Birkhoff's theorem in a new scalar-tensor theory

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It is shown that an analogue of Birkhoff's theorem in general relativity exists in a new scalar-tensor theory of gravitation proposed by Schmidt, Greiner, Heinz, and Müller for the special case when the scalar field is massless and independent of time.

I. INTRODUCTION

Recently, Schmidt *et al.*[1] proposed a new scalar-tensor theory of gravitation, where the gravitational constant depends on a scalar field which itself couples to the surrounding masses through the curvature scalar. The idea was to obtain a possible stable configuration as a final situation in the history of a collapsing object due to the generation of a strong scalar field. The result was contrary to what was expected. In the new theory a mass term was added and an arbitrary coupling constant between the scalar field ϕ and the curvature invariant R was also allowed. The theory was subsequently applied to a Friedmann-Robertson-Walker universe by Banerjee and Santos.[2] Further, Singh and Singh[3] have shown that the spatially homogeneous stationary perfect-fluid cosmological model in this theory cannot include the radiation-filled universe or the empty universe at the limit in the presence of a massive scalar field. Very recently, Banerjee, De Oliveira, and Santos[4] discussed a stiff fluid Bianchi type-I cosmological model in this theory, by considering the cosmological constant Λ and the mass term μ both equal to zero.

In this paper we have established that the time invariance of the scalar field is a sufficient condition for the Birkhoff theorem in this more general scalar-tensor theory proposed by Schmidt *et al.*,[1] for the special case when the mass term and the cosmological constant are each equal to zero.

II. BIRKHOFF THEOREM IN A NEW SCALAR-TENSOR THEORY

In general relativity every spherically symmetric vacuum field of gravitation must be static. This fact is known as Birkhoff's[5] theorem. The Birkhoff theorem guarantees that the outer field of any spherical source is static and it does not depend on its inner structure, but only on its Newtonian mass. The possibility of the theorem being valid in other scalar-tensor theories of gravitation has been examined by several authors (Refs. 6–13). They showed that in different forms of scalar-tensor theories, Birkhoff's theorem holds when the scalar field is time independent.

The gravitational field equations in the scalar-tensor theory proposed by Schmidt *et al.*[1] are given by

$$\left(\gamma-\frac{\beta}{12}\phi^2\right)G_{ij}=-\tfrac{1}{2}T_{ij}-\tfrac{1}{2}[\phi_{,i}\phi_{,j}-\tfrac{1}{2}g_{ij}(\phi_{,k}\phi^{,k}-\mu^2\phi^2)]+\frac{\beta}{12}[(\phi^2)_{;ij}-g_{ij}(\phi^2)_{;k}{}^{;k}]\,, \tag{1}$$

and the wave equation is

$$\Box\phi+\left(\mu^2+\frac{\beta}{6}R\right)\phi=0\,. \tag{2}$$

Here μ is the mass of the scalar field, β is an arbitrary coupling constant, and $\gamma=c^2/16\pi G$ is half of the inverse gravitational constant. The effective inverse gravitational coupling in this theory becomes

$$\gamma_{\rm eff}=\gamma-\frac{\beta}{12}\phi^2$$

and the effective mass of the scalar field now is

$$\mu_{\rm eff}=\left(\mu^2+\frac{\beta}{6}R\right)^{1/2}.$$

We consider the spherically symmetric metric in the form

$$ds^2=e^{\nu}\,dt^2-e^{\lambda}\,dr^2-r^2(d\theta^2+\sin^2\theta\,d\Phi^2)\,, \tag{3}$$

where $\lambda=\lambda(r,t)$, $\nu=\nu(r,t)$, with the scalar field $\phi=\phi(r,t)$.

The field equations (1) and (2) for the metric (3) read in a vacuum as

$$\left(\gamma-\frac{\beta}{12}\phi^2\right)\left[-e^{-\lambda}\left(\frac{\nu'}{r}+\frac{1}{r^2}\right)+\frac{1}{r^2}\right]=\frac{1}{4}e^{-\nu}\dot{\phi}^2+\frac{1}{4}e^{-\lambda}\phi'^2-\frac{\mu^2\phi^2}{4}+\frac{\beta}{12}\left[e^{-\lambda}(\phi^2)'\left(\frac{\nu'}{2}+\frac{2}{r}\right)+e^{-\nu}(\phi^2)^{\cdot}\frac{\dot{\nu}}{2}-e^{-\nu}(\phi^2)^{\cdot\cdot}\right], \tag{4}$$

$$\left(\gamma-\frac{\beta}{12}\phi^2\right)\left[-e^{-\lambda}\left(\frac{\nu''}{2}-\frac{\lambda'\nu'}{4}+\frac{\nu'^2}{4}+\frac{\nu'-\lambda'}{2r}\right)+e^{-\nu}\left(\frac{\ddot{\lambda}}{2}+\frac{\dot{\lambda}^2}{4}-\frac{\dot{\lambda}\dot{\nu}}{4}\right)\right]$$
$$=\frac{1}{4}(e^{-\nu}\dot{\phi}^2-e^{-\lambda}\phi'^2-\mu^2\phi^2)-\frac{\beta}{12}\left[e^{-\lambda}\left(\frac{(\phi^2)'}{2}(\lambda'-\nu')-(\phi^2)''-\frac{(\phi^2)'}{r}\right)+e^{-\nu}\left(\frac{(\phi^2)^{\cdot}}{2}(\dot{\lambda}-\dot{\nu})+(\phi^2)^{\cdot\cdot}\right)\right], \tag{5}$$

$$\left(\gamma-\frac{\beta}{12}\phi^2\right)\left[e^{-\lambda}\left(-\frac{\lambda'}{r}-\frac{1}{r^2}\right)+\frac{1}{r^2}\right]=-\frac{1}{4}(e^{-\nu}\dot{\phi}^2+e^{-\lambda}\phi'^2+\mu^2\phi^2)+\frac{\beta}{12}\left[-e^{-\lambda}\left(\frac{(\phi^2)'}{2}\lambda'-(\phi^2)''-\frac{2(\phi^2)'}{r}\right)-e^{-\nu}\frac{(\phi^2)^{\cdot\cdot}}{r}\lambda^{\cdot}\right], \tag{6}$$

$$-e^{-\lambda}\frac{\dot{\lambda}}{r}\left(\gamma-\frac{\beta}{12}\phi^2\right)=\frac{e^{-\lambda}}{2}\phi'\phi^{\cdot}+\frac{\beta}{12}e^{-\lambda}\left(\frac{(\phi^2)'\lambda^{\cdot}}{2}-(\phi^2)^{\cdot\prime}+(\phi^2)^{\cdot}\frac{\nu'}{2}\right), \tag{7}$$

$$e^{-\lambda}\left(\frac{(\phi^2)'}{2}(\lambda'-\nu')-(\phi^2)''-\frac{2(\phi^2)'}{r}\right)+e^{-\nu}\left(\frac{(\phi^2)^{\cdot}}{2}(\dot{\lambda}-\dot{\nu})+(\phi^2)^{\cdot\cdot}\right)$$
$$=\frac{2\phi\beta}{12\gamma+\beta(\beta-1)\phi^2}\left(-\frac{6}{\beta}\gamma\mu^2-\frac{1}{2}\mu^2\phi^2+\frac{(1+\beta)}{2}e^{-\nu}\dot{\phi}^2-\frac{(1-\beta)}{2}e^{-\lambda}\phi'^2\right), \tag{8}$$

where an overdot denotes a derivative with respect to the coordinate t, and a prime denotes a derivative with respect to r. When massless scalar field ϕ (i.e., $\mu=0$) is a function of r only, that is

$$\dot{\phi}=0, \tag{9}$$

then from Eq. (7) we have

$$\dot{\lambda}\left(\gamma-\frac{\beta}{12}\phi^2+\frac{\beta r}{24}(\phi^2)'\right)=0, \tag{10}$$

which implies that either

$$\dot{\lambda}=0 \tag{11}$$

or

$$\gamma-\frac{\beta}{12}\phi^2+\frac{\beta r(\phi^2)'}{24}=0, \tag{12}$$

i.e.,

$$\phi^2=\phi_0 r^2+\frac{12\gamma}{\beta},\quad \phi_0=\text{const}.$$

Now, from Eqs. (4), (6), (9), and (12) it follows that

$$e^{\lambda}=2-r(\lambda'-\nu'). \tag{13}$$

Using (9) and (12) in Eq. (8) we get

$$\lambda'-\nu'=\frac{\gamma\beta(\beta-1)}{12\gamma\left(\phi_0 r^2+\frac{12\gamma}{\beta}\right)^{1/2}+\beta(\beta-1)\left(\phi_0 r^2+\frac{12\gamma}{\beta}\right)^{3/2}}+\frac{6}{r}. \tag{14}$$

Substituting this value in Eq. (13) we have

$$e^{\lambda}=\frac{r^2\beta(1-\beta)}{12\gamma\left(\phi_0 r^2+\frac{12\gamma}{\beta}\right)^{1/2}+\beta(\beta-1)\left(\phi_0 r^2+\frac{12\gamma}{\beta}\right)^{3/2}}-4. \tag{15}$$

Thus λ is a function of r only. Therefore

$$\dot{\lambda}=0$$

Now differentiation of (14) with respect to t along with the use of Eq. (11) gives

$$\dot{\nu}=0. \tag{16}$$

From this, one has

$$\nu=f(r)+g(t), \tag{17}$$

where f and g are arbitrary functions of r and t, respectively. Then, a simple redefinition of the time coordinate[14] leads ν into a function of r only. This together with (11) reduces the metric (3) to the static case. Thus, we conclude that the outer field of a spherically symmetric nonstatic source is always a static spherically symmetric field, which proves the Birkhoff theorem in a new scalar-tensor theory proposed by Schmidt *et al.*,[1] for the special case of a massless scalar field.

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