path is about 2 fm. Taking the hadronization time to be $\tau_h > 30$ fm, a number which we shall later show is at the low end of values appropriate for uranium-uranium nucleus collisions, we find $l_{\text{dif}} \sim 8$ fm. If the droplet size at closest packing is taken to be a fm, then the diffusion distance is larger than the separation when the droplets disappear. In this case, we expect that any thermal gradients which are generated due to the density difference between plasma droplet and pion gas will be largely smoothed out by diffusion. Even if the droplets are as large as 2 fm at formation, diffusion in one expansion time should be sufficient to largely smooth out density inhomogeneities.

To make this case more firm, it would however be useful to carry out a detailed cascade computation. At the least, there is no reason to assume that the system does

not expand to a good approximation as a mixed phase with only small energy density inhomogeneities in average collisions of large nuclei. The situation is less clear if the nuclei are smaller so that the time for longitudinal and transverse expansion is shorter, or if the droplets of plasma are larger than a few fm at formation. For very small nuclei, we would in fact expect that if a quark-gluon plasma is formed in a collision, there may typically be a good deal of spatial inhomogeneity generated by nucleation. A cascade computation would determine the extent of such inhomogeneities for arbitrary size nuclei.

In the analysis which we shall present, we shall assume that the quark-gluon plasma smoothly turns into a hadronic gas. This might occur if the plasma converted to a hadron gas through a mixed phase, and the nucleation time for the formation of hadron matter was short compared to the expansion time. Also, in the van Hove scenario, if the droplets formed as matter falls apart quickly rehomogenize themselves in a hadronic gas, again the scenario we describe applies. Finally, if the transition from quark-gluon matter is only a rapid transition and not a true first-order phase transition, then the dynamics of the transition region is well approximated by an equation of state with a mixed phase region corresponding to the region of rapid transition. In this latter scenario, large-scale density fluctuations are not expected. As our explicit computations indicate, the matter formed in a high-energy collision seems to spend a large amount of time in a mixed phase, compared to natural hadronic time scales, and in the absence of strong first-order phase transitions, which might generate strong supercooling, the mixed phase scenario is probably appropriate.

Since, as we shall show, the matter takes a long time to get out of the mixed phase, and because the hydrodynamic expansion is power law the matter expands slowly as a hadronic gas. The freeze-out occurs therefore at a very late time and low temperature. In Sec. IV of this paper we shall describe in detail our algorithm for decoupling.

III. THE INITIAL CONDITIONS

In this section we shall consider in detail the initial conditions for perfect-fluid hydrodynamic equations which should be appropriate for high-energy collision processes. We begin by studying spherically symmetric initial conditions, and their relevance to a class of fluctuations in high energy $\bar{p}p$ collisions. We then turn to initial conditions which are boost invariant along the collision axis, and discuss their relevance to both fluctuations in $\bar{p}p$ collisions and ultrarelativistic nuclear collisions. We begin by reviewing the one-space one-time scaling solutions proposed by Bjorken.[4] Then following the analysis of Baym *et al.*,[23] we generalize these considerations to include the central region of collisions of finite nuclei.[23] We analyze various possibilities for the initial transverse entropy and velocity profile. We also relate the initial time of matter to the initial temperature. We also present methods of extracting some of the parameters which characterize initial conditions from experimental data.

In some fluctuations in high-energy $\bar{p}p$ collisions, matter may initially form in a region which is spherically symmetric. For example, in a high-energy gluon-gluon collision, the gluons in the colliding hadrons may undergo a central collision and radiate a large number of gluons which are on the average at rest in the center-of-mass frame of the two gluons. Such a situation has been proposed in the Pokorski—van Hove model.[35]

In a spherical fluctuation, we shall show that the fluctuation is in a limited region of rapidity, and determine the shape of the distribution. This fluctuation is centered on the rapidity of the "fireball" produced by the gluon collision. In the frame comoving with this fireball, particle distributions are spherically symmetric. A first question we must ask is how the spherical nature of the fluctuation is reflected in the rapidity distribution. To understand how this might be done, we make a massless pion approximation, and identify pseudorapidity with rapidity. In this limit,

$$y = -\ln \tan(\theta/2) , \tag{24}$$

where θ is the angle relative to the beam axis. Suppose that the fluctuation is centered at zero rapidity. Then for a spherical fluctuation,

$$\frac{dN}{dy} = \frac{dN}{d\theta}\frac{d\theta}{dy} = \frac{dN}{d\theta}\frac{1}{\cosh y} \sim \frac{dN}{d\Omega}\frac{1}{\cosh^2 y} . \tag{25}$$

Since, by assumption, for spherical expansion the angular distribution $dN/d\Omega$ is uniform, a spherically symmetric fluctuation is characterized by a $\cosh^{-2} y$ falloff centered around the total rapidity of the fluctuation. Such a fluctuation is shown in Fig. 2.

Since the fluctuations are in general not spherically symmetric, it is essential to use the predicted rapidity distribution to select those fluctuations which are in fact spherical. It is also important to note that the initial radial velocity distribution for the fireball is not in general zero throughout the matter. Since we expect that the matter is initially randomly distributed throughout the fireball, it is plausible to assume that on the average, the outward initial velocity will be zero throughout the matter. There will of course be fluctuations in these initial conditions, but we shall only study the generic typical fireball.

Since matter distributions with sharp edges are difficult to use in numerical simulations, we shall smooth out the matter distribution at the surface of the initial fireball. (For any reasonable physical system, this smoothing

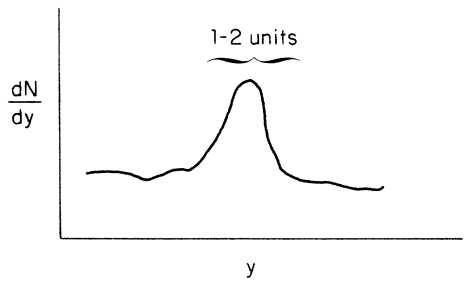

FIG. 2. An example of a rapidity fluctuation which might occur in a $\bar{p}p$ collision which might produce high enough energy density to yield a quark-gluon plasma.

length is naturally the mean free path for particle interactions.) We therefore take an initial matter energy density profile to be a Fermi-Dirac distribution:

$$\epsilon=\epsilon_0\frac{1}{e^{(r-R)/\delta}+1} . \tag{26}$$

In this equation, the radius of the initial matter distribution is taken to be R. The surface thickness is δ, a parameter which we shall make sufficiently small so that the results of our numerical simulations tend to a uniform limit.

For the hydrodynamic numerical simulation which we shall later use, it is also not convenient to take the velocity to be zero everywhere. We shall choose the initial fluid velocity to be zero inside the matter, but to initially approach v_0 outside where there is no matter. We shall later probe the sensitivity of this assumption to arbitrary choices for v_0. (For a physical system, it is plausible to assume that the initial velocity in the diffuse region outside the matter distribution is of the order of a typical particle transverse velocity appropriate for average multiplicity $\bar{p}p$ collisions.) The specific choice which we make for the fluid velocity profile is

$$v=v_0\left[1-\frac{1}{e^{(r-R)/\delta}+1}\right] . \tag{27}$$

We have not yet estimated the radius R of the fireball arising as a fluctuation. The uncertainty principle suggests that the initial size of the matter distribution is $\sim 1/p_t$ where p_t is the typical transverse momentum of particles in the initial fireball. This value should also arise in any scale-invariant description of the initial fluctuation process; that is, $1/p_t$ is the only quantity with the dimensions of a length. It is necessary in this connection to assume that these fluctuations are rare enough so that overlapping fluctuations are not important. Also, unless $R \ll 1$ fm, finite-size effects due to the size of the hadrons which constitute the beam, are important. The proportionality constant which relates R and p_t is difficult to estimate without a more detailed description of the formation process.

Using the above assumptions about the nature of the region in which the matter initially forms, and the assumption that the matter expands according to the laws of perfect-fluid hydrodynamics, it is possible to abstract the correlation between multiplicity and transverse momentum of pions.[36–38] To find this correlation, we must use an equation of state. We shall also need to study decoupling, that is, how the matter freezes out from a flowing nonviscous fluid into a free streaming hadron gas.

To make a relation between initial and final distributions of particles, consider the quantities E/S and E/V (Ref. 39). The energy per unit entropy is conserved in perfect-fluid hydrodynamic expansion since both total energy and total entropy are conserved. The energy per unit entropy is roughly speaking a measure of temperature, and is computed by standard techniques. The energy per unit volume is determined by the conserved total energy and by the volume which is an initial condition.

The theoretical thermodynamical correlation between E/S and E/V is straightforward to understand. At low temperatures where there is an ideal pion gas, and high temperatures where there is an ideal quark-gluon plasma, E/S is $3T/4$. At temperatures where the system rapidly crosses over between a hadronic gas and quark-gluon plasma, the temperature and E/S remain approximately constant. At low temperatures, the energy density changes less rapidly with increasing temperature than at high temperatures since the degrees of freedom of a pion gas are less than those of quark-gluon plasma. At the phase transition, the energy density changes while the temperature remains constant. In Fig. 3, E/S is plotted versus E/V for a bag-model equation of state, with a bag constant of $B^{1/4}=200$ MeV. The flat region where E/S is constant as E/V changes by an order of magnitude is indicative either of a rapid crossover or a phase transition between hadron gas and plasma.

It is experimentally straightforward to measure the total energy E of particles in the initial plasma fireball. This is measured in the rest frame of the fireball and is proportional to dE/dy the transverse energy per unit rapidity, $dE/dy \sim p_t dN/dy$. Since the volume scales as p_t^{-3}, the energy density is, up to an undetermined numerical constant κ,

$$\frac{E}{V}=\kappa p_t{}^4\frac{dN}{dy} . \tag{28}$$

The energy per degree of freedom is more difficult to extract. Experimentally, the energy per particle, or average transverse momentum, is measured. The average energy per particle and the transverse momentum are related as $p_t=(\pi/4)E/N$, which follows only from the assumed spherical symmetry of the matter distribution. To relate the total number of particles to entropy requires a brief study of decoupling. As the temperature decreases below the phase transition temperature, at some temperature heavy mesons are no longer important and there is an almost ideal gas of massless pions. For a range of temperatures where this is true, the total entropy and total number of pions are both conserved and are related as $S \sim 3.7N$. Before this temperature is reached, pion-number-changing processes, which involve four-body collisions, have frozen out. This happens much before there are significant modifications of the equation of state due to finite pion mass, since the pion-number-changing processes involve four powers of Boltzmann factors, $e^{-m/T}$.

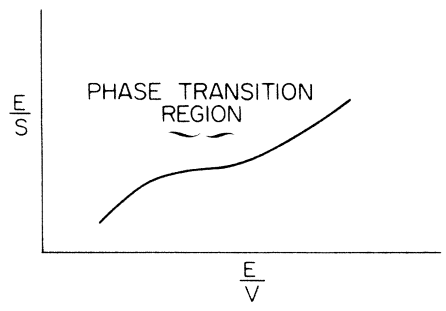

FIG. 3. E/S vs E/V for the bag model.

With the relation between S and N, we may now find that between p_t and dN/dy. The phase transition temperature, and the ratio of energy densities below and above the phase transition, which is the ratio of degrees of freedom of the pion gas and the quark-gluon plasma, therefore follow in a model-independent way from a plot of p_t vs $p_t^4 dN/dy$.

In this analysis, the initial energy stored in thermal fluctuations reappears in the transverse momentum of pions. The thermal energy is converted into energy of collective radial flow. Since the system is in isolation, this is required by energy and entropy conservation.

Some features of the hydrodynamic expansion may not be extracted from the general considerations presented above. For example, if the initial energy density is large, the transverse momentum of large mass particles is expected to be enhanced by a much greater factor than that of pions. This follows since the transverse-momentum enhancement for pions arises from collective radial flow of a fluid. Heavy-mass particles with the same outward flow velocity acquire a larger transverse momentum. It is also difficult to extract the lifetime of the fireball, or rates for dilepton and photon emission without a detailed computation.

Average ultrarelativistic nuclear collisions, and fluctuations in $\bar{p}p$ collisions of high multiplicity, but which are uniform over a wide rapidity interval, must be treated differently from the case of spherical fluctuations. For head-on nuclear collisions, and probably for uniform rapidity high-multiplicity $\bar{p}p$ collisions, the matter forms more or less uniformly over a transverse area which is the geometrical cross section of the colliding nuclei or hadrons, and the geometry of the collision is cylindrically symmetric. To simply analyze this problem, we first assume uniform matter distribution in the transverse direction. Following Bjorken, we also assume that the distributions are uniform in rapidity. Fluctuations may be found which satisfy this criterion, and for average ultrarelativistic nuclear collisions between nuclei of equal A, this criterion should be approximately satisfied for rapidities not too far from the central region.

If the particle distributions are uniform in rapidity, the local comoving distributions of particles are Lorentz invariant. The fluid velocity vector u must therefore be a Lorentz form invariant vector under transformations along the collision's axis, which is only a function of x. Since $u^2=-1$, u must be of the form

$$u^\mu = x^\mu/\tau \,, \tag{29}$$

where

$$\tau=(t^2-z^2)^{1/2} \tag{30}$$

is the proper time. The space-time rapidity variable is

$$\eta=\tfrac{1}{2}\ln\left[\frac{t+z}{t-z}\right] . \tag{31}$$

For the Lorentz-invariant situation we consider here, the fluid rapidity θ and the space-time rapidity η are equal:

$$\eta=\theta \,. \tag{32}$$

The scalar energy density ϵ, the pressure P, and the entropy density σ are Lorentz scalars and are therefore functions only of τ.

As τ becomes smaller, earlier times are probed in the collision. As discussed in the first section, at too early a time, the perfect fluid hydrodynamic description must break down. This happens at some time which is of the order of a scattering time τ_c appropriate for the matter at the time τ. Before this time, entropy-producing effects are important, as is particle production of the matter which produced the quark-gluon plasma.

If the initial energy density is sufficiently large, it is possible to relate the initial temperature and the formation time τ_i. In this context, formation time is meant as the earliest time when it is a good approximation to treat the evolution of the matter as a perfect fluid. To estimate this time in terms of the temperature, we must relate the collision time to the temperature. In a scale-invariant theory, a good approximation if the initial energy density is sufficiently large, the scattering time is[40–44]

$$\tau_c=\kappa/T_i \,, \tag{33}$$

where κ is an as-yet undetermined constant of order one. It should be noted that this relationship is also typical of uncertainty principle relationships between the formation time and the typical energy scale of the matter.

The constant of proportionality may be estimated by a variety of means. Phenomenological analysis of Japanese-American Cooperative Emulsion Experiment (JACEE) data, which we shall soon discuss, and various theoretical estimates suggest that if the collision time is chosen to be a fm/c, then the initial temperature is approximately 250 MeV. We therefore have an approximate relationship of the form

$$\tau_i \ (\mathrm{fm}/c)=\frac{250 \ \mathrm{MeV}}{T_i} \,. \tag{34}$$

The uncertainties in this relationship are probably of order 50%.

The solution to the perfect-fluid hydrodynamic equations for the scale-invariant longitudinal expansion is especially simple and has been discussed by Bjorken.[4] For

$$P=\tfrac{1}{3}\epsilon \,, \tag{35}$$

the energy density is

$$\epsilon=\epsilon_i\left[\frac{\tau_i}{\tau}\right]^{4/3} , \tag{36}$$

the temperature is

$$T=T_i\left[\frac{\tau_i}{\tau}\right]^{1/3} , \tag{37}$$

and the entropy density is

$$\sigma=\sigma_i\left[\frac{\tau_i}{\tau}\right] . \tag{38}$$

The entropy density may be related to the multiplicity per unit rapidity. Since the total entropy S is

$$S=\int \tau\, dy\, d^2r_\perp \sigma \tag{39}$$

we have that the entropy per unit rapidity is

$$\frac{dS}{dy}=\pi R^2 \tau\sigma=\pi R^2 \tau_i \sigma_i \ . \tag{40}$$

The entropy rapidity density is therefore invariant in time, a feature which is generally true independent of the equation of state, and which follows from the isentropic nature of the perfect-fluid hydrodynamic equations.

As was argued above for the case of spherical expansion, the entropy density and pion multiplicity may be related as

$$\frac{dS}{dy}=3.7\frac{dN}{dy} \tag{41}$$

so that the multiplicity distribution may be used to abstract the entropy density. Since the initial entropy density is related to the initial temperature as

$$\sigma_i=\frac{4}{3}\frac{\pi^2}{30}N_{\rm DF}T_i^3 \ , \tag{42}$$

where $N_{\rm DF}$ is the number of degrees of freedom (DF) of the quark-gluon plasma,

$$N_{\rm DF}=(10.5N_f+16)\sim 40 \tag{43}$$

if the number of participating quark flavors is approximated as 2.5. We have therefore that

$$T_i\sim 0.6\tau_i^{-1/3}\left[\frac{1}{\pi R^2}\frac{dN}{dy}\right]^{1/3} \ . \tag{44}$$

If we assume that T_i and τ_i are related according to Eq. (34), then we can determine τ_i from experimental data. With $\kappa\sim 250$ MeV, we have

$$\frac{1}{\tau_i}\sim 0.4\kappa^{-3.2}\left[\frac{1}{\pi R^2}\frac{dN}{dy}\right]^{1/2} \ . \tag{45}$$

This form of the functional relation between multiplicity and formation time has been observed in string models of high-energy collisions, and is probably more general than its derivation here. If we take the results from the JACEE experiment as

$$\left.\frac{dN}{dy}\right|_{\rm ch}\sim 4A \tag{46}$$

then the relation between time and A becomes[40–44]

$$\tau_i\sim 1-2A^{-1/6} \ . \tag{47}$$

For large nuclei such as uranium, this formula suggests that the formation time might be as small as 0.3 fm/c, with a temperature as high as 700–800 MeV. For small nuclei, Eq. (47) most surely breaks down, as does a hydrodynamic analysis for average collisions, and τ_i probably saturates at a fixed value of $\tau_i\sim 1$ fm/c.

With this analysis in hand, we now proceed to an analysis of collisions taking into account the transverse matter profile. As was the case for spherical expansion, we shall take the transverse matter profiles as given by Eqs. (26) and (27). The only difference is that the transverse radial coordinate, not the radial coordinate, appears.

If we have a fully three-dimensional situation, we would in general expect the perfect-fluid hydrodynamic equations to involve t, z, and the transverse coordinate r. This is not the case if the central region multiplicity density dN/dy is independent of y, since if we express the hydrodynamic equations in terms of τ, η, and r with τ and η as given by Eqs. (30) and (31), then the Lorentz covariance of the equations allow scalar quantities not to depend on η. This is not the case when dN/dy is not independent of y, as would happen if we applied this analysis to the fragmentation region. In this case, the lack of boost invariance of the assumed multiplicity distribution induces a lack of boost invariance of the hydrodynamic equations. The longitudinal boost invariance together with $u^2=-1$ requires that the fluid four-velocity be of the form

$$u=\gamma_r(\tau,r)(t/\tau,v_\perp(\tau,r),z/\tau) \ , \tag{48}$$

where γ_r is

$$\gamma_r=(1-v_\perp^2)^{-1/2} \ . \tag{49}$$

The Lorentz-scalar quantities such as the pressure, energy density, $v_\perp$, and entropy density are functions only of τ and r, with no dependence upon η. Notice that as was the case for the (1 + 1)-dimensional expansion, the fluid and space-time rapidity are equal:

$$\eta=\theta \ , \tag{50}$$

where the fluid rapidity is defined to be

$$\theta=\tfrac{1}{2}\ln\left[\frac{1+v^z}{1-v^z}\right] \ . \tag{51}$$

The perfect-fluid hydrodynamic equations for the stress-energy tensor may be written in a simple form. If we use

$$T^{00}=(\epsilon+P)u^0u^0-P \tag{52}$$

and

$$T^{01}=(\epsilon+P)u^0u^1 \tag{53}$$

the hydrodynamic equations are

$$\partial_\tau T^{00}+\frac{1}{r}\partial_r(rT^{01})+\frac{1}{\tau}(T^{00}+P)=0 \tag{54}$$

and

$$\partial_\tau T^{01}+\frac{1}{r}\partial_r[r(T^{00}+P)v_r^2]+\frac{1}{\tau}T^{01}+\partial_r P=0 \ . \tag{55}$$

It should be noted that the hydrodynamic equations for spherical expansion are the same as those above with the trivial modification that all terms proportional to $1/\tau$ are dropped and everywhere $r\to r^2$. The initial conditions for the spherical and Lorentz-invariant cases differ only in that the initial time can be taken to be zero for the spherical expansion, and is finite for the Lorentz-invariant case.

IV. DECOUPLING

As the matter expands according to the laws of perfect-fluid hydrodynamics, at some point it achieves a sufficiently low density that in the entire future history of the fluid, a typical particle may be expected to scatter less than once. In the future, it should therefore be a good approximation to treat the particles as free particles. To analyze this problem, we must follow through the history of the fluid and determine roughly when the fluid freezes out. We must also determine the entropy production at freezeout. Since in our analysis, freezeout occurs at a late time when the local excitation energy is small and when most of the work has been done on particles, the amount of entropy production should be small compared to the total entropy, and in our analysis, we shall ignore this contribution. We therefore make the approximation that the system expands as a perfect fluid until freezeout, which happens instantaneously. Finally, we need an algorithm for determining particle distributions if the fluid velocity and density is known immediately prior to freezeout.

As we discussed in the second section, at some time after the collision which forms the matter, the expansion time becomes large compared to the collision time. This may be seen from the power-law nature of the expansion, combined with the assumed slow variation of the collision time. The collision time should be slowly varying until a phase transition between quark-gluon plasma and hadron gas is initiated.

We can estimate the time at which the plasma begins to become a hadron gas. We shall first consider the case of Lorentz-invariant expansion. At the initial time, the temperature is $T_i \sim \kappa/\tau_i$. The system expands as $T = T_i(\tau_i/\tau)^{1/3}$, so that

$$\tau \sim \frac{\kappa^3}{\tau_i^2 T^3} . \tag{56}$$

If an initial time of $\tau_i \sim 1$ fm/c, then τ is about 2 fm/c at a temperature of 200 MeV. If, on the other hand, the formation time was as small as 0.3 fm/c, as it might very well be for ultrarelativistic uranium collisions, then the time is 20 fm/c at the phase transition temperature. For an initial temperature 500 MeV, corresponding to an initial time of 0.5 fm/c, the time is about 10 fm/c at the transition.

For such large times, we might legitimately worry about whether transverse rarefaction might disrupt the system. For large nuclei such as uranium, except for the highest initial temperatures, this is probably not the case for two reasons. First, if the system were to transverse rarefact with a rarefaction wave with sound velocity typical of an ideal gas, $v = 3^{-1/2} = 0.6$, the rarefaction time is 10 fm/c. On the other hand, as the matter expands it reaches the critical temperature at the edge. Then the pressure gradient disappears and this part of the wave is not accelerated any more. Closer to the collision axis only the forward edge (that is, the slow part) of the rarefaction wave reaches the matter before it is cooled to mixed phase and the transverse acceleration ceases. As a result at time $\tau = \tau_q$ a large part of the system is in the mixed phase which extends radially beyond the original radius R_A and expands slowly transversally. A shock wave propagates slowly inward at the interface of the mixed phase and hadron gas and in the inner parts the expansion is almost one dimensional until the time $\tau = \tau_h$ when longitudinal expansion alone had diluted the energy density to ϵ_H, the energy density of the pion gas at the critical temperature.

For spherical expansion, long times are also required before the system begins to convert into hadronic matter. This follows because the rarefaction must proceed against an expanding fluid into a region of mixed phase where the sound velocity is zero. As we shall see in the next paper, for modest values of the temperature, we find that extremely long times ~ 2–20 fm/c are required before the system begins to hadronize.

After the quark-gluon plasma reaches the phase transition temperature, it must convert the entropy stored in the plasma into the entropy of a hadron gas. Since the degrees of freedom of the plasma are an order of magnitude larger than those of a pion gas, this conversion takes a long time. If the system is only longitudinally expanding during this time, the ratio of the time at which the plasma began the phase transition, τ_q to that at which it completes the phase transition, τ_h is given by the ratios of these degrees of freedom

$$\tau_h/\tau_q \sim N_{\rm pl}/N_\pi \sim 15 . \tag{57}$$

This time might be anywhere in the range of 30–300 fm/c. If the time is so large as 300 fm/c, the assumption that transverse rarefaction may be ignored has surely broken down. Nevertheless, the time it takes to complete the transition is long both compared to a natural time such as a fm, which controls the rate of pion scattering, and long compared to the time that it takes the system to reach the phase-transition temperature. The system spends a very long time in a mixed phase at a temperature close to the phase-transition temperature. Dilepton and photon emission must surely be affected by the long time spent in a mixed phase. There should be a contribution to the emission spectrum of dileptons of the form e^{-M_t/T_p} where M_t is the transverse mass and T_p is the phase transition temperature. This contribution might dominate the emission spectra for moderate values of the transverse mass. This might be studied at low mass and moderate p_t so as to avoid problems with background due to resonance decays and still have a significant thermal signal.

The situation is also probably quite similar for spherical expansion. The time the system spends in a mixed phase is probably large. The system emerges from the mixed phase expanding slowly compared to natural hadronic time scales.

If the system is expanding slowly when it reaches the hadron gas phase, it probably maintains itself in equilibrium with respect to pion number, and expands to a fair approximation isentropically for some time after completing the transition. As the system cools, at some temperature the effects of finite pion mass begin to become important. Also the density of pions is decreasing and interactions begin to decrease in magnitude. This effect of low density probably begins to show up first in the interactions which change pion number. These interactions involve four pion

collisions, and as the density decreases, these interactions rapidly shut off. The reverse reaction of two pions goes to four shuts off because the reaction takes a lot of energy and the tails of the two-pion distribution are sampled. We assume that both $2\rightarrow 4$ and $4\rightarrow 2$ pion-number-changing reactions shut off at the same time. As this occurs, the pion number becomes frozen at some fixed value. We assume that this freezeout occurs at a temperature which is sufficiently large that the massless pion approximation is still valid. This seems to be the case from explicit studies of pion-number-changing processes.[45] The entropy and particle number are related as $N\sim 3.7S$.

Since this freezeout happens rather late, when the expansion is gentle and most of the work has been done in expanding the quark-gluon plasma and converting into a hadron gas, the subsequent evolution of the system does not probably much affect distributions of particles. If we approximate the system as frozen out at this time, and compute again assuming a freezeout at a later time, we do not expect that physical quantities will change much. We shall verify this in our computations.

It is useful nevertheless to be convinced that the freezeout does not occur until very low densities at very late times. To see this, consider the mean free path for pion scattering

$$\lambda\sim 1/\epsilon\sigma\ , \tag{58}$$

where ϵ is the pion number density and σ is the pion-pion scattering cross section. Since the pion number density goes as $1/\tau$, corresponding to a conserved total number of pions, for the Lorentz-invariant one-dimensional expansion, and σ is roughly constant until the pions are nonrelativistic, the mean free path scales as τ. For power-law expansion, the rate of expansion is proportional to $1/\tau$. Thus as long as the one-dimensional expansion scenario is valid, the pions stay in local thermal equilibrium until they become nonrelativistic. Of course, once the three-dimensional nature of the expansion is important, the mean free path grows as τ^3 and the system rapidly freezes out. This again should happen only at quite late times, and again the system is quite cool.

Our conjecture is therefore that the systems which we consider do not freeze out until very late times when the system has done almost all the work it can do to generate particle distributions. The results we compute for particle distributions should therefore be fairly insensitive to the details of freezeout, such as the time the system freezes out, and the chosen freezeout temperature. With this in mind, we shall consider results for momentum distributions of pion for a variety of freezeout temperatures. Most of our results will be for a freezeout temperature of 140 MeV, a number which is arbitrarily chosen, but will not much affect our results.

We have still not presented our algorithm for freezeout. This algorithm is essentially that of Cooper and Fry.[46] We repeat their considerations here. We begin by deriving an expression for the number of particles passing through the freezeout surface which is parametrized as σ^μ. This is a three-dimensional space-time surface which is determined by a freezeout condition such as $T(t,\mathbf{r})=T_0$. It can be visualized as a moving spatial surface $\mathbf{S}(T)$ describing the position of matter which at the time t has reached the freeze-out temperature. If this condition is reached simultaneously in a certain spatial volume, then these regions of space belong to σ^μ.

If dN is the number of particles passing through the surface element $d\sigma^\mu$ and if $f(x,p)$ is the Wigner distribution function which describes the probability that a particle of momentum p and energy $E=(\mathbf{p}^2+m^2)^{1/2}$ is at the space-time coordinate x, then

$$dN=f(x,p)d^3p\{\mathbf{v}\,dt-d\mathbf{x}\}\cdot\hat{\mathbf{n}}\,d^2S\ . \tag{59}$$

In this equation, the particle velocity is $\mathbf{v}$, the normal to the surface is $\hat{\mathbf{n}}_s$. The first term in this equation is the current of particles with momentum d^3p through the surface element $\hat{\mathbf{n}}_s d^2S$ in the time interval dt when the surface element is in the fixed position. The volume term $d^3x=d\mathbf{x}\cdot\hat{\mathbf{n}}_s d^2S$ takes into account the change of the flux through $d\mathbf{S}$ due to the displacement $d\mathbf{x}$ of the surface element in the time interval dt. Thus, spatial volumes where the decoupling condition is reached at a given instant t, may be included in this volume term. This situation can take place around the symmetry axis in cylindrical expansion if the initial-temperature distribution is flat and low enough so that the decoupling temperature is reached before the transverse rarefaction has time to propagate to the axis.

As with all breakup criteria, this criterion is not perfect. At the decoupling surface, particles from either side may flow in and out. This allows in principle the possibility that the momentum distribution might become negative for some values of the momentum. For the situation where we apply this code, the system is outwardly expanding, and we do not expect that this will happen. In our hydrodynamic simulations we see no evidence of such behavior.

The above relation for dN may be written in the Lorentz-invariant form as

$$E\,dN/d^3p=f(x,p)p\cdot d\sigma\ , \tag{60}$$

where the surface element

$$d\sigma^\mu=(d^3x,dt\,\hat{\mathbf{n}}_s d^2S) \tag{61}$$

transforms under Lorentz transformations as a Lorentz four-vector. The expression for $E\,dN/d^3p$ is therefore Lorentz invariant, as it must be.

The particle distributions after the decoupling are obtained from Eq. (60) by integrating over the entire freeze-out surface σ^μ. We shall assume that at freezeout the distribution function $f(x,p)$ is that of an ideal fluid

$$f(x,p)=\frac{g}{(2\pi)^3}(e^{-\beta(x)u^\mu(x)p_\mu}\pm 1)^{-1}\ . \tag{62}$$

Here $\beta(x)=1/T(x)$, $u^\mu(x)$ is the fluid four-velocity, and g is the number of degrees of freedom, $g=N_{\mathrm{DF}}$, for the particles in question. If the freezeout condition is $T(x)=T_{\mathrm{dec}}$, then $\beta(x)=1/T_{\mathrm{dec}}=$constant for the entire integration.

Before going into the details of the calculation of the transverse-momentum distributions, we shall consider the simpler task of computing the average transverse momen-

tum due to collective flow as a function of the particle multiplicity. This computation isolates only the contribution to transverse momentum due to the collective fluid flow of the system. This should be equal the average total transverse momentum if the system evolves from a high temperature to a low temperature, so that at the low temperature of decoupling thermal motion is very small, and the transverse momentum is given entirely in terms of that of collective flow. Realistic cases are handled by a more complicated algorithm which we shall soon discuss.

In the case of longitudinal boost invariance, we can compute the multiplicity distribution dN/dy in terms of the entropy density dS/dy which is known from the local values of entropy density and pressure. Similarly, if the local boost invariance holds, the total transverse momentum of the particles arising from collective flow in a given rapidity interval equals the total transverse momentum of the fluid with flow rapidity equal to ordinary rapidity in the same interval. This transverse momentum can be computed as

$$P_t = \int T^{r\mu} d\sigma_\mu \ , \tag{63}$$

where $T^{r\mu}$ is the transverse μ component of the stress-energy tensor, and the integration is over that part of the freezeout surface σ^μ where the flow rapidity is in the considered interval.

In the case of spherical expansion, the average p_t is determined simply by the ratio of total energy (assuming massless particles) and total entropy. Since the total energy is always conserved, it may be computed from the initial conditions. On the other hand, entropy may be produced during expansion, as in the case of formation of shocks, and should be computed at the freezeout as

$$S = \int_{\sigma^\mu} s \cdot d\sigma \ , \tag{64}$$

where $s^\mu = \sigma u^\mu$ is the entropy four-current

We will now consider details of the decoupling integrals only for the cylindrical expansion. The spherical case can be worked out similarly. With the longitudinal boost invariance, the freezeout condition is of the form $F(\tau,r)=\text{const}$. The equation for the decoupling surface is therefore of the form $\tau=\tau_d(r)$, and is independent of η. On the other hand, the surface element of any space-time surface with cylindrical symmetry may be written as

$$d\sigma^\mu = r\,dr\,d\phi(dr\,dz, \hat{\mathbf{k}}\,dt\,dr, \hat{\mathbf{r}}\,dt\,dz, 0) \ . \tag{65}$$

In the decoupling integrals for the densities, Eqs. (63) and (64), the integrands do not depend on the angle ϕ and we get

$$u \cdot d\sigma = 2\pi r\tau [dr \cosh(y_t) - d\tau \sinh(y_t)] d\eta \tag{66}$$

and

$$g^{r\mu} d\sigma_\mu = d\sigma^r = 2\pi r\tau d\tau\, d\eta \tag{67}$$

leading to

$$\frac{dP_t}{d\eta} = 2\pi \int_{\tau=\tau_d(r)} r\tau d\tau \{ w \sinh(y_t)\cosh(y_t) dr - [\epsilon \sinh^2(y_t) + P \cosh^2(y_t)] \} \tag{68}$$

for the total transverse momentum due to collective flow in the interval $d\eta$. The transverse rapidity is here y_t. The integral for $dS/d\eta$ can be similarly expressed. It should be noted that $\langle p_t \rangle = (dP_t/d\eta)/(3.7 dS/d\eta)$ holds only in the situation where the boost invariance is a reasonable approximation, as it probably is in the central region of heavy-ion collisions. Otherwise, the total transverse momentum of particles and of flow do not match in a finite rapidity interval.

In the above notation, the momentum distribution of the final particles has the form

$$\frac{dN}{dy\, d^2p_t} = \frac{g}{(2\pi)^3} \int_{\tau=\tau_d(r)} r\tau\, d\phi\, d\eta \frac{m_t \cosh(\eta - y) - p_t \cos(\phi)}{e^{p\cdot u/T_{\text{dec}}} \pm 1} \ , \tag{69}$$

where

$$p \cdot u = m_t \cosh(y_t)\cosh(\eta - y) - p_t \sinh(y_t)\cos(\phi) \ . \tag{70}$$

Equation (69) is still a three-dimensional integral. If the denominator of the integrand is expanded as a geometrical series, the integrals over ϕ and η can be performed. The resulting series contains products of modified Bessel functions and converges quite rapidly when $T_{\text{dec}} \lesssim 140$ MeV $\sim m_\pi$. Such an expansion is therefore useful for the numerical integration. Including the first term of the series corresponds to replacing distribution functions by Boltzmann distributions.

Equation (69) is useful for computing the transverse-momentum distribution including the effects of thermal motion. Included in the distribution of particles in this equation is this effect. To compute $\langle p_t \rangle$ we simply integrate over these distributions with weight p_t.

If one wants to study the spectra in the fragmentation region, the condition $v_z = z/t$ does not hold. Also the parametrization of the decoupling surface then depends upon η. As a result, the integration over η must be done numerically. Since we consider here the hydrodynamics in the boost invariant situation only, we will not go into details of decoupling in the fragmentation region.

In our analysis we shall choose surfaces of decoupling corresponding to temperatures in the range $T \sim 100$–140 MeV. These temperatures are high enough so that the massless pion-gas approximation is still probably good, but low enough so that most of the work involved in transverse expansion has been completed. To go to lower temperatures, the massless pion-gas approximation must

be discarded and a pion chemical potential must be introduced. The pion chemical potential is needed since pion number is conserved, and if masses are nonzero, entropy conservation no longer guarantees pion number conservation. With such corrections, it is possible to go to lower temperature than we present here. However, since there is little work done at such low temperatures, and since the entropy production at such a low temperature due to freezeout should be small, our computations should probably give a good approximation to a more thorough and complete treatment.

V. NUMERICAL METHODS

The numerical method which we use to integrate the hydrodynamic equations is a relativistic extension of the flux-corrected transport (FCT) algorithm proposed by Boris and Book.[47] The extension we employ has in large part already been developed by the Frankfurt group.[48,49] For completeness of this paper, and since in the case of rarefaction of shocks, the algorithm does not automatically yield unique solutions, we shall outline the method here.

Hydrodynamics is governed by a system of nonlinear partial differential equations which can possess discontinuous shock wave solutions. In such a situation, many types of solution algorithms, such as the method of characteristics, which has been used to consider the expansion of a quark-gluon plasma in the absence of phase transition, become inapplicable. In our problem, shock waves occur as rarefaction shocks when the matter transversely expands into vacuum. They are allowed to exist because of the first-order phase transition which is incorporated into the bag-model equation of state.[26]

When the rarefaction takes place into vacuum, there is no external constraint which would fix the velocity of the shock front or equivalently of the matter behind the shock. Instead they can vary within a certain range in which the conservation of energy and momentum fluxes can be satisfied.[26] From the point of view of the numerical calculations, it is useful to think of the ratio of entropy fluxes, $R=\sigma\gamma v/\sigma_0\gamma_0 v_0$, as a parameter which differentiates between the possible rarefaction shocks. In the actual computations, we must then be able to ensure the right amount of entropy production across the shock front in order to have the desired solution. We shall return to this question after introducing general features of the FCT technique.

The FCT is an algorithm that can be used to improve the shock-handling properties of many of the usual finite-difference transport schemes.[50,51] To illustrate the method, let us consider a continuity equation

$$\partial\rho/\partial t=-\partial(v\rho)/\partial x\ , \tag{71}$$

where ρ is a positive-definite density and v is a given velocity field. (In the actual problem of hydrodynamics, v is also defined by the system of equations.)

Assume next that a finite-difference approximation can be written in the conserving form

$$\rho_i^{n+1}=\rho_i^n-(F_{i+1/2}-F_{i-1/2})\ , \tag{72}$$

where the F's are called transportive fluxes and are functions of ρ and v at one or several time levels t^n and several spatial grid points x_i. The explicit form of these fluxes depends on the particular difference scheme which is used. In general, high-order schemes provide accurate solutions when ρ is smoothly varying but produce erroneous oscillations near steep gradients. These ripples are due to numerical dispersion (phase error) characteristic to high-order methods and can result in a failure of numerical computation. On the other hand, low-order methods do not suffer from these oscillations, but instead from extensive numerical diffusion which tends to smooth out any shock fronts.

The basic idea of the FCT is to construct the transportive fluxes F as a weighted average of fluxes F^L and F^H which are computed according to some low-order and higher-order scheme, respectively. The weighting is done, point by point, in such a way that the higher-order fluxes will be used only to the extent where no dispersive ripples arise. The procedure to do this is called flux correcting or flux limiting and is the key point of the method.

In practice the computation proceeds through the following steps.[5] At the first step, called the convective or transport stage, one computes a tentative solution $\bar{\rho}^{n+1}$ using the low-order fluxes $F=F^L$ in Eq. (72). Then the antidiffusive fluxes

$$A=F^H-F^L \tag{73}$$

are defined in order to cancel the strong diffusion of the transport stage. If these fluxes were used as such, the result would be that of the higher-order scheme. However, in order to avoid the undesirable ripples, these antidiffusive fluxes are corrected or limited to

$$A^C_{i\pm1/2}=C_{i\pm1/2}A_{i\pm1/2},\quad 0\le C_{i\pm1/2}\le 1\ , \tag{74}$$

and the final values of ρ at the time level t^{n+1} are calculated as

$$\rho^{n+1}=\bar{\rho}^{n+1}-(A^C_{i+1/2}-A^C_{i-1/2})\ . \tag{75}$$

The criterion for the flux correction is such that ρ^{n+1} as calculated from Eq. (75) must not have extrema which are not already present in $\bar{\rho}^{n+1}$. This criterion appears to be very efficient in determining the correct balance between the low- and high-order terms so that no dispersive ripples appear. It also has the important property that the antidiffusive stage, Eq. (75), maintains the positivity of ρ. Further, it is seen that the whole scheme conserves the total quantity associated with the density ρ, any flux subtracted somewhere is added somewhere else. Several flux limiters and specific algorithms using FCT have been developed since its invention.[47,50] The particular algorithm which we are using is the one introduced by Boris and Book in Ref. 47 and is called SHASTA. In this early application of FCT, the high-order scheme is not explicitly stated. Instead, the antidiffusive fluxes are defined in such a way that for the uniform velocity case they cancel the diffusion caused by the low-order scheme at the transport stage. The algorithm may however be cast in the general format presented above, and doing so the version of SHASTA which we have used is defined in the following way.

The low-order fluxes are

$$F^L_{i+1/2}=\tfrac{1}{2}[(1-Q_i)^2\rho^n_i-Q_i{}^2\rho^n_{i+1}]\ , \tag{76}$$

where

$$Q_i=(\tfrac{1}{2}+\epsilon_i)/(1+\epsilon_{i+1}-\epsilon_i) \tag{77}$$

and

$$\epsilon_i=v^{n+1/2}_i\frac{\delta t}{\delta x}\ . \tag{78}$$

The quantities δx and δt are the grid differences and $v^{n+1/2}_i$ are the centered velocities. The positivity of ρ at the transport stage

$$\bar\rho^{\,n+1/2}_i=\rho^n_i-F^L_{i+1/2}+F^L_{i-1/2} \tag{79}$$

is maintained if $|\epsilon_i|<1/2$.

The transport stage of SHASTA can be represented pictorially. In Fig. 4, the density ρ is shown at different steps in the transport computation. The transport is initiated by representing the density with straight lines connecting the ρ values of adjacent grid points, Fig. 4(a). The fluid elements defined by the straight-line sections are then moved to new positions $x_i+\delta t v^{n+1/2}_i$, Fig. 4(b), in such a way that the area under the curve is conserved. Finally, the new values of the density, $\bar\rho^{\,n+1/2}_i$, are computed by assigning the part of the fluid element left from $x_i+\frac{1}{2}\delta x$ to the cell $(x_i-\frac{1}{2}\delta x,\ x_i+\frac{1}{2}\delta x)$ leading to Eqs. (78) and (79). Clearly $F^L_{i+1/2}$ is the flux through the cell boundary $x_i+\frac{1}{2}\delta x$.

As already mentioned, the high-order fluxes are not used explicitly. Instead, the antidiffusive flux is defined directly as

$$A_{i+1/2}=\eta(\bar\rho^{\,n+1}_{i+1}-\bar\rho^{\,n+1}_i)\ , \tag{80}$$

where $\eta=\frac{1}{8}+\epsilon^2/2$ is called the antidiffusive coefficient. This form is obtained by demanding that the diffusion of the transport stage is completely canceled in the case of constant velocity. The corresponding high-order flux $F^H_{i+1/2}=F^L_{i+1/2}+A_{i+1/2}$ is a rather complicated four-point formula which we need not be concerned about since only the quantities A are needed in calculations.

The crucial step of flux correcting is achieved in the SHASTA algorithm by a simple formula

$$A^C_{i+1/2}=S\max[0,\min(\Delta_{i+1/2}S,|A_{i+1/2}|,S\Delta_{i+3/2})]\ , \tag{81}$$

where

$$\Delta_{i+1/2}=\bar\rho^{\,n+1}_{i+1}-\bar\rho^{\,n+1}_i \tag{82}$$

and

$$S=\mathrm{sgn}(\Delta_{i+1/2})\ . \tag{83}$$

It can be verified that this equation is a realization of the general flux-limiting criterion. It ensures that the corrected antidiffusive fluxes cannot push the final values of the density at x_i

$$\rho^{n+1}_i=\bar\rho^{\,n+1}_i-A^C_{i+1/2}+A^C_{i-1/2} \tag{84}$$

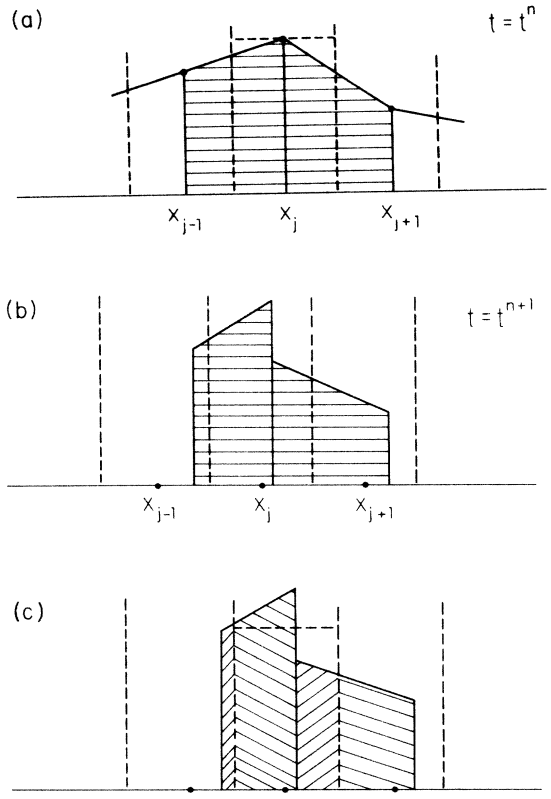

FIG. 4. The transport stage of SHASTA. Two fluid elements (solid walls) and three grid cells (dashed walls) are shown. (a) Initial location of fluid elements. (b) Location and shape after transport. (c) Determination of new values of ρ on the grid.

beyond the values $\bar\rho^{\,n+1}_{i\pm1}$ at the neighboring points.

We now turn to a discussion of handling the shocks. Usually the propagation of shocks is accompanied by entropy production which means that in the solutions of the ideal fluid flow, the shocks appear as discontinuities. Physically, the shocks have a finite thickness and even though the perfect-fluid description may be a good approximation in the smooth regions of flow, viscosity becomes important in the region of shocks where the velocity gradients are very large. When a numerical method is applied to solve ideal-fluid equations in which shocks arise it must either be capable of fitting the shock discontinuities or of producing enough entropy to allow the formation of shocks.[52] The standard procedure in the second alternative is the inclusion of an artificial viscosity with which shocks can be resolved reliably in a few grid points.

In the FCT technique, the entropy production is due to the residual diffusion which is left uncanceled because of the flux correction. It seems to have the remarkable property of being able to produce the right amount of entropy within one or two grid points when the strength of the shock is fixed externally for example by appropriate boundary conditions. We have tested it for relativistic compression shocks up to shock velocities, $v_{\mathrm{shock}}=0.95$

(corresponding to piston velocity 0.9) when the entropy flux ratio $R=1.34$, and it reproduces a sharp shock front with only transient ripples.

In the case of rarefaction shocks which occur (if allowed by the equation of state) when matter expands into the vacuum, there is no external fixing of the strength of the shock from the boundary conditions because the continuous rarefaction wave which joins the shock to vacuum can adjust to any shock strength. What solution is then produced by a given numerical algorithm (if any) depends on the diffusive and viscous properties of that algorithm.

In the case of SHASTA the amount of residual diffusion can be controlled by the antidiffusion coefficient η in Eq. (80). To ensure the condition $|\epsilon| < \frac{1}{2}$ one usually takes $\delta t/\delta x < \frac{1}{2}$. Then $\frac{1}{2}\epsilon^2 \ll \frac{1}{8}$ for most parts of the fluid flow, and one can disregard the velocity dependence of η and take $\eta = \frac{1}{8}$. It turns out that even with the extra diffusion resulting from this approximation, the SHASTA algorithm produces a rarefaction shock with almost no entropy production. For these solutions, the matter behind the shock is supersonic and they can be argued to be unstable by general stability criteria.[53,54] On the other hand, subsonic shocks are not stationary in the expansion to the vacuum.[23] Thus the desirable solution, usually referred to as the Jouguet shock, corresponds to a situation where the velocity of the outflowing matter with respect to the shock front equals the velocity of sound c_s. It also corresponds to a maximum entropy production for a given initial density of matter.[26]

We have not tried to implement the condition of sonic velocity since this would lead to very complicated procedures for the actual calculations with different initial conditions. Instead we assure that the code produces enough entropy to be able to find the Jouget shock. In SHASTA this can be achieved at least in two different ways: artificial viscosity can be added to the algorithm, or the value of η can be changed. We have tested both possibilities for a one-dimensional case with constant initial density where we know the shock part of the solution.[26] Both procedures give the desired result and at least in the one-dimensional case, equivalent results. The value η needs to be reduced below 0.1 to achieve the maximum entropy shock. For the calculations of spherical and cylindrical expansion we have used the value 0.08.

VI. SUMMARY

In this paper we have discussed the limits of the validity of a hydrodynamical description of high-energy hadronic collision processes. We have formulated the hydrodynamical equations for high-energy nucleus-nucleus collisions, and for fluctuations in $\bar{p}p$ collisions. We have outlined the methods required to extract transverse-momentum distributions of hadrons. We also have presented an explicit method to treat systems with phase transitions using the flux-corrected transport hydrodynamic code of Boris and Book. In a later paper, we shall present solutions of these equations and explicitly determine transverse-momentum distributions.

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