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Field theory of paths with a curvature-dependent term

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A novel theory of paths, in which configurations are weighted according to their curvature as well as their length, is developed.

Theories of manifolds $\mathcal{M}$ embedded in higher-dimensional spaces $\mathcal{S}$ are a subject of much current interest. The guiding principle in these theories is invariance under arbitrary reparametrizations of the coordinates used to label $\mathcal{M}$.

Usually, a further assumption is made—that only intrinsic properties of $\mathcal{M}$ enter; then the corresponding action must be proportional to the invariant volume of $\mathcal{M}$. It is conceivable, however, that the manner in which $\mathcal{M}$ is embedded in $\mathcal{S}$ is of consequence, for there are always terms which depend upon $\mathcal{M}$ in an extrinsic fashion, and yet are still general coordinate invariant. Perhaps it is worthwhile to consider the most general theory possible, as long as it respects other physical principles such as renormalizability.

For the case in which $\mathcal{M}$ is a surface, such an "extrinsic" theory has been studied by Polyakov,[1] and separately by Helfrich, and Peliti and Leibler.[2] In this Rapid Communication I follow their lead to develop an analogous theory of paths.

In the theory I propose, the action of a path involves both its curvature and its length. The coupling of the curvature term is dimensionless, so I begin by showing that this coupling is asymptotically free. I then assume that the dimension d of $\mathcal{S}$ is very large, and calculate in a large-d expansion.

This theory of paths might be relevant to polymer physics. Models with a term involving the square of the curvature are well known in polymer systems,[3] they are applicable to long chains for which the effects of self-repulsion can be neglected. It is not difficult to imagine how terms in the curvature squared can arise in an expansion of the free energy F. If k is the curvature and $F\sim(a+bk^2+\cdots)^{1/2}$, for small k $F\sim a+bk^2/2+\cdots$. In contrast, for the model I study, F has a term which is linear in the curvature. This can only occur at an isolated point in the phase diagram where $a\sim 0$ so $F\sim\sqrt{b}k$. Thus, at best my model is only applicable to polymers at those critical points for which $a\sim 0$. I do not know whether such points can be reached in physical systems.

In this paper my interest in the model is merely field theoretic, for as a field theory it is rather curious. To begin with, it is unusual to find asymptotic freedom in one dimension. The quantum geometry of the model is also distinctive—to calculate the small fluctuations about some background path, it is necessary to assume that the path has nonzero curvature along its entire length. This assumption is justified by the large-d expansion, for the paths which dominate the functional integral at large d are uniformly curved. I should also mention that while they are not apparent at first, there are close similarities between this theory and a nonlinear σ model with long-range interactions.[4]

Let $\bar{x}$ represent a path in d flat, Euclidean dimensions, $\mathcal{S}=R^d$. The arc length s is defined by the relation $(d\bar{x}/ds)^2=1$. Then any action formed from just $\bar{x}$ and s is automatically reparametrization invariant. I choose

$$S=\frac{1}{\alpha}\int k\,ds+m\int ds\ , \tag{1}$$

where k is the curvature, $k^2=(d^2\bar{x}/ds^2)^2$. The coupling α is dimensionless, and m is a mass, $m\geq 0$.

This action is uniquely determined by two requirements. The first is that the action density (over ds) be a polynomial in k or like terms. The other is that there be no couplings with dimensions of inverse mass, which will ensure renormalizability. The effects of couplings which do not appear to be renormalizable will be discussed later.

The similarities between Eq. (1) and the models of Polyakov[1] and Helfrich[2] are obvious. For surfaces, the term with the dimensionless coupling involves the square of the (mean) curvature; there is also a term for the area of the surface.

With a given parametrization $\bar{x}=\bar{x}(t)$,

$$S=\frac{1}{\alpha}\int\frac{[\dot{\bar{x}}^2\ddot{\bar{x}}^2-(\dot{\bar{x}}\cdot\ddot{\bar{x}})^2]^{1/2}}{\dot{\bar{x}}^2}dt+m\int(\dot{\bar{x}}^2)^{1/2}dt\ , \tag{2}$$

$\dot{\bar{x}}=d\bar{x}/dt$, $\ddot{\bar{x}}=d^2\bar{x}/dt^2$. S has a local gauge symmetry—it is invariant under $t\rightarrow f(t)$, where $f(t)$ is an arbitrary differentiable function of t. Notice also that the curvature term is invariant under scale transformations, $\bar{x}(t)\rightarrow\sigma\bar{x}(t)$; since $s\rightarrow\sigma s$, the mass term is not. For each value of t, the model has $d-1$ degrees of freedom: $\bar{x}(t)$ contributes d, minus one for the gauge symmetry. For paths in a plane ($d=2$) the single degree of freedom is trivial, so I presume that $d\geq 3$.

Of course when $\alpha^{-1}=0$, S is the action for a relativistic particle, so that t can be viewed as the time. This connection is lost when $\alpha^{-1}\neq 0$, but the model is still perfectly sensible as a type of statistical mechanics, as long as t is treated just as a parameter.

While t is not really a time, a canonical analysis is still of use. The Lagrangian $\mathcal{L}(S=\int\mathcal{L}dt)$ depends on $\dot{\bar{x}}$ and $\ddot{\bar{x}}$, so two canonical momenta must be introduced:[5]

$$\bar{p}_1=\frac{\delta\mathcal{L}}{\delta\dot{\bar{x}}}-\frac{d}{dt}\frac{\delta\mathcal{L}}{\delta\ddot{\bar{x}}},\ \bar{p}_2=\frac{\delta\mathcal{L}}{\delta\ddot{\bar{x}}}\ . \tag{3}$$

$\bar{p}_1$ is conjugate to $\dot{\bar{x}}$, and $\bar{p}_2$ to $\ddot{\bar{x}}$. The (gauge-dependent) equations of motion are $d\bar{p}_1/dt=0$. The Hamiltonian

$\mathcal{H}$ is

$$\mathcal{H} = \bar{p}_1 \cdot \dot{\bar{x}} + \bar{p}_2 \cdot \ddot{\bar{x}} - \mathcal{L} \ ; \tag{4}$$

$\mathcal{H}$ vanishes identically, as a familiar consequence of general coordinate invariance. While in principle $\mathcal{H}$ is a function of $\dot{\bar{x}}$, $\bar{p}_1$, and $\bar{p}_2$,[5] in this theory they are not all independent variables, and must satisfy the constraints $\bar{p}_1 \cdot \dot{\bar{x}} = m(\dot{\bar{x}}^2)^{1/2}$ and $\bar{p}_2 \cdot \dot{\bar{x}} = 0$.

Initially, one might wonder whether the model has any interesting dynamics, for with fixed end points the classical path which minimizes the action of Eq. (1) is just a straight line.

This overlooks an essential property of the model—its gauge invariance. It is convenient to choose a gauge in which the parameter t is the arc length s. Henceforth I work in this arc-length gauge; then $\dot{\bar{x}}^2 = 1$, from which $\dot{\bar{x}} \cdot \ddot{\bar{x}} = 0$, and so on.

In the arc-length gauge, nothing depends on m, at least as far as quantities which are local in s are concerned. The only place in which m enters is in the total free energy, and there its appearance is a trivial factor of m times the total arc length. (This differs from the theory of surfaces,[1,2] where the surface tension does appear as a term in a local Lagrangian for the conformal mode.)

For this reason, in the arc-length gauge the Lagrangian can be taken as

$$\tilde{\mathcal{L}} = \frac{1}{\alpha}(\ddot{\bar{x}}^2)^{1/2} + \frac{\omega}{2\alpha}(\dot{\bar{x}}^2 - 1) \ . \tag{5}$$

$\omega = \omega(s)$ is a constraint field which enforces the gauge condition. In contrast with Eq. (1), which has two couplings of which one has the dimensions of mass, the local Lagrangian of Eq. (5) has only a single dimensionless coupling constant.

Equation (5) has the form of a nonlinear σ model with a peculiar square-root-type kinetic energy for a d-component σ field $\dot{\bar{x}}(s)$. In the language of the σ model, straight paths correspond to an ordered state, $\dot{\bar{x}}(s)$ = a constant vector. Thus, the analysis of the large-d expansion, which shows that the dominant paths are curved, corresponds to a disordered vacuum state for the σ model. It is well known that a σ model with the usual kinetic energy is always disordered at or below two dimensions;[4] for the square-root kinetic energy of Eq. (5), this happens in one dimension.

To calculate the renormalization-group equations to one-loop order, I use a background-field method—with $\bar{x} = \bar{x}_{\rm cl} + \bar{x}_{\rm qu}$, $\omega = \omega_{\rm cl} + \omega_{\rm qu}$, $\tilde{\mathcal{L}} = \tilde{\mathcal{L}}_{\rm cl} + \tilde{\mathcal{L}}_1 + \tilde{\mathcal{L}}_2 + \cdots$, $\tilde{\mathcal{L}}_{\rm cl}$ is just $\tilde{\mathcal{L}}$ with $\bar{x}_{\rm cl}$ and $\omega_{\rm cl}$ replacing $\bar{x}$ and ω. $\tilde{\mathcal{L}}_1$ generates the equations of motion for $\bar{x}_{\rm cl}$ and $\omega_{\rm cl}$, and is assumed to vanish. $\tilde{\mathcal{L}}_2$ is

$$\tilde{\mathcal{L}}_2 = \frac{1}{2\alpha}\frac{1}{(\ddot{\bar{x}}_{\rm cl}^2)^{1/2}}\left[\ddot{\bar{x}}_{\rm qu}^2 - \frac{1}{\ddot{\bar{x}}_{\rm cl}^2}(\ddot{\bar{x}}_{\rm cl} \cdot \ddot{\bar{x}}_{\rm qu})^2\right] + \frac{1}{2\alpha}\omega_{\rm cl}\dot{\bar{x}}_{\rm qu}^2 + \frac{1}{\alpha}\omega_{\rm qu}\dot{\bar{x}}_{\rm cl} \cdot \dot{\bar{x}}_{\rm qu} \ . \tag{6}$$

In order for $\tilde{\mathcal{L}}_2$ to be well defined, it is necessary to assume that the background field is everywhere curved—$\ddot{\bar{x}}_{\rm cl}^2(s) \neq 0$ for all s. Paths which have isolated points at which $\ddot{\bar{x}}_{\rm cl}^2 = 0$ appear to have infinite action, and so make no contribution to the functional integral. As a σ model, the assumption that $\ddot{\bar{x}}_{\rm cl}^2 \neq 0$ means that the theory is only sensible in a disordered phase; i.e., at or below its critical dimension of one.

To one-loop order in the arc-length gauge, the contribution of ghosts is independent of $\dot{\bar{x}}_{\rm cl}$ and $\omega_{\rm cl}$, and so can be ignored. Integrating over $\omega_{\rm qu}$ just gives factors of $\delta(\dot{\bar{x}}_{\rm cl} \cdot \dot{\bar{x}}_{\rm qu})$ in the measure of the functional integral. This δ function can be rewritten as a term in the Lagrangian $\sim(\dot{\bar{x}}_{\rm cl} \cdot \dot{\bar{x}}_{\rm qu})^2/(2\alpha\xi)$, in the limit $\xi \to 0$. The remaining integral over $\bar{x}_{\rm qu}$ yields the effective action $\Delta\tilde{S}$:

$$\Delta\tilde{S} = \lim_{\xi \to 0} \operatorname{tr} \ln \tilde{\mathcal{G}}^{-1} \ , \tag{7}$$

$$\tilde{\mathcal{G}}_{ij}^{-1} = D^2 \frac{1}{(\ddot{\bar{x}}_{\rm cl}^2)^{1/2}}\left(\delta_{ij} - \frac{\ddot{\bar{x}}_{\rm cl}^i \ddot{\bar{x}}_{\rm cl}^j}{(\ddot{\bar{x}}_{\rm cl}^2)^{1/2}}\right) D^2 - \frac{1}{\xi} D \dot{\bar{x}}_{\rm cl}^i \dot{\bar{x}}_{\rm cl}^j D - \delta_{ij} D \omega_{\rm cl} D \ ,$$

$D \equiv d/ds$. Since all I want from $\Delta\tilde{S}$ are the counterterms to one-loop order, I take some shortcuts in evaluating $\Delta\tilde{S}$. $\omega_{\rm cl}$ is treated as a constant. Within the trace, which includes an integration over the momentum p, I take $\ddot{\bar{x}}_{\rm cl} = D\dot{\bar{x}}_{\rm cl} \sim (p + p_{\rm cl})\dot{\bar{x}}_{\rm cl}$, $(\ddot{\bar{x}}_{\rm cl}^2)^{1/2} \sim |p + p_{\rm cl}|$, where $p_{\rm cl}$ is the momentum of the background $\bar{x}_{\rm cl}$ field. Expanding $\Delta\tilde{S}$ in $p_{\rm cl}$ and $\omega_{\rm cl}$, the only ultraviolet divergences arise from the terms linear in $|p_{\rm cl}|$ and $\omega_{\rm cl}$. These represent renormalizations of $(\ddot{\bar{x}}_{\rm cl}^2)^{1/2}$ and $\omega_{\rm cl}$, and generate the counterterm Lagrangian $\Delta\tilde{\mathcal{L}}$. The renormalized Lagrangian, $\tilde{\mathcal{L}}_{\rm ren} = \tilde{\mathcal{L}}_{\rm cl} + \Delta\tilde{\mathcal{L}}$, is found to be

$$\tilde{\mathcal{L}}_{\rm ren} = \frac{1}{\alpha}(\ddot{\bar{x}}_{\rm cl}^2)^{1/2}\left(1 - \frac{(d-1)}{2\pi}\alpha \ln\Lambda\right) + \frac{\omega_{\rm cl}}{2\alpha}\left[\dot{\bar{x}}_{\rm cl}^2 - \left(1 - \frac{(d-1)}{\pi}\alpha\ln\Lambda\right)\right] \ , \tag{8}$$

where Λ is an ultraviolet cutoff. Let Z be the wave-function renormalization constant,

$$Z = 1 - \frac{(d-1)}{\pi}\alpha\ln\Lambda \ ,$$

and $\bar{x}_r = \bar{x}_{\rm cl}/\sqrt{Z}$ and $\omega_r = \omega_{\rm cl}$ the renormalized fields. Then $\tilde{\mathcal{L}}_{\rm ren}$ can be written as

$$\tilde{\mathcal{L}}_{\rm ren} = \frac{1}{\alpha_r}(\ddot{\bar{x}}_r^2)^{1/2} + \frac{1}{2\alpha_r}\omega_r(\dot{\bar{x}}^2 - 1) \tag{9}$$

if $\alpha_r = \alpha/Z$. The β function of α is

$$\beta = -\frac{\partial}{\partial \ln\Lambda}\alpha_r = -\frac{(d-1)}{\pi}\alpha^2 + \cdots \tag{10}$$

and the theory is asymptotically free. To leading order in α, the renormalization of charge is determined entirely by that of the wave function.

Brézin, Zinn-Justin, and Le Guillou[4] have studied nonlinear σ models with long-range interactions, including an example with a critical dimensionality of one. For this model, in momentum space the propagator of the σ field is $\sim 1/|p|$. This is essentially how the propagator for $\dot{\bar{x}}$ behaves at large momenta, so it is not surprising to find

that the renormalization-group equations for these two models agree to leading order in the coupling constant. I do not believe this identity persists to higher order. For the long-range σ models, wave-function and charge renormalization are always equal.[4] In the present model, beyond leading order, ghosts associated with the gauge invariance can enter in a nontrivial way and so violate this equality.

At this point, it is worth mentioning what happens if the action is generalized to include terms whose couplings have inverse dimensions of mass. Usually, these terms will generate new vertices for the $\dot{\bar{x}}$'s, e.g., $\int k^n ds$ for $n > 2$. In this case, renormalization will induce an infinite set of counterterms, and so these terms are nonrenormalizable.

There are terms, though, which are quadratic in $\bar{x}$ in the arc-length gauge:

$$\int k^2 ds = \int (d^2\bar{x}/ds^2)^2 ds, \ \int (d^3\bar{x}/ds^3)^2 ds$$

and so forth.[5] These terms only contribute to the $\bar{x}$ propagator, and thus do not generate new terms under renormalization. In fact, they render the model finite.

What, then, is the use of analyzing the action of Eq. (1)? Let me make the customary and reasonable assumption that any term with a coupling of inverse mass dimension arises solely as a result of effects at short distances. Then these couplings should all be proportional just to (inverse) powers of the cutoff Λ, and never to those of the mass m. This means that for momenta $\ll\Lambda$, we can take the action to be that of Eq. (1).

The infrared behavior of an asymptotically free theory such as this is bound to be involved. For paths in the quantum theory to be well defined, they must be curved—so exactly how do paths "curl up" over large distances? To develop some insight, I solve the model for $d \to \infty$ with αd fixed. The large-d expansion is based on the identity

$$\int_{-\infty}^{+\infty} d\lambda \exp\left[-a\lambda^2 - \frac{b}{\lambda^2}\right] = \left[\frac{\pi}{a}\right]^{1/2} \exp(-2\sqrt{ab}) \ , \quad (11)$$

$a,b > 0$. Consequently, from a Lagrangian $\mathcal{L}'$,

$$\mathcal{L}' = \frac{1}{2\alpha}\lambda^2 + \frac{1}{2\alpha}\frac{\ddot{\bar{x}}^2}{\lambda^2} + \frac{1}{2\alpha}\omega(\dot{\bar{x}}^2 - 1) \ , \quad (12)$$

integration over the constraint field $\lambda = \lambda(s)$ gives, identically, the Lagrangian $\mathcal{L}$ of Eq. (5). For this to be true, it is crucial that b, which is $\sim\ddot{\bar{x}}^2$, enters only in the exponential on the right-hand side of Eq. (11) and not in the prefactor. If b did appear in the prefactor, after introducing $\lambda(s)$ it would be necessary to keep track of curvature-dependent terms in the measure of the functional integral. Such terms in the measure can be ignored when the total arc length is infinite, but since I shall also consider the case of finite total arc length, it is essential to be certain that nothing in the measure has been overlooked.

In Eq. (12) $\bar{x}$ only appears quadratically, and so $\bar{x}$ can be integrated out to give the effective Lagrangian $\mathcal{L}_{\rm eff}$:

$$\mathcal{L}_{\rm eff}(\lambda,\omega) = \frac{1}{2\alpha}\lambda^2 - \frac{1}{2\alpha}\omega + \frac{d}{2}\,\mathrm{tr}_p \ln G^{-1}(\lambda,\omega) \ ,$$
$$G^{-1}(\lambda,\omega) = D^2\left[\frac{1}{\lambda^2}\right]D^2 - D\,\omega D \ . \quad (13)$$

[In Eq. (13), the trace is only over the momentum p, $D^2 = -p^2$.] I expand $\lambda(s)$ and $\omega(s)$ about fields $\lambda_{\rm cl}$ and $\omega_{\rm cl}$: $\lambda = \lambda_{\rm cl} + \lambda_{\rm qu}$, $\omega = \omega_{\rm cl} + \omega_{\rm qu}$. If $\lambda_{\rm cl}$ and $\omega_{\rm cl}$ are to give the true ground state at large d, then the terms linear in $\lambda_{\rm qu}$ and $\omega_{\rm qu}$ must vanish. This gives the equations

$$\lambda_{\rm cl} d\left[\frac{1}{\alpha d} - \mathrm{tr}_p\left[\frac{p^2}{\lambda_{\rm cl}^4}\frac{1}{p^2/\lambda_{\rm cl}^2 + \omega_{\rm cl}}\right]\right] = 0 \ , \quad (14a)$$

$$-\frac{d}{2}\left[\frac{1}{\alpha d} - \mathrm{tr}_p\left[\frac{1}{p^2/\lambda_{\rm cl}^2 + \omega_{\rm cl}}\right]\right] = 0 \ , \quad (14b)$$

where I assume that $\lambda_{\rm cl}$ and $\omega_{\rm cl}$ commute with p.

What is the solution of Eq. (14)? A first guess—$\lambda_{\rm cl}$ and $\omega_{\rm cl}$ both constant—does not work, since then the trace in Eq. (14a) is ultraviolet divergent as $\sim\Lambda$. This divergence does not appear with dimensional regularization, but even so there is no consistent solution for real $\lambda_{\rm cl}$ and $\omega_{\rm cl}$.

Equation (14) does have an obvious solution, if one is willing to take a bit of a leap. With $\Lambda_{\rm cl} = (-D^2)^{1/4}$, $p^2/\lambda_{\rm cl}^2 = |p|$ within the trace over momenta, both equations give

$$\frac{1}{\alpha d} - \mathrm{tr}_p \frac{1}{|p| + \omega_{\rm cl}} = 0 \ . \quad (15)$$

This equation can always be solved by a constant $\omega_{\rm cl}$; the value of the constant will depend on the boundary conditions applicable.

This solution has a simple physical interpretation. From Eq. (12), $\lambda^4 \sim \ddot{\bar{x}}^2$, so $\lambda_{\rm cl} \neq 0$ shows that the dominant paths at large d are curved. Indeed, since $\lambda_{\rm cl}^4 \sim -D^2$ is an operator, this solution for $\lambda_{\rm cl}$ introduces nonlocal interactions into the theory; in some sense, this nonlocality is how one part of a path "communicates" its curvature to another.

As illustrated by Eq. (15), with this $\lambda_{\rm cl}$ the propagator for $\dot{\bar{x}}$ is $\sim 1/(|p| + \omega_{\rm cl})$. Consequently, to leading order in d^{-1}, the results for the large-d expansion are, up to some minor differences, the same as for a one-dimensional nonlinear σ model with long-range interactions.[4] For reasons that will be explained below, this is *not* true beyond leading order in d^{-1}.

It is simplest to solve Eq. (15) for infinite arc length. I define α_r as the renormalized coupling at a scale μ:

$$\frac{1}{\alpha_r d} = \frac{1}{\alpha d} - \frac{1}{\pi}\ln\left[\frac{\Lambda}{\mu}\right] \ . \quad (16)$$

From Eq. (15), the value of $\omega_{\rm cl}$ for infinite $s_t, \omega_{\rm cl}^\infty$, is

$$\omega_{\rm cl}^\infty = \mu \exp\left[-\frac{\pi}{\alpha_r d}\right] \ . \quad (17)$$

The renormalization of charge in Eq. (16) is consistent (at large d) with that found perturbatively in Eq. (10). From Eq. (12), $\delta_{ij} G(\lambda_{\rm cl}, \omega_{\rm cl})$ is the $\bar{x}$ propagator at large d. For example,

$$\langle \dot{\bar{x}}(s) \cdot \dot{\bar{x}}(s') \rangle \sim \exp(-\omega_{\rm cl}^\infty |s - s'|) \ , \quad (18)$$

as $|s - s'| \to \infty$. Thus $\omega_{\rm cl}^\infty$ acts like a dynamically generated mass scale for correlations of $\dot{\bar{x}}(s)$. These results are all typical of nonlinear σ models in their critical dimension.[6]

Thinking of the physics of a polymer, it is natural to ask what happens when a strand of finite length is snipped and allowed to wiggle into thermal equilibrium. Let the total arc length be s_t, and assume that the end points are held fixed, $p=\pi n/s_t$, $n=1, 2, 3, \ldots$. I use zeta-function regularization, replacing the trace in Eq. (15) by

$$\sum_{n=0}^{\infty}\frac{1}{(n+x)^{1+\varepsilon}}\underset{\varepsilon\to 0}{\sim}\frac{1}{\varepsilon}-\psi(x)+O(\varepsilon)\ , \tag{19}$$

$\psi(x)\equiv d\ln\Gamma(x)/dx$. To recover Eq. (17) for infinite s_t, I take $\varepsilon^{-1}=\ln(\Lambda s_t/\pi)$. Then Eq. (15) becomes

$$\frac{1}{\alpha_r d}+\frac{1}{s_t\omega_{\rm cl}}+\frac{1}{\pi}\psi\left(\frac{\omega_{\rm cl}s_t}{\pi}\right)=\frac{1}{\pi}\ln\left(\frac{\mu s_t}{\pi}\right)\ . \tag{20}$$

In Eq. (20), $\omega_{\rm cl}$ is implicitly a function of s_t. For large s_t,

$$\omega_{\rm cl}(s_t)\underset{\mu s_t\gg 1}{\sim}\omega_{\rm cl}^{\infty}\left[1-\frac{\pi}{2\mu s_t}\exp\left(\frac{\pi}{\alpha_r d}\right)+\cdots\right]\ . \tag{21a}$$

$\omega_{\rm cl}(s_t)$ decreases as s_t does. There is a critical value of s_t, s_t', at which $\omega_{\rm cl}(s_t')$ vanishes: $s_t'=\pi\exp(-\gamma)/\omega_{\rm cl}^{\infty}$. $\omega_{\rm cl}(s_t)$ is negative when $s<s_t'$, and it approaches $-\pi/s_t$ as $s_t\to 0$:

$$\omega_{\rm cl}(s_t)\underset{\mu s_t\ll 1}{\sim}-\frac{\pi}{s_t}\left[1+\frac{1}{\ln(\mu s_t/\pi)}+\cdots\right]\ . \tag{21b}$$

Another quantity of physical interest for a polymer is how the free energy behaves as a function of the arc length. At large d, the total free energy of Eq. (1) is the sum of the effective action $S_{\rm eff}^{\rm cl}\equiv\int ds\,\mathcal{L}_{\rm eff}(\lambda_{\rm cl},\omega_{\rm cl})$ and ms_t. For large arc length, $S_{\rm eff}^{\rm cl}$ should produce some constant and presumably positive mass per unit length, but its behavior should be much more involved as s_t decreases.

I shall be interested in the terms $O(d)$ in $S_{\rm eff}^{\rm cl}$, so the contribution of ghosts, which is $O(d^0)$, can be neglected. $\int\lambda_{\rm cl}^2 ds\sim\sum_n|n|\sim\zeta(-1)$ by zeta-function regularization, and since $\zeta(-1)=-\pi/12$, this term just gives a constant, independent of s_t, which can be ignored. Then

$$S_{\rm eff}^{\rm cl}=-\frac{1}{2\alpha}\omega_{\rm cl}s_t+\frac{d}{2}s_t\,{\rm tr}\ln G^{-1}(\lambda_{\rm cl},\omega_{\rm cl})\ . \tag{22}$$

$\mathrm{tr}\ln(|p|+\omega_{\rm cl})$ can be found by integrating $\mathrm{tr}(|p|+\omega_{\rm cl})^{-1}$, which was needed in Eq. (20), with respect to $\omega_{\rm cl}$. The constant of integration is given by

$$\mathrm{tr}\ln|p|\sim-\ln(s_t)\zeta(0)\equiv\tfrac{1}{2}\ln(\mu s_t)\ . \tag{23}$$

μ is introduced into Eq. (23) as a matter of convention, to make the dimensions work out right. The final result is

$$S_{\rm eff}^{\rm cl}=\frac{d}{2}\left\{\omega_{\rm cl}s_t\left[-\frac{1}{\alpha_r d}+\frac{1}{\pi}\ln\left(\frac{\mu s_t}{\pi}\right)\right]-\ln\left[\frac{\omega_{\rm cl}}{\sqrt{\mu^3 s_t}}\Gamma\left(\frac{\omega_{\rm cl}s_t}{\pi}\right)\right]\right\}\ . \tag{24}$$

For large s_t,

$$S_{\rm eff}^{\rm cl}\underset{\mu s_t\gg 1}{\sim}\frac{d}{2}\left(\frac{\omega_{\rm cl}^{\infty}s_t}{\pi}+\ln(\mu s_t)+O(s_t^0)\right)\ . \tag{25a}$$

While it was not obvious from Eq. (22), Eq. (25a) shows that $\omega_{\rm cl}^{\infty}$ does act like a positive, mass-type term for large s_t. As $s_t\to s_t'$, $\omega_{\rm cl}(s_t)\to 0$, and $S_{\rm eff}^{\rm cl}\to 3d\ln(\mu s_t'/\pi^{2/3})/4$. $S_{\rm eff}^{\rm cl}$ is well defined and differentiable about s_t', so at least to leading order in d^{-1}, $s_t=s_t'$ is not a point of any phase transition. $S_{\rm eff}^{\rm cl}$ remains real for $s_t<s_t'$, since $x\Gamma(x)>0$ for $-1<x<0$. For small s_t,

$$S_{\rm eff}^{\rm cl}\underset{\mu s_t\ll 1}{\sim}\frac{d}{2}\left\{-\ln\left[\ln\left(\frac{\pi}{\mu s_t}\right)\right]+\tfrac{1}{2}\ln(\mu s_t)+O(s_t^0)\right\}\ . \tag{25b}$$

One can also calculate for the case of end points which are free to move, by taking the allowed momenta to be $p=2\pi m/s_t$, $m=0, \pm 1, \pm 2, \ldots, \omega_{\rm cl}^{\infty}$ is unchanged, but $\omega_{\rm cl}(s_t)$ increases for decreasing s_t; as $s_t\to 0$, $\omega_{\rm cl}(s_t)\sim-\pi/[s_t\ln(\mu s_t)]$. The leading terms in $S_{\rm eff}^{\rm cl}$ are the same as in Eq. (25), although the nonleading ones differ.

The identity between the results for the theory of paths and a one-dimensional σ model with long-range interactions breaks down beyond leading order in d^{-1}. For the σ model, the corrections in d^{-1} are given by expanding the constraint field $\omega=\omega_{\rm cl}+\omega_{\rm qu}$ in small fluctuations about $\omega_{\rm cl}$; $\langle(\omega_{\rm qu})^2\rangle\sim d^{-1}$. To any finite order in d^{-1}, this expansion is infrared finite.

For the theory of paths, there are two constraint fields about which to expand, $\lambda=\lambda_{\rm cl}+\lambda_{\rm qu}$ and ω. The fluctuations in $\lambda_{\rm qu}$ are not only unique to the theory of paths, but they dramatically alter the physics. From Eq. (13), $1/\lambda^2$ appears in $G^{-1}(\lambda,\omega)$, so expansion in $\lambda_{\rm qu}$ will inevitably produce terms $\sim 1/\lambda_{\rm cl}^2=1/|p|$; no higher powers of $1/\lambda_{\rm cl}$ seem to. These factors of $1/|p|$ introduce infrared divergences for finite d that do not occur at infinite d. For $d<\infty$, the end points must be held fixed so that any momenta are always nonzero. Even with fixed end points, the corrections in d^{-1} will bring in a logarithmic sensitivity to s_t for large s_t.

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