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Mean-field renormalization-group technique for Z_N gauge theories

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The mean-field renormalization-group technique which has been applied to spin systems with considerable success is adapted to lattice gauge theories. Because of the simplicity of our block-spin choice, results are not as successful as they are for the spin systems. Taking larger blocks is expected to improve the results.

I. INTRODUCTION

Wilson's[1] lattice regularization of divergences by defining any field theory on a lattice reduces the renormalization problem to determining the form of dependence of the couplings on the lattice spacing. The lattice formalism consequently enables one to use the real-space renormalization-group methods,[2] which were initially introduced for the spin systems, in any field theory.

The success of the renormalization-group method lies in the choice of transformation prescription. Methods which are developed for the spin systems cannot be directly used in gauge theories. However, there are very successful prescriptions[3,4] which are used in Monte Carlo renormalization-group calculations of lattice gauge theories. The major drawback of the Monte Carlo renormalization-group method[5] is its requirement of large computers and extremely extensive computer time.

Mean-field techniques[6] provide a successful alternative to Monte Carlo methods for lattice gauge theories despite their problems with gauge invariance. The success of this method gives rise to hope for the application of Kinzel's mean-field renormalization-group technique[7] to lattice gauge theories.

Kinzel's mean-field renormalization-group technique combines the mean-field techniques with real-space renormalization-group transformations. This combination brings considerable improvement to the computation of critical indices and of critical points. The values are closer to the exact values rather than the mean-field values for the two-dimensional Ising spin system.[7] In this paper we have combined the simplest mean-field ideas in lattice gauge theories with Swendson's renormalization-group transformation prescription[3] in order to apply the mean-field renormalization-group ideas to lattice gauge theories.

In Sec. II we will outline the mean-field renormalization-group technique as it is applied in Z_N gauge theories. Section III is devoted to discussions of the results.

II. MEAN-FIELD RENORMALIZATION-GROUP TECHNIQUE FOR Z_N LATTICE GAUGE THEORIES

The essence of Kinzel's mean-field renormalization-group technique lies in blocking the system such that after the real-space renormalization-group transformation each block transforms as a "block spin." The interactions between the spins of a given block are calculated exactly while interactions between the spins of different blocks are taken as a mean-field effect.

Blocking procedures have some concrete rules for spin systems. In gauge systems, this blocking procedure is not so clear. It is impossible to define distinct blocks in which links will be treated exactly, since each link is used by $d(d-1)$ plaquettes. In this work, we have the simplest block of link variables: namely, three links which lie on the straight path between the sites A and B (Fig. 1). All surrounding links are taken as mean values. In effect, the surrounding links are decisive on the renormalization-group transformation. We have employed Swendson's renormalization-group prescription,[3] such that after one transformation these three links will be replaced by one link of the renormalized system. The lattice-gauge-theory action, in the Wilson form,

$$S=\sum_{\text{plaquettes}} K\sigma_1\sigma_2^\dagger\sigma_3\sigma_4^\dagger \ , \tag{1}$$

under the renormalization group transformation, becomes

$$\begin{aligned} S'(\sigma_l') &= \ln \mathrm{Tr} P(\sigma_l',\sigma_{lv})e^{-s} \\ &= \sum_{\text{new plaquettes}} K'\sigma_1'\sigma_2'^\dagger\sigma_3'\sigma_4'^\dagger + K_0' \ . \end{aligned} \tag{2}$$

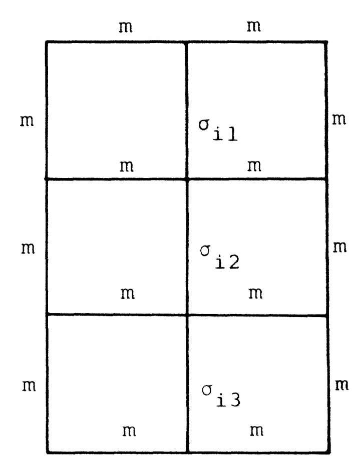


FIG. 1. Ordered configuration for two-dimensional lattice gauge system. Since the same pattern repeats itself, after the renormalization transformations the system remains ordered.

Here, the σ_l''s are defined by the renormalization-group transformation $P(\sigma_l', \sigma_{l\nu})$ which couples the new link variables σ_l' to all link variables $\sigma_{l\nu}$ in the block i, where K is the original coupling constant, and K' and K_0' are the renormalized coupling constants.

In Kinzel's prescription one takes two configurations: one ordered and one disordered. In transformation, the properties of these two systems give the renormalized coupling constants as a function of the original coupling constants:

$$K' = \frac{S'(+) - S'(-)}{2Nd(d-1)/2} , \tag{3a}$$

$$K_0 = \frac{S'(+) - S'(-)}{2N} , \tag{3b}$$

where $S'(+)$ and $S'(-)$ are the actions of ordered and disordered configurations. For a gauge-invariant system the ground state is not unique, but all of the ground states are related to each other by gauge transformations. Consequently, the choice of any one of the ground states will be sufficient for the ordered configuration. The same is also true for the disordered state. In Figs. 1 and 2 we have given our choice of ordered and disordered states for two-dimensional Z_N gauge theory. For any Z_N gauge theory, the link variables σ_1, σ_2, and σ_3, are the elements of the gauge group being studied and m's are the mean values of the link variables:

$$m = \frac{\sum_l P(\sigma_l', \sigma_{l\nu})\sigma_{l\nu} \exp\left[\sum_{l\nu} S(\sigma_{l\nu}, m)\right]}{\sum_l P(\sigma_l', \sigma_{l\nu}) \exp\left[\sum_{l\nu} S(\sigma_{l\nu}, m)\right]} . \tag{4}$$

Here

$$S(\sigma_{l\nu}, m) = K\frac{d(d-1)}{2}\sigma_{l\nu} m^3$$

is the usual mean-field lattice-gauge-theory action. If $\sigma_{l\nu}$'s are chosen such that the majority of the plaquette action yields a positive value, Eq. (4) defines the mean value of the link variables for the ordered state which is given in Fig. 1. Our choice of link variables and plaquette actions for an ordered configuration in Z_2 and Z_3 gauge theories are given in Table I.

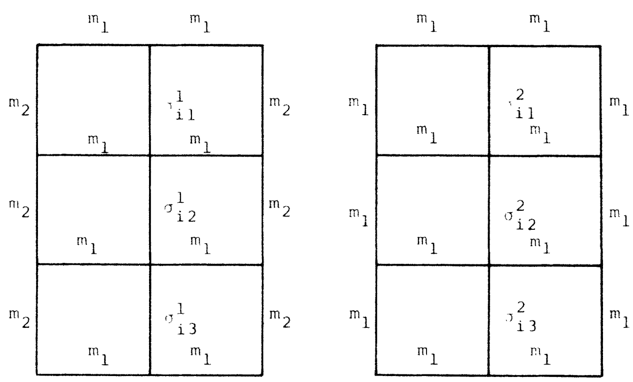
Since the disordered phase is made up of link variables with unequal values we need to define at least two different mean values: m_1 and m_2. To calculate these mean values one needs to define two distinct sets of group elements. Disordered configurations are obtained from these two sets of group elements in such a way that after the renormalization-group transformation the transformed system will still be in the disordered phase. Our choice of configuration is given by Figs. 2(a) and 2(b). The mean values m_1 and m_2 are calculated by solving

$$m_1 = \frac{\sum_l \sigma_{l\nu}^1 P(\sigma_l^1, \sigma_{l\nu}^1) \exp\left[\sum_{l\nu} S(\sigma_{l\nu}^1, m_l^2, m_2)\right]}{\sum_l P(\sigma_l^1, \sigma_{l\nu}^1) \exp\left[\sum_{l\nu} S(\sigma_{l\nu}^1, m_l^2, m_2)\right]} \tag{5a}$$

and

$$m_2 = \frac{\sum_l \sigma_{l\nu}^2 P(\sigma_l^2, \sigma_{l\nu}^2) \exp\left[\sum_{l\nu} S(\sigma_{l\nu}^2, m_l^3)\right]}{\sum_l P(\sigma_l^2, \sigma_{l\nu}^2) \exp\left[\sum_{l\nu} S(\sigma_{l\nu}^2, m_l^3)\right]} \tag{5b}$$

simultaneously. $\sigma_{l\nu}^1$ and $\sigma_{l\nu}^2$ are defined as two distinct sets of group elements. The plaquette actions and our choice of the two sets of group elements are given by Table II for Z_2 and Z_3 gauge theories. If the original action represents the ordered (disordered) state, after successive transformations the action of the transformed system will still be representing the ordered (disordered) system since the new system will be a replica of the original system. Hence, using Eqs. (3a) and (3b) one can calculate iteratively how the system goes to either the low- or high-temperature fixed points. If a finite-temperature fixed point exists, the same iterative procedure locates its position. In Fig. 3, the renormalized coupling constant K of 3-, 4-, and 5-dimensional Z_2, Z_3, Z_4, and Z_6 gauge theories are plotted versus the coupling



(a) (b)

FIG. 2. Disordered configuration for two-dimensional lattice gauge system: (a) The first set of link variables $\sigma_{l\nu}^1$, of which the mean value is m_1, is disordered by the existence of the mean value m_2; (b) the second set of link variables $\sigma_{l\nu}^2$ with surrounding links with mean values m_1 defines a different disordered system. Juxtaposition of last row of the plaquettes of (a) and the first row of (b) gives the actual pattern of the disordered configuration.

TABLE I. A choice of link variables $\sigma_{l\nu}$ and corresponding actions are given for the ordered configurations of Z_2 and Z_3 gauge theories. The link variables $\sigma_{l\nu} = \exp[i(2\pi/N)n_\nu]$ are defined by the integer numbers n_ν, where $n_\nu = 0, 1, \ldots, N-1$ for any Z_N gauge theory.

	Z_2 gauge theory		
n_1	n_2	n_3	$S/[Kd(d-1)/2]$
0	0	0	$3m^3$
0	0	1	
0	1	0	m^3
1	0	0	
	Z_3 gauge theory		
n_1	n_2	n_3	$S/[Kd(d-1)/2]$
0	0	0	$3m^3$
0	0	1	
0	1	0	$3/2m^3$
1	0	0	
0	0	2	
0	2	0	$3/2m^3$
2	0	0	

TABLE II. Two distinct sets of link variables $\sigma^1_{l\nu}$ and $\sigma^2_{l\nu}$ for the disordered system with corresponding actions $S(\sigma^1_{l\nu}, m_l^2, m_2)$ and $S(\sigma^2_{l\nu}, m_l^3)$ are given for the gauge groups Z_2 and Z_3. Here, again, the link variables are represented by the integer numbers $n^k_{l\nu}$ as in Table I.

Z_2 gauge theory							
n^1_1	n^1_2	n^1_3	$S/[Kd(d-1)/2]$	n^2_1	n^2_2	n^2_3	$S/[Kd(d-1)/2]$
0	0	0	$3m_1^2m_2$	1	1	1	$-3m_1^3$
0	0	1		1	1	0	
0	1	0	$m_1^2m_2$	1	0	1	$-m_1^3$
1	0	0		0	1	1	

Z_3 gauge theory							
n^1_1	n^1_2	n^1_3	$S/[Kd(d-1)/2]$	n^2_1	n^2_2	n^2_3	$S/[Kd(d-1)/2]$
0	0	0	$3m_1^2m_2$	1	1	1	$-3/2m_1^3$
0	0	1		1	1	0	
0	1	0	$3/2m_1^2m_2$	1	0	1	0
1	0	0		0	1	1	
0	0	2		1	1	2	
0	2	0	$3/2m_1^2m_2$	1	2	1	$-3/2m_1^3$
2	0	0		2	1	1	

constant K of the original system. For $K'=K$ the system reproduces itself and this point is the critical point. In Table III the critical points are listed for the above-mentioned gauge theories. A quick inspection will show that these values obtained by using real-space renormalization-group techniques are slightly better than the most simple mean-field approximation. Nevertheless, 2-dimensional Z_N gauge theories exhibit phase transitions.

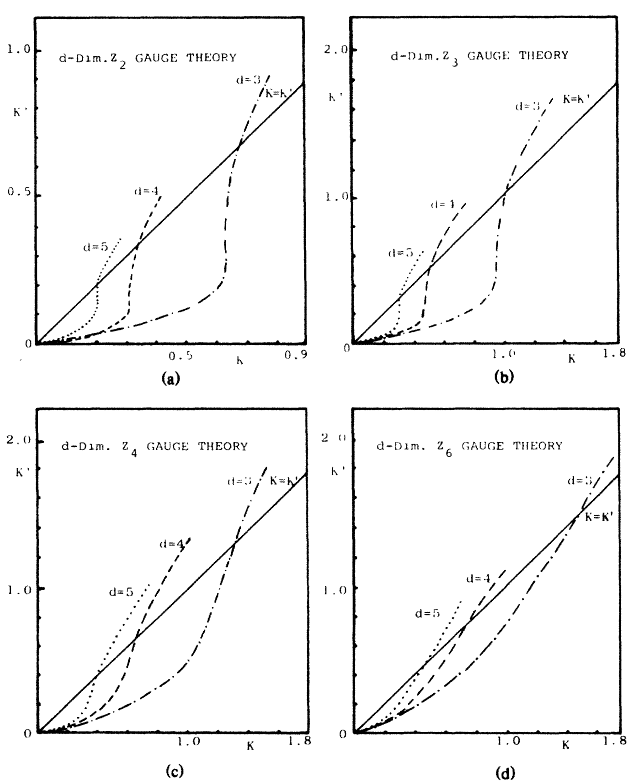


FIG. 3. Renormalized couplings K' vs original couplings K for (a) Z_2, (b) Z_3, (c) Z_4, and (d) Z_6 gauge theories for d-dimensional space-time.

It is well known in the literature[6] that the mean-field approximation in lattice gauge theories possesses a discontinuity in free energy and consequently a first-order phase transition is exhibited for any Z_N gauge theory. In the real-space renormalization-group approach the critical indices can be calculated from the ratio[7]

$$\nu = \log_3 \frac{\delta K'}{\delta K} .$$

From Fig. 3 one can calculate the critical indices. These values indicate continuous transitions. It may be seen from the slopes of the graphs in Fig. 3 that for Z_N gauge theories an increasing N results in a smaller value of ν.

III. DISCUSSIONS

Mean-field renormalization-group techniques have successfully been applied to Z_N gauge theories; however, our results possess the major defects of the standard mean-field approximation: namely, Z_N gauge theories exhibit a phase transition in two dimensions. Moreover, our calculations have not shown any improvement in the values of the critical points compared with the standard mean-field approach.

TABLE III. The critical points of 2-, 3-, 4-, and 5-dimensional Z_2, Z_3, Z_4, and Z_6 gauge theories. Some of the known critical points (Ref. 9) are given below the mean-field results.

Gauge group	Dimensionality of the system 2	3	4	5
Z_2	2.05	0.68 (0.7613)	0.34 (0.4407)	0.21
Z_3	3.05	1.02 (1.085)	0.51 (0.670)	0.30
Z_4	4.10	1.33 (1.5226)	0.66 (0.8814)	0.40
Z_6	4.37	1.46 ($\sim$2.8)	0.72	0.44

This is in contrast with spin systems where the mean-field renormalization-group calculations are considerably better than the mean-field results. The problem, we believe, lies in the fact that in the case of gauge theories a large number of ground and disordered states exist, but not all of the ground states or disordered states are reproduced by the renormalization-group transformation.

One advantage of the mean-field renormalization-group approach is probably the ability of calculating the critical indices, in which case it is not successful due to our choice of block spins. Although in this approach the saddle-point approximation and related improvements are possible, unless larger blocks are taken neither the critical indices nor the values of the critical points may be expected to have any significant improvement. We believe that if one takes larger regions of the lattice as the basic blocks and takes all other links as the mean value the major drawbacks of the mean-field theory will be suppressed, and the power of the renormalization-group techniques may be more transparent. The mean-plaquette[8] approach may be of help in this direction of the studies.

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