Superconformal anomalies from the path-integral measure

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The one-loop superconformal anomalies for a chiral superfield in background $N=1$ super Yang-Mills and $N=1$, $n=-\frac{1}{3}$ supergravity fields are derived by considering the dependence of the chiral measure on the supergravity compensator superfield. The superspace ζ-function method is used for regularization.

A nongraphical technique for calculating one-loop chiral and trace anomalies for matter fields in background Yang-Mills and gravitational fields has been given by Fujikawa.[1–3] This involves considering the variation of the functional-integral measure of the matter fields under the chiral and local scale transformations. We show here that one-loop superconformal anomalies may also be calculated from the functional-integral measure. The system considered consists of a chiral superfield η, coupled to a background $N=1$, super Yang-Mills field described by a real superfield V, and coupled to background $N=1$, $n=-\frac{1}{3}$ supergravity, described by a Hermitian axial-vector superfield $H_{\alpha\dot{\alpha}}$, and chiral compensator superfield Φ.

First define the one-loop generating functional (we work in Euclidean space)

$$Z[H_{\alpha\dot{\alpha}},V,\Phi,\tilde{\Phi}]=\int D\mu\, D\tilde{\mu}\exp(S[H_{\alpha\dot{\alpha}},V,\Phi,\tilde{\Phi},\eta,\tilde{\eta}])\,, \tag{1}$$

where the action S is

$$S[H_{\alpha\dot{\alpha}},V,\Phi,\tilde{\Phi},\eta,\tilde{\eta}]\equiv\int d^8z\,\hat{E}^{-1/3}(1\cdot e^{-\overleftarrow{H}})^{1/3}\times(\tilde{\Phi}\tilde{\eta})(\Phi\eta)\,, \tag{2}$$

and the functional-integral measures are

$$D\mu=D(\Phi^{3/2}\eta)\,,\quad D\tilde{\mu}=D(\tilde{\Phi}^{3/2}\tilde{\eta})\,. \tag{3}$$

Notation and conventions are those of Ref. 4. $\tilde{\eta}(\tilde{\Phi})$ is the super Yang-Mills- and supergravity-covariant conjugate of $\eta(\Phi)$. The integration variable $\Phi^{3/2}\eta$ is the superspace version of the integration variable $g^{1/4}\Phi$ which is used for an ordinary scalar field $\Phi(x)$ in a background gravitational field $g_{\mu\nu}(x)$ ($g\equiv\det g_{\mu\nu}$). This latter choice of variable is fixed by the requirement that there be no gravitational anomaly.[5] Note that $\Phi^3=e^{-1}+(\theta$ terms) where $e\equiv\det e_a{}^{\mu}$ (see Ref. 4).

We will now assume that there is no local supersymmetry anomaly for the system we are considering (see the comment at the end of this paper). This means that the generating functional Z defined by Eq. (1) is invariant under the L-gauge transformations of super Yang-Mills plus supergravity. Thus

$$0=\underset{\sim}{\Delta}_L\ln Z=(\underset{\sim}{\Delta}_L H_{\alpha\dot{\alpha}})\frac{\underset{\sim}{\Delta}\ln Z}{\underset{\sim}{\Delta}H_{\alpha\dot{\alpha}}}+\left[(\underset{\sim}{\Delta}_L\Phi)\frac{\underset{\sim}{\Delta}\ln Z}{\underset{\sim}{\Delta}\Phi}+\text{c.c.}\right]\,, \tag{4}$$

where $\underset{\sim}{\Delta}_L$ is the super Yang-Mills- and supergravity-covariant infinitesimal variation with respect to L, defined in Ref. 4. The variation of $\ln Z$ with respect to V simply serves to make the variations with respect to $H_{\alpha\dot{\alpha}}$, Φ, and $\tilde{\Phi}$ super Yang-Mills gauge covariant. One has (see Ref. 4)

$$\begin{aligned}\underset{\sim}{\Delta}_L H_{\alpha\dot{\alpha}}&=\underset{\sim}{\nabla}_\alpha L_{\dot{\alpha}}-\underset{\sim}{\overline{\nabla}}_{\dot{\alpha}}L_\alpha\,,\\ \underset{\sim}{\Delta}_L\Phi&=\underset{\sim}{\delta}_L\Phi^3=(\underset{\sim}{\overline{\nabla}}^2+R)[\underset{\sim}{\nabla}_\alpha(\Phi^3L^\alpha)]\,.\end{aligned} \tag{5}$$

$\underset{\sim}{\nabla}_A=(\underset{\sim}{\nabla}_\alpha,\underset{\sim}{\nabla}_{\dot{\alpha}},\underset{\sim}{\nabla}_{\alpha\dot{\alpha}})$ are the super Yang-Mills- and supergravity-covariant derivatives. From here on we will drop the wiggly underlines and ∇_A will stand for $\underset{\sim}{\nabla}_A$. Define the one-loop supercurrent and supertrace by

$$\langle J^{\alpha\dot{\alpha}}\rangle\equiv\frac{\underset{\sim}{\Delta}\ln Z}{\underset{\sim}{\Delta}H_{\alpha\dot{\alpha}}}\,,\quad\langle J\rangle\equiv\frac{\underset{\sim}{\Delta}\ln Z}{\underset{\sim}{\Delta}\Phi}\,. \tag{6}$$

Notice that as Φ is covariantly chiral, so is $\langle J\rangle$. From Eq. (4), then,

$$\int d^8z\,E^{-1}(\nabla_\alpha L_{\dot{\alpha}}-\overline{\nabla}_{\dot{\alpha}}L_\alpha)\langle J^{\alpha\dot{\alpha}}\rangle=-\int d^6z\,\Phi^3(\overline{\nabla}^2+R)[\nabla_\alpha(\Phi^3L^\alpha)]\langle J\rangle+\text{c.c.} \tag{7}$$

Now,

$$\langle J\rangle=\frac{\underset{\sim}{\Delta}\ln Z}{\underset{\sim}{\Delta}\Phi}=\frac{\delta\ln Z}{\delta\Phi^3}=\frac{1}{Z}\frac{\delta}{\delta\Phi^3}\int D(\Phi^{1/2}\eta)D(\tilde{\Phi}^{1/2}\tilde{\eta})\times\exp(S'[H,V,\eta,\tilde{\eta}])\,, \tag{8}$$

where

$$S'[H_{\alpha\dot{\alpha}},V,\eta,\tilde{\eta}]\equiv\int d^8z\,\hat{E}^{-1/3}(1\cdot e^{-\overleftarrow{H}})^{1/3}\tilde{\eta}\eta\,. \tag{9}$$

To evaluate Eq. (8) we need to define the measure. We will not do this with any rigor here, but will just make such assumptions as seem necessary to give the result one might (naively) expect [Eq. (22) below]. A proof would require a full discussion of analysis in superspace which, to our knowledge, is not yet available.

First let $C=\{\eta:\overline{\nabla}_{\dot\alpha}\eta=0\}$ be the space of left covariantly chiral superfields. Take η to have no spinor indices for simplicity. Assume firstly that the appropriate "inner product" on C is given by

$$(\eta,\xi)\equiv\int d^6z\,\Phi^3\eta\xi \tag{10}$$

for $\eta,\xi\in C$. Then assume that there exists a complete set of superfields $\eta_i\in C$, orthonormal with respect to the "inner product" of Eq. (10), which are the eigenstates of the kinetic operator $\Box_+$ with real positive eigenvalues, i.e.,

$$\Box_+\eta_i=\lambda_i^2\eta_i\ , \tag{11}$$

with λ_i real and $\Box_+$ being $(\overline{\nabla}^2+R)(\nabla^2+\overline{R})$, in the form it takes when acting on superfields in C:

$$\Box_+=\tfrac{1}{2}\nabla^{\alpha\dot\alpha}\nabla_{\alpha\dot\alpha}-\tfrac{1}{2}R\nabla^\alpha\nabla_\alpha-iW^\alpha\nabla_\alpha-\tfrac{1}{2}iG^{\alpha\dot\alpha}\nabla_{\alpha\dot\alpha}$$
$$-\tfrac{1}{2}(\nabla^\alpha R)\nabla_\alpha+R\overline{R}+(\overline{\nabla}^2\overline{R})-\tfrac{1}{2}i(\nabla^\alpha W_\alpha)\ . \tag{12}$$

Notice that $\Box_+$ is self-adjoint with respect to the definition, Eq. (10), i.e., $(\Box_+\eta,\xi)=(\eta,\Box_+\xi)$. The proof uses covariant integration by parts (see Ref. 4, Sec. 5.3b), as well as the relation

$$\int d^6z\,\Phi^3(\overline{\nabla}^2+R)=\int d^8z\,E^{-1}\ .$$

For left covariantly chiral (antichiral) η ($\tilde\eta$), expand

$$\Phi^{1/2}\eta=\sum_i a_i\eta_i\ ,$$
$$\tilde\Phi^{1/2}\tilde\eta=\sum_i\tilde b_i\tilde\eta_i\ , \tag{13}$$

and define

$$D(\Phi^{1/2}\eta)D(\tilde\Phi^{1/2}\tilde\eta)\equiv\prod_{i,j}da_i d\tilde b_j\ . \tag{14}$$

Now, under a variation $\delta\Phi^3$ of Φ, one has

$$\delta(\Phi^{1/2}\eta)=\tfrac{1}{6}\Phi^{-5/2}(\delta\Phi^3)\eta\ . \tag{15}$$

This is equivalent to considering instead a variation of η:

$$\delta\eta=\tfrac{1}{6}\Phi^{-3}(\delta\Phi^3)\eta\ . \tag{16}$$

The variation of the measure, Eq. (14), under Eq. (16) can then be found. Define $\eta'=\eta+\delta\eta$ and let a_n' be the coefficients in the expansion of $\Phi^{1/2}\eta'$ in terms of the η_i. By the completeness and orthonormality of the η_i one has

$$\sum_i\eta_i(z)\eta_i(z')=\Phi^{-3}(z')\delta^6(z-z') \tag{17}$$

and hence

$$a_i'=\sum_j(\eta_i,[1+\tfrac{1}{6}\Phi^{-3}(\delta\Phi^3)]\eta_j)a_j\ . \tag{18}$$

Thus

$$\prod_i da_i'=\det(\eta_j,[1+\tfrac{1}{6}\Phi^{-3}(\delta\Phi^3)]\eta_k)\left[\prod_i da_i\right]$$
$$=\exp\left[\sum_j(\eta_j,\tfrac{1}{6}\Phi^{-3}(\delta\Phi^3)\eta_j)\right]\left[\prod_i da_i\right] \tag{19}$$

or

$$\delta\left[\prod_i da_i\right]=\sum_j(\eta_j,\tfrac{1}{6}\Phi^{-3}(\delta\Phi^3)\eta_j)\left[\prod_i da_i\right]\ . \tag{20}$$

Then

$$\delta D(\Phi^{1/2}\eta)=\left[\sum_i\int d^6z\,\Phi^3\eta_i\tfrac{1}{6}\Phi^{-3}(\delta\Phi^3)\eta_i\right]$$
$$\times D(\Phi^{1/2}\eta) \tag{21}$$

and so

$$\frac{\delta D(\Phi^{1/2}\eta)}{\delta\Phi^3}=\tfrac{1}{6}\Phi^{-3}(\mathrm{tr}1)_z D(\Phi^{1/2}\eta)\ , \tag{22}$$

where for an operator $A:C\to C$ we define

$$(\mathrm{tr}A)_z\equiv\sum_i\eta_i(z)A\eta_i(z) \tag{23}$$

[this requires regularization—see below Eq. (32)]. From Eq. (8) we now conclude that

$$\langle J\rangle=\tfrac{1}{6}\Phi^{-3}(\mathrm{tr}1)_z\ . \tag{24}$$

One may derive the antichiral versions of Eqs. (23) and (24), with $(\tilde J,\tilde\eta,\tilde\Phi)$ replacing (J,η,Φ) in a similar way. The first term on the right-hand side of Eq. (7) can be written, using Eq. (24), as

$$-\int d^6z\,\Phi^3(\overline{\nabla}^2+R)[\nabla_\alpha(\Phi^3L^\alpha)]\tfrac{1}{6}\Phi^{-3}(\mathrm{tr}1)_z$$
$$=-\tfrac{1}{6}\int d^8z[\nabla_\alpha(\Phi^3L^\alpha)](\mathrm{tr}1)_z$$
$$=-\tfrac{1}{6}\int d^8z\,\Phi^3L^\alpha\nabla_\alpha(\mathrm{tr}1)_z$$
$$=-\tfrac{1}{6}\int d^8z\,E^{-1}L^\alpha\nabla_\alpha(\mathrm{tr}1)_z\ . \tag{25}$$

From Eq. (7) one concludes that

$$\overline{\nabla}^{\dot\alpha}\langle J_{\alpha\dot\alpha}\rangle=\tfrac{1}{6}\nabla_\alpha(\mathrm{tr}1)_z\ . \tag{26}$$

To calculate $(\mathrm{tr}A)_z$, as defined by Eq. (23), one can use the ζ-function method in superspace. Define

$$\zeta(z,z';s)\equiv\sum_i\eta_i(z')(\lambda_i)^{-2s}\eta_i(z)\ . \tag{27}$$

(We will use chiral superspace coordinates so that $\overline{\nabla}_{\dot\alpha}=\overline{\partial}_{\dot\alpha}$.) Then

$$(\mathrm{tr}A)_z=\lim_{\substack{s\to0\\ z'\to z}}[A_z\zeta(z,z';s)]\ . \tag{28}$$

One has

$$\zeta(z,z';s)=\frac{1}{\Gamma(s)}\int_0^\infty dt\,t^{s-1}\hat E^{-1/2}h(z,z';t)$$
$$\times(\hat E')^{-1/2}\ , \tag{29}$$

where the superspace heat kernel h satisfies

$$\Box_+ h(z,z';t) = -\frac{\partial}{\partial t} h(z,z';t) ,$$
$$\lim_{t\to 0} h(z,z';t) = \hat{E}\delta^6(z-z') . \qquad (30)$$

The superspace heat kernel has been discussed recently in Ref. 6. h must be covariantly chiral on z and z'. $\hat{E}$ is the superdeterminant of the supervielbein restricted to chiral superspace, and $\hat{E}' \equiv \hat{E}(z')$. h is a density of weight $\frac{1}{2}$ with respect to both z and z'. Equation (29) may be proved as follows—formally, from Eqs. (30):

$$h(z,z';t) = \langle z' | e^{-t\Box_+} | z \rangle ,$$
$$\langle z' | z \rangle = \hat{E}\delta^6(z-z') , \qquad (31)$$

where we have introduced a Hilbert space, spanned by the covariant basis densities (of weight $\frac{1}{2}$) $|z\rangle$. Then

$$\zeta(z,z';s) = \sum_i \eta_i(z')(\lambda_i)^{-2s}\eta_i(z)$$
$$= \frac{1}{\Gamma(s)} \int_0^\infty dt\, t^{s-1} \sum_i \eta_i(z') e^{-t\lambda_i^2} \eta_i(z)$$
$$= \frac{1}{\Gamma(s)} \int_0^\infty dt\, t^{s-1} (\hat{E})^{-1/2} h(z,z';t) (\hat{E}')^{-1/2} \qquad (32)$$

from Eq. (31). The ζ function provides a regularization for the sum in Eq. (23) in the sense that one regulates by selecting only the *finite* part of the right-hand side of Eq. (28). Now expand the coincidence limit of h as

$$h(z,z;t) = \sum_{n=0}^{\infty} s - b_n(z,z) t^{(n-4)/2} . \qquad (33)$$

We will assume that the operator A_z has no free derivatives, so that the coincidence limit in Eq. (28) may be taken directly on the ζ function. When this is not the case, one must expand $h(z,z';t)$ in a supersymmetric generalization of the Schwinger-DeWitt expansion[7] (see also Ref. 8).

Substituting Eq. (33) into the coincidence limit of Eq. (29), which is to be put in Eq. (28), one sees that the regulated definition of $(\mathrm{tr} A)_z$ is then

$$(\mathrm{tr} A)_z^{\mathrm{reg}} \equiv A_z s - b_4(z,z) , \qquad (34)$$

hence,

$$(\mathrm{tr} 1)_z^{\mathrm{reg}} = s - b_4(z,z) . \qquad (35)$$

McArthur[9–11] has calculated $s-b_4(z,z)$ for the system we are considering (he has already noted that this coefficient is proportional to the one-loop supertrace $\langle J \rangle$). If one repeats his calculation using our conventions, one obtains

$$16\pi^2 s - b_4(z,z) = \tfrac{1}{2} W^\alpha W_\alpha + \tfrac{1}{24} W_{\alpha\beta\gamma} W^{\alpha\beta\gamma} , \qquad (36)$$

where the background fields have been taken to be on-shell for simplicity. Using Eqs. (35) and (36) in Eq. (26) one thus obtains the superconformal anomaly equation,

$$\bar{\nabla}^{\dot\alpha} \langle J_{\alpha\dot\alpha} \rangle = \frac{1}{16\pi^2} \frac{1}{3} \nabla_\alpha \left(\tfrac{1}{4} W^\beta W_\beta + \tfrac{1}{48} W_{\beta\gamma\delta} W^{\beta\gamma\delta} \right) , \qquad (37)$$

which agrees with the results obtained by graphical methods (see Refs. 4 and 12 and references therein).

It is anticipated that the appropriate superfield integration variables for all superfields interacting with $n=-\frac{1}{3}$ supergravity can be fixed by requiring the local supersymmetry to be anomaly-free. This could be shown by a superfield version of the work of Ref. 5. [This would also justify nongraphically the assumption we made above that our system has no local supersymmetry anomaly, which is expressed by Eq. (4).] Using these measure variables, one should be able to show that the one-loop supertrace $\langle J \rangle$ is proportional to $(\mathrm{tr} 1)_z^{\mathrm{reg}}$, evaluated in the appropriate functional space, for any superfield in an $N=1$ matter-gauge-minimal supergravity system. Given this result, one can prove the conjecture that only chiral superfields with undotted spinor indices (and their conjugates) contribute to one-loop superconformal anomalies. This has been shown by McArthur[9,11] and occurs because $(\mathrm{tr} 1)_z^{\mathrm{reg}}$, i.e., $s-b_4(z,z)$, vanishes for all but these superfields.

Finally, we remark that it should also be possible to investigate local supersymmetry anomalies in the new and nonminimal supergravities by considering the variation of the appropriate functional-integral measure under supersymmetry transformations.

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