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# Anomalies in conservation laws in the Hamiltonian formalism

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We present an analysis of anomalies and its relation with the Ehrenfest theorem from the Hamiltonian point of view. It is shown that when an operator becomes anomalous it is due to the fact that it does not keep invariant the domain of definition of the Hamiltonian, proving a theorem which states that an operator which keeps invariant such a domain cannot have anomalies.

In general, we say that an anomaly appears whenever a symmetry of a classical field theory is not preserved in the corresponding quantum theory. The history of anomalies is very long; in fact, the existence of an axial anomaly and its relevance for current algebra and the PCAC (partial conservation of axial-vector current) hypothesis was discovered by studying certain Feynman diagrams and further in the analysis of the $\pi^0 \rightarrow 2\gamma$ decay process and in the Schwinger model.[1] Since then, similar anomalies have been discovered in non-Abelian gauge theories. We now know that such anomalies lead many chiral theories to be inconsistent, which provides strong constraints for model building.

Though this phenomenon is now well known, it remains somewhat mysterious why the full set of classical symmetries cannot be preserved in any of the many possible quantization schemes; in fact, in the Hamiltonian version we see the origin of the anomaly as a consequence of the spectral flow induced as one orbits the gauge configuration space once.[2] Then, after being regularized, some charges depend on the background gauge potential, and this causes the anomaly.[3] But in the Hamiltonian formalism, we also know that the Ehrenfest theorem ensures that the classical equations of motion remain true at the quantum level once we take expectation values, thus preventing the appearance of anomalies; then it is not clear why the Ehrenfest theorem is broken when an anomaly is present. Another point not yet fully clarified is which kind of operators are expected to become anomalous.

In this Rapid Communication we present an analysis of the anomalies and their relation with the Ehrenfest theorem from the Hamiltonian point of view. It is shown that when an operator becomes anomalous it is due to the fact that it does not keep invariant the domain of definition of the Hamiltonian. Then a theorem is obtained which states that an operator which keeps invariant such a domain cannot have anomalies. We analyze some quantum-mechanical anomalies and the axial anomaly in the vector, axial-vector, and chiral Schwinger models, showing that chiral electrodynamics is an ill-defined theory.

Let us start with a classical field theory given by the Hamiltonian

$$H=\int [dx]H(\Pi_\Phi,\Phi,\Pi_A,A,\psi^\dagger,\psi) \ . \tag{1}$$

In order to quantize it, we shall impose the canonical commutation (anticommutation) relations among the bosonic (fermionic) fields and their canonical momenta. In doing so, we need to fix, at the same time, the domain $D_H$ of definition of the Hamiltonian $H$, such that acting on $D_H$, $H$ is a self-adjoint operator and keeps $D_H$ invariant ($D_H$ is dense in $L^2$):

$$\begin{aligned} &H=H^\dagger \ , \\ &HD_H=D_H \ . \end{aligned} \tag{2}$$

Then the Schrödinger equation gives the time evolution of any state belonging to the Hilbert space as

$$\xi[A,\psi,\Phi,t]=e^{-iHt}\xi[A,\psi,\Phi,0] \ , \tag{3}$$

where $\hbar=1$. Let $B$ be an operator acting on the Hilbert space. The time derivative of the expectation value of such an operator in the physical state given by $\xi$ is

$$\frac{d}{dt}\langle B\rangle = i[(H\xi\,|\,B\xi)-(\xi\,|\,BH\xi)] \ , \tag{4}$$

where $(\xi\,|\,\xi')$ means the scalar product defined in our Hilbert space of the vector states $\xi$ and $\xi'$. Normally, one says that the right-hand side of (4) is just the expectation value on the state $\xi$ of the commutator of $H$ with $B$. Then one writes

$$\frac{d}{dt}\langle B\rangle = i\langle [H,B]\rangle \ , \tag{5}$$

that is, the Heisenberg equation. The problem is that Eq. (5) is not true, in general, since in going from (4) to (5) we have used the fact that $H$ is self-adjoint; but this is true only when $H$ acts on vector states $\xi\in D_H$, so we are implicitly assuming that

$$\forall \xi\in D_H \Longrightarrow B\xi\in D_H \ , \tag{6}$$

which cannot be the case. So, in general, we would have

$$\frac{d}{dt}\langle B\rangle = i\langle [H,B]\rangle + i[(H\xi\,|\,B\xi)-(\xi\,|\,HB\xi)] \ , \tag{7}$$

where the "anomalous" second term on the right-hand side of (7) is different from zero only for those states $\xi\in D_H$ such that $B\xi\notin D_H$. Then we can see that the anomalous equation for the operator $B$ comes because it does not keep $D_H$ invariant. So we can write (7) as

$$\frac{d}{dt}\langle B\rangle = i\langle [H,B]\rangle + \langle \mathcal{A}\rangle \ , \tag{8}$$

where

$$\langle \mathcal{A}\rangle = i[(H\xi\,|\,B\xi)-(\xi\,|\,HB\xi)] \ . \tag{9}$$

 

Then a theorem emerges: Any operator which leaves invariant the domain of definition of the Hamiltonian cannot have anomalies.

This point of view is important especially for those quantum-mechanical problems where we are dealing with compact-support spaces, and in particular, for gauge quantum field theories. In both cases we must identify our wave functions (or functionals) at different points, that give nontrivial boundary conditions to be imposed on the vectors belonging to $D_H$.[4] Let us see that point in some examples.

The first is the simplest one that one can consider in quantum mechanics. Consider a classical particle moving on a circumference of unit radius, whose Lagrangian is given by

$$L=\tfrac{1}{2}m\dot{\theta}^2 \; . \tag{10}$$

The classical equations of motions are

$$\dot{P}_\theta=0,\; P_\theta=m\dot{\theta} \; . \tag{11}$$

The quantum Hamiltonian corresponding to that problem is

$$H=-\frac{1}{2m}\frac{d^2}{d\theta^2} \tag{12}$$

defined on

$$D_H=\{f\in L^2[0,2\pi]\,|\,f(0)=f(2\pi)\;;\; f'(0)=f'(2\pi),f''\in L^2[0,2\pi]\} \; . \tag{13}$$

Taking a general state defined as

$$\xi(\theta)=\sum_{n=-\infty}^{+\infty} c_n\varphi_n(\theta) \; , \tag{14}$$

where $\varphi_n(\theta)$ are the eigenfunctions of $H$, it is easy to obtain

$$\frac{d}{dt}\langle\theta\rangle=-\sum_n\sum_{k\neq n}\frac{c_k^*c_n}{2m}(n+k) \times\exp[-i(n^2-k^2)t/(2m)] \; , \tag{15}$$

$$i\langle[H,\theta]\rangle=\langle P_\theta\rangle=\sum_k k\,|c_k|^2/m \; , \tag{16}$$

$$\langle\mathcal{A}\rangle=-\sum_k\sum_n\frac{c_k^*c_n}{2m}(n+k)\exp[-i(n^2-k^2)t/(2m)] \; ,$$

where in (16) we have used the extension of $H$ to the entire Hilbert space. It is clear now that it is Eq. (8) which is satisfied rather than Eq. (5). The problem here arises because given a periodic function $f(0)=f(2\pi)$, then $\theta f(\theta)$ is no longer a periodic function; in fact, the anomaly can be written as a surface term

$$\langle\mathcal{A}\rangle=[\theta J(\theta)]_{2\pi}^{0} \; ,$$
$$J(\theta)=\frac{i}{2m}\left[\xi^*(\theta)\frac{d}{d\theta}\xi(\theta)-\left(\frac{d}{d\theta}\xi^*(\theta)\right)\xi(\theta)\right] \; . \tag{17}$$

It must be noted that in order to avoid the anomaly we cannot define

$$D_H=\{f\in L^2[0,2\pi]\,|\,f(0)=f(2\pi)=0 \; , \; f'(0)=f'(2\pi)=0\}$$

since $P_\theta$ would then be symmetric but not self-adjoint.

The above problem is similar to the case of a charged particle moving on a two-torus and coupled to an Abelian gauge field. This problem was analyzed by Manton.[5] As in Ref. 5, we suppose the mass and the charge of the particle to be unity and specify the gauge field, saying that it acts as a magnetic field normal to the torus and that its total flux must be an integer multiple of $2\pi$. After fixing the gauge, this leads to the Hamiltonian

$$H=-\frac{1}{2}\frac{\partial^2}{\partial x^2}-\frac{1}{2}\left(\frac{\partial}{\partial y}+2\pi i x\right)^2 \; , \tag{18}$$

where $x\in[0,1]$ and $y\in[0,1]$ and the points $x=0$ and $x=1$ ($y=0$ and $y=1$) must be identified. The problem now arises because $x$ is an angular variable, whereas $H$ depends on $x$ and $x^2$, so we cannot define $D_H$ as periodic functions at the points $(0,y)$, $(1,y)$ and $(x,0)$, $(x,1)$, since then Eqs. (2) are not satisfied. Then we must identify the functions of $D_H$ at the points $(x,0)$ and $(x,1)$, $(0,y)$ and $(1,y)$ up to a unitary general transformation

$$\begin{aligned}&\xi(0,y)=e^{if(y)}\xi(1,y)\;,\\&\partial_x\xi(0,y)=e^{if(y)}\partial_x\xi(1,y)\;,\\&\xi(x,0)=e^{ig(x)}\xi(x,1)\;,\\&\partial_y(x,0)=e^{ig(x)}\partial(x,1)\;,\end{aligned} \tag{19}$$

where $f(y)$ and $g(x)$ are functions to be determined imposing (2). It is easy to obtain $f(y)=k_1+2\pi y$ and $g(x)=k_2$, where $k_1$ and $k_2$ are constants and therefore irrelevant in our discussion; so we shall take them to be zero in what follows. Now defining $D_H$ as in (19) we have a well-defined Hamiltonian, but the price we must pay is that again $P_y$ does not keep $D_H$ invariant,

$$P_y\xi(x,y)\,|_{x=0}\neq e^{if(y)}P_y\xi(x,y)\,|_{x=1} \; , \tag{20}$$

and will have anomalous equations. Related to the definition of $D_H$ is the problem of the spectral flow[5] which causes a permutation in the eigenfunctions by the operation of circling the torus in the $x$ direction. In any case, it then is easy to realize on a general state

$$\frac{d}{dt}\langle P_y\rangle=\pi\int_0^1\left[\frac{\partial}{\partial x}\xi^*(x,y)\xi(x,y)-\xi^*(x,y)\frac{\partial}{\partial x}\xi(x,y)\right]_{x=0}dy \; , \tag{21}$$

so the anomaly is a surface term due to the boundary conditions (19) that we must impose on $D_H$. In a similar way, it is obvious that the $x$ and $y$ operators will also have anomalous equations of motion since, as in the previous example, they are coordinates on a torus.

Finally, let us revise the Schwinger model; as in Ref. 3 the space will be a circle of length $2\pi$, since then it is easy to clarify certain properties of the model without changing the physics with respect to the usual versions in any significant way. Furthermore, it is straightforward to take the space as a circle of length $2\pi L$ and recover the standard Schwinger model as $L\rightarrow\infty$. Working in the temporal gauge, since in the Hamiltonian version the operators are defined on hypersurfaces at constant time, then doing a topologically trivial transformation, we can make $A_1(x)$ constant; by a further topologically nontrivial transformation $[\Lambda(0)=\Lambda(2\pi)+2\pi n]$ we can make $A_1$ vary in the interval [0,1] where the points $A_1=0$ and $A_1=1$ are equivalent, making in this case the gauge field itself an angular variable. The Hamiltonian is

$$H=-\frac{1}{4\pi}\frac{\partial^2}{\partial A_1{}^2}+i\int_0^{2\pi}\bar{\psi}\gamma^\mu(\partial_\mu+ieA_1)\psi dx\ , \quad (22)$$

or equivalently, absorbing the coupling constant in the gauge field and transforming to momentum space

$$H=-\frac{e^2}{4\pi}\frac{\partial^2}{\partial A_1{}^2}+\sum_P(a_{1,P}^\dagger a_{1,P}-a_{2,P}^\dagger a_{2,P})(P+A_1)\ , \quad (23)$$

where

$$\psi_j(x)=\frac{1}{\sqrt{2\pi}}\sum_k a_{j,k}e^{ikx}\ \ (j=1,2)\ . \quad (24)$$

Again, since $A_1\in[0,1]$ and we want Eqs. (2) to be satisfied, we must be careful with the boundary conditions on the functions of $D_H$. As before we cannot define periodic boundary conditions at $A_1=0$ and $A_1=1$ since $H$ has a term linear with $A_1$, so periodicity up to a phase must be imposed on the functionals of $D_H$:

$$\xi[A_1,\psi^\dagger,\psi]\,|_{A_1=0}=e^{iF[\psi^\dagger,\psi]}\xi[A_1,\psi^\dagger,\psi]\,|_{A_1=1}\ , \quad (25)$$

$$\partial_{A_1}\xi[A_1,\psi^\dagger,\psi]\,|_{A_1=0}=e^{iF[\psi^\dagger,\psi]}\partial_{A_1}\xi[A_1,\psi^\dagger,\psi]\,|_{A_1=1}\ , \quad (26)$$

where $F$ is a functional to be determined imposing (2). Let us define the operators

$$Q_P^{\pm}=a_{1,P}^\dagger a_{1,P}\pm a_{2,P}^\dagger a_{2,P}\ ,$$
$$Q=\sum_P Q_P^+\ , \quad (27)$$
$$Q_5=\sum_P Q_P^-\ .$$

If Eqs. (2) are satisfied, then

$$\sum_P Pe^{-iF}Q_P^- e^{iF}\xi[A_1=1]=\sum_P(P+1)Q_P^-\xi[A_1=1]\ ,$$

that is,

$$e^{-iF}Q_P^- e^{iF}=Q_{P-1}^-\ . \quad (28)$$

This implies that $e^{-iF}$ acting on $\xi[A_1,\psi^\dagger,\psi]\,|_{A_1=0}$ decreases the momenta of all particles by one unit. Then it is a fact that $H$ is linear on a cyclic coordinate which forces the existence of nontrivial periodicity on the amplitudes; this fact is related also to the spectral flow which appears as analyzed in Ref. 3.

The physical states will be linear combinations, with complex amplitudes depending on $A_1$, of those states subject to Fermi statistics, in which each energy level is either filled (there is a particle in such a state) or empty (there is no particle), and in which almost all (except a finite number) of the negative-energy levels are filled, and almost all of the positive-energy levels are empty. Let us see one of these states as represented in Fig. 1, where we have regularized taking a cutoff on the momenta $(-k\le p\le k)$. Since $e^{iF}$ must be a unitary operator, by doing the momentum regularization we shall identify

$$a_{2,k+1}^\dagger=a_{1,-k}^\dagger\ ,$$
$$a_{2,k}^\dagger=a_{1,-(k+1)}^\dagger\ .$$

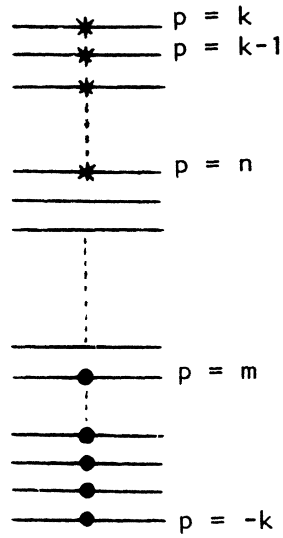

FIG. 1. An unexcited basis state. The dots (stars) mean a level filled by left-handed (right-handed) particles.

Then it is easy to find

$$Q\xi\,|_{A_1=0}=e^{iF}Q\xi\,|_{A_1=1}\ ,$$
$$Q_5\xi\,|_{A_1=0}=e^{iF}Q_5\xi\,|_{A_1=1}+2e^{iF}\xi\,|_{A_1=1}\ , \quad (29)$$

showing that $QD_H\in D_H$, whereas $Q_5D_H\notin D_H$, and correspondingly, $Q$ will have a normal equation of motion and $Q_5$ has an anomaly.

The $Q_5$ anomaly can be evaluated and it yields a surface term as a consequence of the quasiperiodic boundary conditions, explicitly,

$$\langle\xi|\mathcal{A}|\xi\rangle=-i\frac{e^2}{\pi}\sum_P[\xi^*(A_1,a_{1,P},\ldots)\partial_{A_1}\xi(A_1,a_{1,P},\ldots)]$$
$$=2\langle\xi|E^{\rm tr}|\xi\rangle\ , \quad (30)$$

where $E^{\rm tr}$ is the transverse part of the electric field.

Here the anomaly results from the gauge invariance which forces the equivalence of some configurations. It is then impossible to have a well-defined self-adjoint Hamiltonian and an axial-vector current which keeps invariant the domain of definition of the Hamiltonian.

In the case of the axial Schwinger model we can repeat the previous analysis and see that again we must define quasiperiodic boundary conditions for the functionals of

$D_H$, but now

$$e^{-iF}a^\dagger_{1,P}a_{1,P}e^{iF}=a^\dagger_{1,P-1}a_{1,P-1} ,$$
$$e^{-iF}a^\dagger_{2,P}a_{2,P}e^{iF}=a^\dagger_{2,P+1}a_{2,P+1} . \tag{31}$$

So, after changing the role of the vector and axial-vector currents, everything remains analogous, that is,

$$\frac{d}{dt}\langle Q_5\rangle=i\langle[H,Q_5]\rangle ,$$
$$\frac{d}{dt}\langle Q\rangle=i\langle[H,Q]\rangle+\langle\mathcal{A}\rangle . \tag{32}$$

Then, it is clear that in the chiral Schwinger model it is impossible to define $D_H$ in such a way that Eqs. (2) are satisfied, since Eqs. (28) and (31) are not compatible; correspondingly, the chiral Schwinger model is an ill-defined theory.[3]

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