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Stability under dilations of nonlinear spinor fields

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The stability problem of the localized solutions for classical Dirac fields with scalar self-interactions is considered in the framework of the Shatah-Strauss formalism. We study the stability and instability under dilations and provide an application to the Soler model.

I. INTRODUCTION

The most basic question about solitary waves is their stability. They are said to be stable if they are not destroyed under the influence of a general perturbation. If we regard them as models of extended particles we would like to know whether or not the particles decay to some other states.

Mathematically, the question of stability is related to the eigenvalues of the linearized operator. This relationship is, however, not direct for the typical wave equations of physics because, in the absence of dissipative mechanisms, most of the spectrum of the dynamical problem lies on the imaginary axis. For the case of the ground state of a nonlinear scalar field, the stability problem has been completely solved by Shatah and Strauss.[1-3]

In the case of spinor fields satisfying a nonlinear Dirac (NLD) equation, there have been two lines of work. The first one, developed mainly by Alvarez[4-6] and Bogolubsky,[7] is devoted to a numerical study of stability. The second one, considered mainly by Mathieu and Morris[8,9] and Werle,[10,11] uses energy minimization as a stability criterion. However, this criterion is not necessarily correct, as is shown by some simple examples with a finite number of degrees of freedom.

II. THE SHATAH-STRAUSS FORMALISM

We consider some localized solutions, or bound states, of lowest energy of the NLD equation in three space dimensions. The equation is

$$i\gamma^0\partial_t\psi+i\gamma^k\partial_k\psi+F'(\overline{\psi}\psi)\psi=0\ , \tag{1}$$

with

$$\gamma^0=\begin{pmatrix} I & 0\\ 0 & -I\end{pmatrix},\ \gamma^k=\begin{pmatrix} 0 & \sigma^k\\ -\sigma^k & 0\end{pmatrix}\ ,$$

where σ^k are the Pauli matrices, $\overline{\psi}=\psi^\dagger\gamma^0$, and F is a real C^1 function. The bound states are solutions of (1) of the form

$$\psi=e^{-i\omega t}\psi_\omega(x)\ , \tag{2}$$

where ω is a real parameter and ψ_ω tends to zero in a suitable sense at $|x|\to\infty$. The three important functionals of the field are the energy, the action (Lagrangian), and the charge, which are defined, respectively, as

$$E(\psi)=-\frac{i}{2}\int(\overline{\psi}\gamma^k\partial_k\psi-\partial_k\overline{\psi}\gamma^k\psi)dx-\int F(\overline{\psi}\psi)dx\ , \tag{3}$$

$$L(\psi,\psi_t)=\frac{i}{2}\int(\psi^\dagger\psi_t-\psi_t^\dagger\psi)dx-E(\psi)\ , \tag{4}$$

$$Q(\psi)=\int\psi^\dagger\psi dx\ . \tag{5}$$

Next we define the *scaling functional:*

$$R(\psi,\psi_t)=\frac{i}{2}\int(\psi^\dagger\psi_t-\psi_t^\dagger\psi)dx+\frac{i}{2}\int(\overline{\psi}\gamma^k\partial_k\psi-\partial_k\overline{\psi}\gamma^k\psi)dx+\int\overline{\psi}\psi F'(\overline{\psi}\psi)dx\ . \tag{6}$$

This functional vanishes if ψ satisfies the Dirac equation. Indeed, we multiply (1) by $\overline{\psi}$ and its adjoint by ψ and integrate to get $R=0$. Finally, we have the *dilation functional*

$$\begin{aligned}K(\psi,\psi_t)&=L(\psi,\psi_t)-\frac{i}{6}\int(\overline{\psi}\gamma^k\partial_k\psi-\partial_k\overline{\psi}\gamma^k\psi)dx\\&=\frac{i}{2}\int(\psi^\dagger\psi_t-\psi_t^\dagger\psi)dx\\&\quad+\frac{i}{3}\int(\overline{\psi}\gamma^k\partial_k\psi-\partial_k\overline{\psi}\gamma^k\psi)dx+\int F(\overline{\psi}\psi)dx\ .\end{aligned} \tag{7}$$

Multiplying (1) by $r\bar{\psi}_r = x_j \partial_j \bar{\psi}$ gives

$$K(\psi,\psi_t) = \frac{\partial}{\partial t}\frac{i}{3}\int r\psi_r^\dagger \psi dx + \frac{i}{2}\int (\psi^\dagger \psi_t + \psi_t^\dagger \psi)dx$$

for solutions of the NLD equation. It vanishes if ψ is a bound state (2) of the NLD equation, as follows directly from this identity or from the virial theorem.[12] Note that by definition we have

$$L(\psi,\psi_t) - R(\psi,\psi_t) = \int [F(\bar{\psi}\psi) - \bar{\psi}\psi F'(\bar{\psi}\psi)]dx \ . \quad (8)$$

It is often the case that the last integral is negative. For instance, if

$$F(\bar{\psi}\psi) = -m\bar{\psi}\psi + \lambda(\bar{\psi}\psi)^2 \quad (m,\lambda > 0)$$

then

$$L - R = -\lambda \int (\bar{\psi}\psi)^2 dx \le 0 \ . \quad (9)$$

A bound state (2) satisfies the stationary equation

$$\omega\gamma^0\psi_\omega + i\gamma^k\partial_k\psi_\omega + F'(\bar{\psi}_\omega\psi_\omega)\psi_\omega = 0 \ . \quad (10)$$

As mentioned above, we have

$$R(\psi_\omega, -i\omega\psi_\omega) = K(\psi_\omega, -i\omega\psi_\omega) = 0 \ .$$

We define

$$d(\omega) = -L(\psi_\omega, -i\omega\psi_\omega) \ .$$

Using (8) and $R=0$ we find that

$$d(\omega) = \int [\bar{\psi}_\omega\psi_\omega F'(\bar{\psi}_\omega\psi_\omega) - F(\bar{\psi}_\omega\psi_\omega)]dx \ . \quad (11)$$

From the definition of L and Q we may also write

$$d(\omega) = E(\psi_\omega) - \omega Q(\psi_\omega) \ . \quad (12)$$

Its derivative is

$$d'(\omega) = \left\langle E'(\psi_\omega) - \omega Q'(\psi_\omega), \frac{d\psi_\omega}{d\omega}\right\rangle - Q(\psi_\omega) \ .$$

But (10) can be written as $E'(\psi_\omega) - \omega Q'(\psi_\omega) = 0$. Hence

$$d'(\omega) = -Q(\psi_\omega) \ . \quad (13)$$

III. BEHAVIOR UNDER DILATIONS

We now present two theorems related to the stability or instability of the bound states under dilational perturbations. These perturbations represent modifications of the shapes of the localized solutions which preserve the charge and the spin.

Theorem 1. Assume $sF'(s) \ge F(s)$ for all s. Fix ω_0 and a bound state $\psi_{\omega_0}(x)$. If $d''(\omega_0) < 0$ then E, subject to the constraint $Q = Q(\psi_{\omega_0})$, does not have a local minimum at ψ_{ω_0}.

Proof. By (11) and the assumption on F, $d(\omega) \ge 0$. Let $\phi_\omega(x) = \psi_\omega[x/\lambda(\omega)]$, where

$$[\lambda(\omega)]^3 = \int \psi_{\omega_0}^\dagger \psi_{\omega_0} dx \Big/ \int \psi_\omega^\dagger \psi_\omega dx \ . \quad (14)$$

Then $Q(\phi_\omega) = Q_0 \equiv Q(\psi_{\omega_0})$. Replacing ψ by ψ_ω in (4) and changing variables, we have

$$-L(\phi_\omega, -i\omega\phi_\omega) = \lambda^3\left[-\omega\int \psi_\omega^\dagger\psi_\omega dx - \int F(\bar{\psi}_\omega\psi_\omega)dx\right] - \lambda^2\frac{i}{2}\int (\bar{\psi}_\omega\gamma^k\partial_k\psi_\omega - \partial_k\bar{\psi}_\omega\gamma^k\psi_\omega)dx$$
$$= -(\lambda^2 - \tfrac{2}{3}\lambda^3)\frac{i}{2}\int (\bar{\psi}_\omega\gamma^k\partial_k\psi_\omega - \partial_k\bar{\psi}_\omega\gamma^k\psi_\omega)dx \ ,$$

since $K(\psi_\omega, -i\omega\psi_\omega) = 0$. By (7) this can be written as

$$-L(\phi_\omega, -i\omega\phi_\omega) = (3\lambda^2 - 2\lambda^3)d(\omega) \le d(\omega) \ , \quad (15)$$

for ω near ω_0. Here we have used the facts that $\lambda(\omega_0) = 1$, that $3\lambda^2 - 2\lambda^3 \le 1$ for ω near ω_0, and that $d(\omega) \ge 0$. By (4) and the concavity of $d(\omega)$, this inequality can be rewritten as

$$E(\phi_\omega) - \omega Q(\phi_\omega) = -L(\phi_\omega, -i\omega\phi_\omega)$$
$$< d(\omega_0) + (\omega - \omega_0)d'(\omega_0) \ , \quad (16)$$

for ω near $\omega_0 (\omega \ne \omega_0)$. By (12) and (13) the last expression is

$$E(\psi_{\omega_0}) - \omega_0 Q(\psi_{\omega_0}) - (\omega - \omega_0)Q(\psi_{\omega_0})$$
$$= E(\psi_{\omega_0}) - \omega Q(\psi_{\omega_0}) \ .$$

Therefore,

$$E(\phi_\omega) < E_0 \equiv E(\psi_{\omega_0}) \ , \quad (17)$$

even though $Q(\phi_\omega) = Q_0$.

Theorem 1 indicates (but does not mathematically imply) the instability under dilations of the bound states, assuming $d(\omega)$ is concave. The next theorem indicates the stability under dilations assuming the convexity of $d(\omega)$ and of a related function.

Theorem 2. Assume $sF'(s) \ge F(s)$ for all s. If $d''(\omega_0) > 0$ and $(d^{-1/2})''(\omega_0) > 0$, then the energy along the dilated curve $E(\phi_\omega)$ has a local minimum at $\omega = \omega_0$.

Proof. As in the preceding proof, we have

$$E(\phi_\omega) - \omega Q(\phi_\omega) = \mu(\omega)d(\omega) \ , \quad (18)$$

where $\mu(\omega) = 3\lambda^2(\omega) - 2\lambda^3(\omega)$. We differentiate and set $\omega = \omega_0$, $\lambda(\omega_0) = 1$, to obtain

$$\mu(\omega_0) = 1 \ ,$$
$$\mu'(\omega_0) = 6(\lambda - \lambda^2)\lambda' = 0 \ ,$$
$$\mu''(\omega_0) = 6(\lambda - \lambda^2)\lambda'' + 6(1 - 2\lambda)(\lambda')^2 = -6[\lambda'(\omega_0)]^2 \ .$$

However, from (13) we have

$$\lambda^3(\omega) = \frac{Q_0}{\int \psi_\omega^\dagger \psi_\omega dx} = \frac{d'(\omega_0)}{d'(\omega)} \ ,$$

so that $\lambda'(\omega_0) = -d''(\omega_0)/3d'(\omega_0)$. So, upon expanding (18) in powers of $\omega - \omega_0$, we obtain

$$E(\phi_\omega)-\omega Q(\phi_\omega)=[1+\tfrac{1}{2}\mu''(\omega_0)(\omega-\omega_0)^2][d(\omega_0)+d'(\omega_0)(\omega-\omega_0)+\tfrac{1}{2}d''(\omega_0)(\omega-\omega_0)^2]+O((\omega-\omega_0)^3)$$
$$=d(\omega_0)+d'(\omega_0)(\omega-\omega_0)+\tfrac{1}{2}a(\omega-\omega_0)^2+O((\omega-\omega_0)^3)\ ,$$

where

$$a=d''(\omega_0)+d(\omega_0)\mu''(\omega_0)$$
$$=d''-\frac{2}{3}\frac{d(d'')^2}{(d')^2}=\frac{4}{3}\frac{d''d^{5/2}}{(d')^2}(d^{-1/2})''\ . \tag{19}$$

Since $d>0$, $d'\neq 0$, $d''>0$, and $(d^{-1/2})''>0$, we have $a>0$. Therefore,

$$E(\phi_\omega)-\omega Q(\phi_\omega)>d(\omega_0)+d'(\omega_0)(\omega-\omega_0)$$

for ω close enough to ω_0. That is,

$$E(\phi_\omega)>E(\psi_{\omega_0})-\omega_0 Q(\psi_{\omega_0})$$
$$-(\omega-\omega_0)Q(\psi_{\omega_0})+\omega Q(\phi_\omega)\ ,$$

so that

$$E(\phi_\omega)>E_0\ . \tag{20}$$

This theorem allows us to find the stability regions under dilations for the localized solutions (2). The unusual assumption that $d^{-1/2}$ is convex may be interpreted as a growth restriction on the nonlinear contribution to the energy.

IV. APPLICATION TO THE SOLER MODEL

The field equation of the Soler model is

$$i\gamma^0\partial_t\psi+i\gamma^k\partial_k\psi-m\psi+2\varepsilon(\bar{\psi}\psi)\psi=0\ . \tag{21}$$

In this case $F(s)=-ms+\varepsilon s^2$. There are localized solutions separable in spherical coordinates of the form

$$\psi(x,t)=e^{-i\omega t}\begin{pmatrix} g(r)\\ 0\\ if(r)\cos\theta\\ if(r)\sin\theta e^{i\phi}\end{pmatrix} \tag{22}$$

provided the radial scalar functions f and g satisfy

$$\frac{dg}{dr}+(1+\omega)f-2(g^2-f^2)f=0\ ,$$
$$\frac{df}{dr}+\frac{2}{r}f+(1-\omega)g-2(g^2-f^2)g=0\ , \tag{23}$$

where $0<\omega<1$ and we have chosen $m=\varepsilon=1$. Localized solutions of (21) were found numerically by Finkelstein *et al.*,[13] but it was Soler[14] who first proposed them as a model of extended fermions. The physical properties of such models are well reviewed by Rañada.[15] We compute

$$E(\psi_\omega)\equiv E(\omega)=2\pi(\omega I_1+\tfrac{1}{2}I_2)\ ,$$
$$Q(\psi_\omega)\equiv Q(\omega)=2\pi I_1,\ d(\omega)=\pi I_2\ ,$$

where

$$I_1=\int_0^\infty (g^2+f^2)r^2dr\ ,$$
$$I_2=\int_0^\infty (f^2-g^2)^2r^2dr\ .$$

Some necessary conditions for the existence of solutions are given by Vázquez[16] and by Mathieu and Morris.[17] Cazenave and Vázquez[18] have proved the existence of localized solutions of (23). Soler[14] computed that $E(\omega)$ and $Q(\omega)$ both have minima at $\omega_c=0.936$. Since $d'(\omega)=-Q(\omega)$ we have

$$\omega<\omega_c\rightarrow Q'(\omega)<0\rightarrow d''(\omega)>0\ ,$$
$$\omega>\omega_c\rightarrow Q'(\omega)>0\rightarrow d''(\omega)<0\ .$$

The condition $(d^{-1/2})''>0$ is equivalent to $3Q^2>-2dQ'$. For $\omega>\omega_c$, $Q'>0$, so it is obvious. For $\omega<\omega_c$, it has been partially checked numerically. So $d^{-1/2}$ appears to be always convex. We note that this convexity implies $d<cQ^{2/3}$.

We infer from Theorem 1 that the localized solutions are dilation unstable for $0.936<\omega<1$. We conjecture that they are dilation stable for $0<\omega<0.936$. This conjecture agrees perfectly with the numerical computations of Alvarez and Soler.[5]

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