Extension of the gauge technique to broken symmetry and finite temperature

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(Received 13 January 1986)

The gauge technique generates vertex functions which automatically satisfy Ward identities and which are good approximations to the true vertices for small momenta. We show how to construct such vertex functions for very general situations, including spontaneously broken non-Abelian gauge symmetries and finite-temperature gauge theory. Potential applications are outlined.

I. INTRODUCTION

This work describes the construction of three-point vertex functions for the gauge mesons of a non-Abelian gauge theory which satisfy the relevant Ward identities by construction, even when the gauge theory is spontaneously broken or at finite temperature. Such gauge-technique vertex functions have not previously been constructed, to the authors' knowledge, except for the work of Delbourgo and collaborators,[1] which emphasizes different aspects of the problem.

The gauge technique has a long history.[2] Its modern era begins[3,4] with the recognition that the propagator equation for a charged spin-0 or -$\frac{1}{2}$ particle becomes linear (thus soluble) if the full vertex is approximated by the gauge-technique vertex and the photon propagator is replaced by the bare propagator. This approximation, exact in the infrared, is not renormalizable but can be made so[5] without sacrificing linearity by adding specific transverse terms to the gauge-technique vertex. More recently numerous authors[6–10] have used the gauge technique for (quarkless) QCD, where the relevant equations are necessarily nonlinear.[11] In these applications the gauge symmetry is unbroken, and Lorentz (or Euclidean) invariance holds. For the applications we have in mind either the gauge symmetry is spontaneously broken with no elementary scalar fields in the Lagrangian (which necessitates the addition of fermions in other than the fundamental representation of the gauge group[12]), or the theory is at finite temperature so Lorentz invariance no longer holds. In either case it is not obvious how to use the standard technique[3] of exploiting the Lehmann representation of the propagator. This is unfortunate, because the spectral-representation form of the gauge technique is convenient for computing in the nonlinear cases of interest. In this work, we show that in the case of spontaneously broken symmetry a gauge-technique vertex can still be constructed from a propagator spectral representation, but now involving a double spectral integral. We also show that for both this case and for finite T, where there is no useful (as far as we can see) Lehmann representation for the propagator, there exists a vertex expressed algebraically in terms of proper self-energies which satisfies the Ward identities. It is generally possible to write the spectral form of the vertex as an algebraic form involving proper self-energies (as King[5] has done for the spectral form[3] of the QED vertex).

The main idea behind our construction is to introduce[12] auxiliary massless scalar fields with the help of which one may construct gauge-invariant mass terms for the gauge potentials. The auxiliary scalars are eliminated by using their equation of motion, and masses replaced by proper self-energies. The result is an effective Lagrangian containing three-point (and higher) vertices which are guaranteed to satisfy the Ward identities, whatever the form of the proper self-energies (which are conserved, according to other Ward identities). At $T=0$ an alternative approach identifies the gauge-meson mass with the integration parameter of the Lehmann representation, and gives rise to the spectral-representation form of the vertices.

In the effective Lagrangian the vector-meson masses are treated as constants, and the resulting gauged nonlinear σ model is not necessarily renormalizable. But the effective Lagrangian is only used as a tool, later discarded, to solve the Ward identities. The renormalizability of the spontaneously broken original Lagrangian depends on the vanishing of dynamically generated masses at large momentum, which is known[7,12] to happen for the models discussed here.

II. THE EXTENDED GAUGE TECHNIQUE

The Ward identities which we wish to "solve" are very complicated, except in ghost-free gauges where they take on the naively expected form. We will use the light-cone gauge throughout this paper, but this is not at all a restriction. The conventional propagator and vertices even in a ghost-free gauge are virtually devoid of physical meaning because they depend nontrivially on the gauge chosen, so it is of the utmost importance to recall[7] that a resummation of the graphs entering a physical, gauge-invariant process yields new proper vertices and proper self-energies which are completely gauge independent in ghost-free gauges[13] and which obey the same Ward identities (but not the same Schwinger-Dyson equations) as the usual self-energy and vertex in a ghost-free gauge. Our considerations apply to both types of vertices and self-energies; it must be kept in mind that the modified self-energy of Ref. 7 depends on only one scalar function instead of two, as occur for the conventional self-energy in noncovariant gauges.

Begin by introducing the gauge potential in the standard anti-Hermitian matrix form

$$A_\mu = \frac{1}{2i}\sum_a \lambda_a A_\mu^a \ , \tag{1}$$

which under a gauge transformation V changes to

$$A'_\mu = VA_\mu V^{-1} - g^{-1}(\partial_\mu V)V^{-1} \ . \tag{2}$$

Introduce[12] also a matrix U depending on angles θ^a as

$$U(\theta) = \exp\left[\frac{i}{2}\sum \lambda_a \theta^a\right] \tag{3}$$

and postulate the gauge-transformation property

$$U' = U(\theta') = VU \ . \tag{4}$$

It then follows[12] that the modified gauge potential

$$B_\mu \equiv U^{-1}A_\mu U - g^{-1}(\partial_\mu U^{-1})U \tag{5}$$

is invariant under the gauge transformation (2), (4), and an invariant-mass term $\sim m^2 \mathrm{Tr} B_\mu B^\mu$ can be added to the usual Lagrangian. Upon variation of the new term with respect to the angles θ^a, equations of motion for these angles are found which can be solved in a power series:[12]

$$\theta^a = g\frac{1}{\Box}\partial_\mu A_a^\mu - \frac{g^2}{\Box}\left[\tfrac{1}{2}(\partial\cdot A)\times\frac{1}{\Box}(\partial\cdot A) + A^\mu \times \partial_\mu \frac{1}{\Box}\partial\cdot A\right]_a + \cdots \ , \tag{6}$$

where

$$(A\times B)_a \equiv f_{abc}A_b B_c \tag{7}$$

in terms of the group structure constants. The gauge transformation law (2) when used in (6) verifies the supposed law for the θ^a given in (4). When (7) is substituted in (6), the mass term in the Lagrangian contains a series of terms, not only of $O(A^2)$, but also with $A^3, A^4, \ldots$ terms. The three-point terms thus found, when added to the three-point functions of the massless theory, yield[7] an effective vertex for the massive theory (see Fig. 1 for notation):

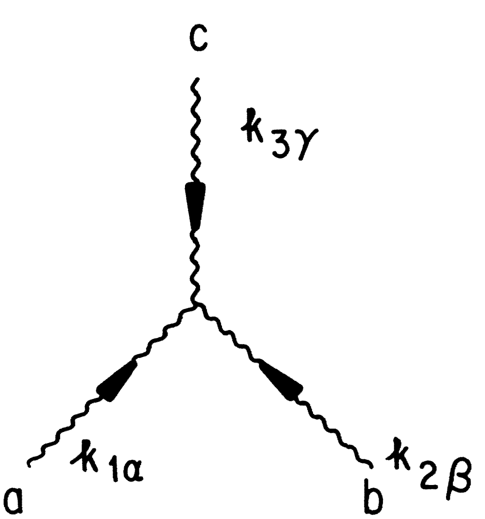

FIG. 1. The three-point Yang-Mills vertex.

$$\Gamma^{abc}_{\alpha\beta\gamma}(k_1,k_2,k_3) \equiv i f_{abc}\Gamma^{(m)}_{\alpha\beta\gamma} \ ,$$
$$\Gamma^{(m)}_{\alpha\beta\gamma} = (k_1-k_2)_\gamma g_{\alpha\beta} + \frac{m^2}{2}\frac{k_{1\alpha}k_{2\beta}(k_1-k_2)_\gamma}{k_1^2 k_2^2} + \text{c.p.} \ , \tag{8}$$

where c.p. refers to cyclic permutations of the indices and momenta. The longitudinally coupled massless particle poles are akin to Goldstone poles, and never appear in any physical process; they are purely gauge degrees of freedom, as (4) makes evident. The massless poles in (8) are, of course, the momentum-space transcription of the $\Box^{-1}$ appearing in (6). Even though use of (6) may appear to introduce an element of nonlocality, this appearance is spurious; we have merely integrated out degrees of freedom originally appearing in a fully local Lagrangian. All properties of local field theory, i.e., the Lehmann representation, still hold. One may similarly find mass corrections to the four-point vertex, which we do not record here. The vertex (8) obeys the Ward identities appropriate (in a ghost-free gauge) to the massive theory:

$$k_1^\alpha \Gamma^{(m)}_{\alpha\beta\gamma} = \Delta^{-1}{}_{\beta\gamma}(k_2) - \Delta^{-1}{}_{\beta\gamma}(k_3) \ , \tag{9}$$

where in the light-cone gauge $n\cdot A = 0$, $n^2=0$,

$$\Delta^{-1}{}_{\beta\gamma}(k) = \left[-g_{\beta\gamma} + \frac{k_\beta k_\gamma}{k^2}\right](k^2-m^2) - \frac{n_\mu n_\nu}{\eta} \ , \tag{10}$$

and η is a parameter to be set equal to zero once a physical process has been calculated. In earlier work[7] on unbroken gauge theories at $T=0$, (9) and (10) were generalized by interpreting m^2 as the integration variable in the Lehmann representation for a modified propagator $\hat{\Delta}_{\mu\nu}$, which is the usual propagator $\Delta_{\mu\nu}$ plus terms arising from higher-point functions. This propagator has the special property that its proper self-energy $\hat{\Pi}_{\mu\nu}$ is completely independent of the gauge and in particular of n_μ in the light-cone gauge, so $\hat{\Pi}_{\mu\nu}(k) = (-k^2 g_{\mu\nu} + k_\mu k_\nu)\Pi(k^2)$ and the Lehmann representation is, in this gauge,

$$\hat{\Delta}_{\mu\nu}(k) = Q_{\mu\nu}\int dm^2 \frac{\rho(m^2)}{k^2-m^2+i\epsilon} \ , \tag{11}$$

$$Q_{\mu\nu} = -g_{\mu\nu} + \frac{n_\mu k_\nu + n_\nu k_\mu}{n\cdot k} \ . \tag{12}$$

[A term in $Q_{\mu\nu}$ which vanishes with η of (10) has been dropped.] Corresponding to the modified propagator there is[7] a modified vertex $\hat{\Gamma}_{\alpha\beta\gamma}$, obeying the Ward identity (9) with $\hat{\Delta}$ in place of Δ. The gauge-technique vertex given in Ref. 7 is implicitly defined by

$$\hat{\Delta}^{\beta\beta'}(k_2)\hat{\Gamma}_{\alpha\beta'\gamma'}\hat{\Delta}^{\gamma\gamma'}(k_3) = \int dm^2 \rho(m^2)\frac{Q^{\beta\beta'}(k_2)\Gamma^{(m)}_{\alpha\beta'\gamma'}Q^{\gamma\gamma'}(k_3)}{(k_2{}^2-m^2+i\epsilon)(k_3{}^2-m^2+i\epsilon)} \tag{13}$$

which obeys the Ward identity only for the index α, and lacks complete Bose symmetry.

We are now ready to give the new results which are the central point of this paper. Evidently, if there is a completely gauge-invariant field like B_μ in (6) we are not restricted, in writing down effective Lagrangians, just to a constant scalar mass matrix which is a multiple of the identity in group space. The most general form of the "mass" term of the effective Lagrangian that we need to consider is

$$\int d^4x\, \delta\mathscr{L} = \int d^4x\, d^4y\, \mathrm{Tr} B^\mu(x) \hat{G}_{\mu\nu}(x-y) B^\nu(y) , \tag{14}$$

where the (Hermitian) functions $\hat{G}_{\mu\nu}$ need have no particular Lorentz-transformation properties, nor are they multiples of the identity in group space. We can now repeat the procedures leading to Eqs. (6) and (7). The equations of motion for the θ^a fields are compactly written as

$$D^\mu(U\{\hat{G}_{\mu\nu}, * B^\nu\}U^{-1}) = 0 , \tag{15}$$

where D^μ is the usual gauge-covariant derivative, and the $*$ operation is simply a convolution integral; note that the arguments of U and U^{-1} in (15) are the same as that of D^μ and do not get convoluted. For general (invertible) $\hat{G}_{\mu\nu}$ the perturbation solutions to these equations involve terms such as in (5) plus corrections if $\hat{G}_{\mu\nu}$ is not a multiple of the identity; the solution when substituted in $\delta\mathscr{L}$ of (14) yields a series of terms from which the three-point vertex can be read off. It turns out that the vertex involves only the transverse projection of $\hat{G}_{\mu\nu}$, so we go to momentum space [using $\hat{G}_{\mu\nu}(k)$ to indicate the Fourier transform of $\hat{G}_{\mu\nu}(x)$] and define, for the unrenormalized self-energy

$$\hat{\Pi}_{\mu\nu}(k) = P_{\mu\alpha}(k) P_{\nu\beta}(k) \hat{G}^{\alpha\beta}(k) , \tag{16}$$

$$P_{\mu\alpha}(k) = -g_{\mu\alpha} + \frac{k_\mu k_\alpha}{k^2} . \tag{17}$$

We give the gauge-technique vertex only for the case when $\hat{\Pi}_{\mu\nu}$ is diagonal in group space, so it could be labeled by a single index, as $\hat{\Pi}^a_{\mu\nu}$. We suppress this group index in the formula below; it is to be inferred from the momentum label. Then the result is, for the unrenormalized vertex,

$$\hat{\Gamma}_{\alpha\beta\gamma} = g_{\alpha\beta}(k_1-k_2)_\gamma - \frac{1}{2}\frac{k_{1\alpha}k_{2\beta}}{k_1{}^2 k_2{}^2}(k_1-k_2)^\mu \hat{\Pi}_{\mu\gamma}(k_3) - [P_\alpha{}^\mu(k_1)\hat{\Pi}_{\mu\beta}(k_2) - \hat{\Pi}_\alpha{}^\mu(k_1)P_{\mu\beta}(k_2)]\frac{k_{3\gamma}}{k_3{}^2} + \text{c.p.} \tag{18}$$

As the notation indicates, we are dealing here with the modified (gauge-invariant) proper self-energies and vertex. It is easy to see that $\hat{\Gamma}_{\alpha\beta\gamma}$ of (18) has complete Bose symmetry and that it obeys a Ward identity on all three legs, e.g.,

$$k_1^\alpha \hat{\Gamma}_{\alpha\beta\gamma} = \hat{\Delta}^{-1}{}_{\beta\gamma}(k_2) - \hat{\Delta}^{-1}{}_{\beta\gamma}(k_3) \tag{19}$$

where, in the light-cone gauge,

$$\hat{\Delta}^{-1}{}_{\beta\gamma}(k) = -g_{\beta\gamma}k^2 + k_\beta k_\gamma - \hat{\Pi}_{\beta\gamma}(k) - \frac{n_\beta n_\gamma}{\eta} . \tag{20}$$

We will see below how renormalization affects the Ward identity (19). Note that satisfaction of the Ward identities depends *only* on the transversality of the $\hat{\Pi}_{\mu\nu}$. In particular, $\hat{\Pi}_{\mu\nu}$ need not be Lorentz covariant, so (18) can be used at finite T.

A special case[14] of (18) in which the $\hat{\Pi}_{\mu\nu}$ represent a diagonal nonsinglet mass matrix,

$$\hat{\Pi}^a_{\mu\nu}(k) = P_{\mu\nu}(k) m_a{}^2 , \tag{21}$$

is useful in giving an alternative gauge-technique vertex at $T=0$, where Lorentz invariance holds and the Lehmann representation is available. The alternate vertex will resemble (13), except that there will be two spectral integrations for the case given below. To be concrete, consider the Georgi-Glashow model: an O(3) gauge group, with symmetry breaking to be driven by fermions in the adjoint representation; we omit all scalar fields. After symmetry breaking there will be two charged vectors $W^\pm$ with the same mass, and a neutral vector of different mass. In Fig. 1, let line 1 represent a W^+, line 2 a Z, and line 3 a W^-. Substitute (21) in (18), which defines a vertex $\hat{\Gamma}_{\alpha\beta\gamma}(m_W{}^2, m_Z{}^2)$. Now we will treat both $m_W{}^2$ and $m_Z{}^2$ as spectral integration parameters, writing (in the light-cone gauge)

$$\hat{\Delta}^{W,Z}_{\mu\nu} = Q_{\mu\nu}\int \frac{dm^2 \rho_{W,Z}(m^2)}{k^2-m^2+i\epsilon} . \tag{22}$$

The resulting W^+W^-Z vertex is defined implicitly, as in (13), by

$$\hat{\Delta}_Z^{\beta\beta'}(k_2)\hat{\Gamma}_{\alpha\beta'\gamma'}\hat{\Delta}_W^{\gamma\gamma'}(k_3) = \int dm_Z{}^2\rho_Z(m_Z{}^2)\int dm_W{}^2\rho_W(m_W{}^2)\frac{Q^{\beta\beta'}(k_2)\hat{\Gamma}_{\alpha\beta'\gamma'}(m_W{}^2,m_Z{}^2)Q^{\gamma\gamma'}(k_3)}{(k_2{}^2-m_Z{}^2+i\epsilon)(k_3{}^2-m_W{}^2+i\epsilon)} . \tag{23}$$

Multiply the left-hand side of (23) by k_1^α to get

$$\hat{Z}_Z \Delta^W_{\beta\gamma}(k_3) - \hat{Z}_W \hat{\Delta}^Z_{\beta\gamma}(k_2) , \tag{24}$$

where

$$\hat{Z}_{Z,W} = \int dm^2 \rho_{Z,W}(m^2) . \tag{25}$$

To the extent that $\hat{Z}_{Z,W}$ differ from unity, we interpret them as renormalization effects. If we consider all the propagators and vertices in (23) to be renormalized, then it is necessary to make the interpretation

$$\hat{Z}_{Z,W} = \frac{\hat{Z}_1}{\hat{Z}_{3Z,W}} \tag{26}$$

in terms of the hatted vertex and wave-function renormalization constants which are the analogs, for $\hat{\Delta}$ and $\hat{\Gamma}$, of the usual renormalization constants; the hatted constants are gauge invariant.[7] In the symmetric case, $\hat{Z}_1 = \hat{Z}_{3Z} = \hat{Z}_{3W}$. Previous work[5–7] shows that the gauge-technique vertex must be supplemented by additional conserved terms, unimportant in the infrared but needed in the ultraviolet regime, in order to renormalize the vertices and propagators consistently. Since we do not take up these additional terms here, it is somewhat premature to discuss renormalization effects.

III. POTENTIAL APPLICATIONS

We would like to describe actual applications, but these are in a rather primitive stage of development because for broken symmetries or at finite T the sheer number of terms in $\hat{\Gamma}_{\alpha\beta\gamma}$ is so large that computations—even with symbolic manipulation programs—are extremely difficult. Instead, we make some qualitative remarks about the role of (electric and magnetic) gluon mass generation at finite T (Ref. 10), especially in connection with the deconfining phase transition. In addition, we give the Schwinger-Dyson equation based on (18) for the symmetric case (e.g., $T=0$ QCD), and contrast the picture of gluon-mass generation thus found with the analogous picture based on the spectral form of the gauge technique.[7] The latter will be taken up first.

A. Gluon-mass generation in QCD at $T=0$

The Euclidean Schwinger-Dyson equation previously derived[7] from the spectral form of the gauge-technique vertex as given in (13) reads, in ostensibly renormalized form,

$$d^{-1}(q^2)=d^{-1}(0)+q^2\left[Kg^2+bg^2\int_0^{q^2/4}dz\,d(z)\left[1-\frac{4z}{q^2}\right]^{1/2}\right]+\frac{bg^2}{11}\int_0^{q^2/4}dx\,d(z)\left[1-\frac{4z}{q^2}\right]^{1/2} . \tag{27}$$

Here b is the lowest-order coefficient in the β function: $\beta=-bg^2+\cdots$, K is chosen so that $d^{-1}(q^2=\mu^2)=\mu^2$ for some large renormalization momentum μ, and $d(q^2)$ is defined in terms of the spectral weight ρ by [see (11)]

$$d(q^2)=\int\frac{dm^2\rho(m^2)}{q^2+m^2} . \tag{28}$$

Actually, as pointed out in Ref. 7, (27) does not obey the expected renormalization-group equations, which tell us that the equation for d, when written in terms of $D=g^2d$, should not contain g^2 explicitly. This would require an extra power of g^2 in the integrals on the right-hand side (RHS) of (27). We accomplish this by multiplying the integrals of (27) by

$$g^2\bar{g}^{-2}(z)=1+bg^2\ln(z/\mu^2) , \tag{29}$$

where $\bar{g}^2(z)$ is the running charge, given to first order in g^2 on the RHS of (29). Thus, instead of actually calculating higher-order corrections to (27) which would lead to a renormalizable equation, we observe that these corrections must respect the renormalization group, which allows at least a plausible guess for getting the correct ultraviolet behavior. Any uncertainty as to whether z or q^2 should be the argument of $\bar{g}^{-2}$ in (29) is settled by the solution of (27) as modified by (29), which shows that z is indeed the correct choice.

Our interest is in (27) at large q. Make[7] the ansatz

$$d^{-1}(q)=[q^2+m^2(q^2)]\times\left[1+bg^2\ln\left[\frac{4m^2(q^2)+q^2}{\mu^2}\right]\right] \tag{30}$$

and put it in (27), with the modification (29). It is legitimate at large q to drop $4m^2$ relative to q^2 in the logarithm in (30); saving it corresponds to a redefinition of μ. We assume that $m^2(q^2)$ vanishes as an inverse power of $\ln q^2$ at large q, which allows us to separate uniquely (modulo the definition of μ and K) the Schwinger-Dyson equation into two parts: one for the kinetic term going like $q^2\ln q^2$, and one for $m^2(q^2)$. At large q^2, the factors $(1-4z/q^2)^{1/2}$ can be dropped, as can be verified by estimating the integrals over z using the ansatz (30). The kinetic term equation yields the expected result $d^{-1}(q)\to q^2(1+bg^2\ln q^2)$, and the mass-term equation is easily rearranged to

$$[1+bg^2\ln(q^2/\mu^2)]m^2(q)=d^{-1}(0)-\frac{bg^2}{11}\int_0^\infty\frac{dz\,m^2(z)}{z+m^2(z)}+bg^2\int_{q^2/4}^\infty\frac{dz\,m^2(z)}{z+m^2(z)}\left[\frac{1}{11}+\frac{q^2}{4z}\right] . \tag{31}$$

Since, by hypothesis, $m^2(q)$ vanishes at $q=\infty$, the first two terms on the RHS of (31) must cancel; we verify this later. Assuming this cancellation, one easily finds that the solution to (31) is

$$m^2(q)\sim(\ln q^2)^{-12/11}[1+O((\ln q^2)^{-1})] . \tag{32}$$

As with (30), we make an ansatz for $m^2(q)$ which is useful down to $q=0$:

$$m^2(q)=m^2\left[1+bg^2\ln\left[\frac{q^2+4m^2}{\mu^2}\right]\right]^{-12/11}\times\left[1+bg^2\ln\left[\frac{4m^2}{\mu^2}\right]\right]^{-12/11} . \tag{33}$$

Thus $m^2=m^2(q=0)$. In view of the relation between g and μ based on the one-loop definition of the renormalization-group invariant mass Λ,

$$bg^2=[\ln(\mu^2/\Lambda^2)]^{-1}\ , \tag{34}$$

and because $m\sim\Lambda$, we see that (33) is renormalization-group invariant.

Consider the cancellation of the first two terms on the RHS of (31). The integral I, where

$$I=\frac{bg^2}{11}\int_0^\infty \frac{dz\, m^2(z)}{z+m^2}\ , \tag{35}$$

is greater than the corresponding integral with $z+m^2$ replaced by $z+4m^2$, whose exact value is $bg^2m^2\ln(4m^2/\Lambda^2)=d^{-1}(0)$. The difference of these two integrals is bounded by dropping the logarithmic dependence, and we finally come to

$$I=d^{-1}(0)+bg^2m^2\epsilon,\ \ 0\le\epsilon<(\ln 4)/11\ . \tag{36}$$

Since[7] $m\simeq 2\Lambda$, I differs from $d^{-1}(0)$ by less than 5%, and it is clear that exact cancellation can be found by a minor adjustment of $m^2(q)$ at small q.

We conclude that the spectral form of the gauge technique, when modified to conform to the renormalization group, leads to the generation of a gluon mass falling as an inverse power of $\ln q^2$, as indicated in Ref. 7. Now turn to the problem of mass generation as described by the new gauge-technique vertex of (18).

As is shown in Ref. 7, $\hat{\Pi}_{\mu\nu}$ has the form

$$\hat{\Pi}_{\mu\nu}=(-g_{\mu\nu}+q_\mu q_\nu q^{-2})\hat{\Pi}(q^2) \tag{37}$$

for $T=0$ QCD. Following the prescriptions of Ref. 7, we have constructed the Schwinger-Dyson equation for $\hat{\Pi}$ by adding to the usual one-loop term, based on $\hat{\Gamma}$ of (18), extra terms arising from the modifications turning Π into $\hat{\Pi}$ and Γ into $\hat{\Gamma}$. In this equation, we drop all terms explicitly dependent on the light-cone gauge vectors n_μ, since we know they must cancel, even before doing momentum integrations.[10] The result explicitly verifies self-consistency of the conserved form (37) for $\hat{\Pi}_{\mu\nu}$, and the Schwinger-Dyson equation is written as a scalar equation for $\hat{\Pi}(q^2)$. We write this below for the Euclidean propagator

$$d^{-1}(q)\equiv q^2+\hat{\Pi}(q^2) \tag{38}$$

and find

$$d^{-1}(q)=q^2Kg^2-\frac{N_cg^2}{(2\pi)^4}\int d^4k\left\{-2d(k)+d(k)d(k+q)\left[3q^2+d^{-1}(q)+\frac{4}{3}\left[k^2-\frac{(k\cdot q)^2}{q^2}\right]\right]\right\}, \tag{39}$$

where again K is introduced to renormalize the equation.

As with (27), (39) is not really renormalization-group invariant, and we simply guess at a solution to this problem by replacing each propagator $d(p)$ on the RHS of (39) by

$$\hat{d}(p)\equiv g^2\bar{g}^{-2}(p)d(p)\ . \tag{40}$$

We identify[6] $g^2\bar{g}^{-2}$ with the second factor on the RHS of (30) and come to

$$\hat{d}(p)^{-1}=p^2+m^2(p)\ . \tag{41}$$

Simple rearrangement of (39) then leads to

$$\begin{aligned} d^{-1}(q)={}&q^2Kg^2-\frac{bg^2q^2}{\pi^2}\int d^4k\,\hat{d}(k)\hat{d}(k+q)+\frac{4bg^2}{11\pi^2}\int d^4k\,\hat{d}(k)+\frac{bg^2}{11\pi^2}\int d^4k\,\hat{d}(k)\hat{d}(k+q)[4m^2(k)-3m^2(q)]\\ &+\frac{bg^2}{11\pi^2q^2}\int d^4k\,\hat{d}(k)\hat{d}(k+q)[m^2(k)-m^2(k+q)]^2+\frac{2bg^2}{11\pi^2q^2}\int d^4k\,\hat{d}(k)[m^2(k)-m^2(k+q)]\ . \end{aligned} \tag{42}$$

If, on the RHS of (42), all momentum-dependent masses $m(p)$ are replaced by constants m, and the integrals are further integrated over the spectral density $\rho(m^2)$ of (28), the result is precisely what would be found using the spectral form of the gauge technique. That is, Eq. (27) would follow from this procedure. This immediately shows the infrared equivalence of (27) and (42) since for small q, k the running masses can be replaced by constants. Not unexpectedly, these two equations are not the same in the ultraviolet, if for no other reason than that we have guessed at modifications to the original forms which make them renormalization-group compatible, and these modifications are not unique. For example, replacing $4m^2(k)-3m^2(q)$ by, e.g., $m^2(k)$ in (42) is equally good both in the infrared regime and for renormalization-group compatibility. But (27) and (42) do lead to the same *lowest-order* behavior of the running mass $m^2(q)$ at large q, as one may see by replacing all m^2 on the RHS of (42) by a bare term $m_0{}^2$ and renormalizing, as was already done[7] for Eq. (27), namely,

$$m^2(q)=m^2(1-\tfrac{12}{11}bg^2\ln q^2)\ , \tag{43}$$

which is, of course, the expansion of the solution (33) at lowest order in g^2. Past perturbation theory, it is interesting to note that (42) may have solutions in which $m^2(q)$ behaves like q^{-2} (up to powers of logarithms) at large q. The RHS of (42) yields, after angular integrations and separating off terms growing like $q^2\ln q^2$,

$$bg^2m^2(q)\ln q^2=\frac{bg^2}{11q^2}\int^{q^2}dk^2[6m^2(k)-3m^2(q)]+\cdots\ , \tag{44}$$

where the omitted terms are small at least by one power of

$\ln q^2$ [we have dropped constants which must cancel, as also appear in the spectral version, Eq. (31)]. Because of the already-mentioned ambiguity in choosing the arguments of the running masses, the logarithmic power in $q^2 m(q)$ is also ambiguous. It is very likely that the correct QCD running mass $m^2(q)$ does fall off like $q^{-2}(\ln q^2)^a$, but to decide between this behavior and the behavior of (32) is beyond our powers, until the gauge technique is further refined in the ultraviolet regime as King[5] has done for QED and the electron mass.

B. QCD at finite temperature

We wish to imitate the $T=0$ procedure of finding a modified propagator $\hat{\Delta}_{\mu\nu}$ whose self-energy is gauge independent. However, at finite T there are two conserved tensor forms for the proper self-energy:

$$\hat{\Pi}_{\mu\nu}(q)=\hat{\Pi}_1(q^2)\left[-g_{\mu\nu}+\frac{q_\mu q_\nu}{q^2}\right]+\hat{\Pi}_2(q^2)\left[U_\mu-\frac{U\cdot q q_\mu}{q^2}\right]\left[U_\nu-\frac{U\cdot q q_\nu}{q^2}\right], \quad (45)$$

where $U_\mu=(1,0)$ signals the distinction between time and space for $T\neq 0$. The kinematic gauge dependence of $\hat{\Delta}_{\mu\nu}$ is more complicated than at $T=0$; for example, in the light-cone gauge we find, instead of (11),

$$\hat{\Delta}_{\mu\nu}=\left[Q_{\mu\nu}-\frac{q^2}{(U\cdot q)^2-q^2}\left[U_\mu-\frac{n\cdot U}{n\cdot q}q_\mu\right]\left[U_\nu-\frac{n\cdot U}{n\cdot q}q_\nu\right]\right]\hat{\Delta}_M(q)+\frac{q^2}{(U\cdot q)^2-q^2}\left[U_\mu-\frac{n\cdot U}{n\cdot q}q_\mu\right]\left[U_\nu-\frac{n\cdot U}{n\cdot q}q_\nu\right]\hat{\Delta}_E(q)\,, \quad (46)$$

where the gauge-invariant electric and magnetic propagators $\hat{\Delta}_E, \hat{\Delta}_M$ are

$$\hat{\Delta}_E{}^{-1}=q^2-\hat{\Pi}_1-\frac{q^2\hat{\Pi}_2}{(U\cdot q)^2-q^2}, \quad \hat{\Delta}_M{}^{-1}=q^2-\hat{\Pi}_1\,. \quad (47)$$

and the functions $\hat{\Pi}_i$ depend on $U\cdot q$ as well as q^2. The dependence on $U\cdot q$ essentially makes the Lehmann representation useless at finite T.

Next, in principle at least, we would use (45) in the gauge-technique vertex (18) and then write down two Schwinger-Dyson equations, one for $\hat{\Delta}_E$ and the other for $\hat{\Delta}_M$, following the rules of Ref. 7. We would find cancellation of all n-dependent terms, before momentum-space integration, and thus we could transfer these equations to Euclidean space and replace the integral over q_4 with the usual finite-T sum. Part of this program has already been carried out:[10] We have calculated $\hat{\Delta}_{E,M}$ in perturbation theory, and looked at the large-T limit of the $\hat{\Delta}_M$ equation using the spectral form of the gauge technique (which is possible since at large T the electric and magnetic sectors decouple[15]). At finite T, the electric and magnetic sectors couple in an elaborate way, and we are still working on the problem. A few qualitative remarks are in order, if only to forestall confusion.

At finite T, there is an electric gluon mass m_E and a magnetic mass m_M. Both have contributions of the sort discussed in Sec. III A, as well as terms $\sim T$; schematically,

$$m_{E,M}{}^2=\Lambda^2 f_{E,M}(T/\Lambda)+T^2 g_{E,M}(T/\Lambda)\,, \quad (48)$$

where the f's and g's are regular at $T=0$, and positive at least judging from nonperturbative finite-T results found in Ref. 10. Only the g_E term can be found in perturbation theory. It would be wrong to expect a signal for the deconfining phase transition of (quarkless) QCD of the sort seen in the behavior of fermionic mass gaps, as in superconductivity. There the fermionic gap goes to zero at the critical temperature T_c, because the gap equation has a kernel $\sim\tanh(\frac{1}{2}\beta\omega)$ which decreases as T increases. But a bosonic mass gap has a kernel $\coth(\frac{1}{2}\beta\omega)$ which instead increases, eventually leading to the dominance of the T^2 terms in (48). Instead, the most remarkable feature of the electric and magnetic gluon masses is that they show no special behavior at all near T_c. This simple feature largely determines the character of the deconfining phase transition, in the absence of quarks.

The phase transition is signaled by the disappearance of certain parts of the vortex condensate which has been argued[7] self-consistently accompanies gluon-mass generation. An individual vortex inhabits a region $\sim m^{-1}$ thick around a closed two-surface or world sheet; this closed surface can link topologically with a Wilson loop, and a condensate of vortices leads to an area law for a Wilson loop. At finite T we consider instead Wilson (or Polyakov) lines, extending in the time direction from $t=0$ to $t=\beta$. Vortices which can link with such lines have world sheets largely perpendicular to the time axis, thus a thickness which extends in the time direction. As T increases βm decreases, because of the Λ^2 terms in (48), and the vortices are squeezed; their action goes up and their entropy goes down so these vortices are energetically disfavored and decondense. But vortices whose world sheets contain the time axis are not squeezed. Both their action and their entropy decrease by losing the time degree of freedom, but the energetics still favor their condensation. These three-dimensional vortices, whose world sheet at large T has been reduced to a spacelike world string, support the magnetic mass gap and give rise to an area law for spacelike Wilson loops (which has nothing to do with confinement).

We are hardly in a position to supply quantitative details for the above scenario, but there is a counterpart calculation which could be carried out, in principle, using the gauge technique developed here. One would calculate the vacuum energy E_{vac} as a function of T, to see when it rises to equal purely perturbative ($\sim T^4$) contributions of thermal gluons. In Ref. 7, a crude estimate was made for E_{vac} at $T=0$ as a function of the gluon mass m, and for $m \simeq 500$ meV rough agreement was found with the known value coming from the trace anomaly and QCD sum rules:

$$E_{vac}(T=0)=\frac{\beta(g)}{8g}\langle G_{\mu\nu}{}^2\rangle \simeq -4\times 10^{-3}\ \text{GeV}^4 . \quad (49)$$

The leading finite-T correction to the rules of Ref. 6 is a term $\sim m^2T^2$, where m depends on T but is not singular or vanishing at $T=T_c$. We will not explicitly exhibit the dependence of m^2 on T. Let us make the crudest of estimates of this term by using a high-temperature expansion of the momentum-space integrals appearing in the expression for E_{vac} given in Eq. (6.22) of Ref. 7, turning frequency integrals into sums in the usual way. In so doing we ignore the distinction between electric and magnetic gluons. The result is, for N_c colors,

$$E_{vac}(T)\simeq E_{vac}(0)+\frac{N_c{}^2-1}{8}m^2T^2+O(T^4) , \quad (50)$$

where (at large T, at least) the T^4 terms represent the usual perturbative terms. The first two terms on the RHS of (50) vanish at a critical temperature T_c given by (for $N_c=3$)

$$T_c=m^{-1}\left[-\frac{\beta(g)}{8g}\langle G_{\mu\nu}{}^2\rangle\right]^{1/2} \simeq 130\ \text{meV} , \quad (51)$$

where the numerical result follows from $m \simeq 500$ MeV. Then, aside from perturbative terms,

$$|E_{vac}(T)| = |E_{vac}(0)|(1-T^2/T_c{}^2) . \quad (52)$$

Of course, we have no reason at all to trust the specific value of T_c given in (51), but the behavior (52) should be reliable. Now $-E_{vac}$ is the bag constant, and in phenomenological models of confinement where the pressure of the vacuum (i.e., the bag constant) balances the pressure of chromoelectric flux confined into a tube, it is well known that the string tension K_F is proportional to $(-E_{vac})^{1/2}$; that is,

$$K_F(T)=K_F(0)(1-T^2/T_c{}^2)^{1/2} . \quad (53)$$

The results (52) and (53) are also found[16] in phenomenological theories of chromoelectric flux tubes thought of as elementary strings.

Our program, then, is to use the gauge-technique vertex (18) along with the finite-T proper self-energy (45) and the Schwinger-Dyson equation rules of Ref. 6 to find the electric and magnetic masses as functions of T (expecting no unusual behavior near T_c), then to use the propagators, vertices, and masses in an expression for E_{vac} to establish quantitatively the results (52) and (53), as well as T_c itself.

ACKNOWLEDGMENTS

We are very grateful to S. Sabharwal for his important contributions at an early stage of this work. This work was supported in part by the National Science Foundation.

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