Chiral anomalies, nonminimal couplings, and Kaluza-Klein theory

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(Received 4 November 1985)

By making use of the heat kernel, the chiral anomalies for two- and four-dimensional Abelian gauge theories with Pauli couplings (such as Kaluza-Klein higher-dimensional models) are derived. These are reduced to the minimal anomalies after adding counterterms to the action and by redefining the axial-vector current; confirmation of these counterterms is provided by diagrammatic analysis. Summing over the fermion modes, both the counterterms and the anomalies disappear.

I. INTRODUCTION

Upon dimensional reduction of an N-dimensional Kaluza-Klein theory coupled to fermions, nonminimal interactions of the resultant gauge fields to the fermionic matter are generated.[1] Known as Pauli terms,[2] these couplings make gauge theories power-counting nonrenormalizable and seem to be of no help in solving long-standing difficulties such as the chiral fermion problem.[3] Such terms are also not invariant under a chiral transformation, making them akin to a mass term.

The purpose of this paper is to study the chiral anomalies of Abelian versions of such theories, using the path-integral approach.[4,5] In particular, we completely solve the two-dimensional case, reproduce the result of Clark and Love[6] in four dimensions, and consider another model, directly motivated by the original Kaluza-Klein theory. For this final model we find, unexpectedly, new terms due solely to the nonminimal interaction. However, these extra contributions can be gotten rid of by gauge-invariantly redefining the axial-vector current and supplementing the effective action with counterterms.

The paper proceeds as follows: The next section is devoted to general considerations: Secs. III and IV contain the two- and four-dimensional cases, respectively; Sec. V reproduces the results through Feynman diagrams and in Sec. VI we discuss the additional contributions to the Kaluza-Klein theory and their removal. The summation over modes eliminates the modifications of the axial-vector current, reducing it to the classical expression, and it also removes the axial anomaly. There is an appendix listing our conventions and some useful identities.

II. GENERAL CONSIDERATIONS

Take the $2N$-dimensional theory described by

$$e^{iW[\Sigma]}=\int d\psi\, d\bar{\psi}\exp\left[i\int \bar{\psi}(i\partial\!\!\!/+\Sigma)\psi\right],$$

where Σ represents some configuration of gauge fields. The chiral anomaly can be regarded as due to the noninvariance of the fermionic measure[4] or as the noninvariance of the action[5] under a chiral transformation. In either case the result is[7,8]

$$-2iN\,\mathrm{Tr}[\gamma_5 a_N(x,x)]/(4\pi i)^N\,, \qquad (1)$$

where the $a_N(x,y)$ come from the series expansion of the heat kernel (for small t),

$$H(x,y;t)=-i(4\pi t)^{-N}e^{-i(x-y)^2/4t}\sum a_n(x,y)t^n$$

and the kernel satisfies the equation

$$-i\partial H(x,y;t)/\partial t=-(i\partial\!\!\!/+\Sigma)^2H(x,y;t)$$

with boundary condition

$$H(x,y;0)=\delta^4(x-y)\,.$$

Expressions for the $a_N(x,x)$ have been constructed.[7] In particular, if one defines a differential operator $\mathscr{D}_\mu$ such that

$$X=D\!\!\!\!/^{\,2}-\mathscr{D}^2,\quad D\!\!\!\!/=i\partial\!\!\!/+\Sigma$$

is not a differential operator, then

$$a_1(x,x)=-iX\,, \qquad (2)$$

$$a_2(x,x)=-X^2/2-Y^{\mu\nu}Y_{\mu\nu}/12-[\mathscr{D}^\mu,[\mathscr{D}_\mu,X]]/6\,, \qquad (3)$$

with $Y_{\mu\nu}=[\mathscr{D}_\mu,\mathscr{D}_\nu]$. In general, for $D\!\!\!\!/=i\partial\!\!\!/+\Sigma$, the choice for $\mathscr{D}$ is

$$\mathscr{D}_\mu=i\partial_\mu+\{\gamma_\mu,\Sigma\}/2$$

from which we get

$$X=i\partial\!\!\!/\Sigma-i\{\gamma^\mu,\partial_\mu\Sigma\}/2+\Sigma^2-\{\gamma_\mu,\Sigma\}\{\gamma^\mu,\Sigma\}/4\,.$$

The formal solution for the flat-space chiral anomaly in two and four dimensions in provided by the coefficients $a_1(x,x)$ and $a_2(x,x)$, respectively. However, as originally discussed by Bardeen,[9] it is possible to add counterterms to the Lagrangian whose variations cancel some of the anomalous terms. This is effectively a redefinition of the action and an appropriate choice leaves us with the minimal anomaly; see Balachandran, Marmo, Nair, and Trahern[8] for such a calculation of the chiral anomaly for non-Abelian vector and axial-vector couplings in four dimensions.

The evaluation of Eq. (1) for the cases of interest, namely $\Sigma=V\!\!\!\!/+A\!\!\!\!/\gamma_5+G\cdot\sigma+iF\cdot\sigma\gamma_5$ in two dimensions and $\Sigma = A\!\!\!/ \gamma_5 + G\cdot\sigma$ or $\Sigma = V\!\!\!/ + iF\cdot\sigma\gamma_5$ in four dimensions, is straightforward. It is the determination of the counterterms that is the crucial part of the next two sections.

III. THE TWO-DIMENSIONAL CASE

For the two-dimensional action, $\int \bar\psi D\!\!\!/ \psi$, Eqs. (2) and (3) give the chiral anomaly to be

$$i\,\mathrm{tr}[\gamma_5(D\!\!\!/^2 - \mathscr{D}^2)]/2\pi \,. \tag{4}$$

Taking $D\!\!\!/ = i\partial\!\!\!/ + V\!\!\!/ + A\!\!\!/\gamma_5 + G\cdot\sigma + iF\cdot\sigma\gamma_5$, which we can rewrite by virtue of Eq. (A1) as

$$D\!\!\!/ = i\partial\!\!\!/ + V\!\!\!/ + A\!\!\!/\gamma_5 + \tilde G\gamma_5 + i\tilde F \,,$$

where $\tilde G = G_{\mu\nu}\epsilon^{\mu\nu}$, $\tilde F = F_{\mu\nu}\epsilon^{\mu\nu}$, we see that the non-minimal terms are similar to mass terms (for this reason we would expect them to give no new anomalies). Now

$$\begin{aligned}D\!\!\!/^2 = &-\partial^2 + V^2 - A^2 + \tilde G^2 - \tilde F^2 + i(\partial\!\!\!/ V\!\!\!/) + 2iV\cdot\partial \\ &+ i\gamma_5(\partial\!\!\!/ A\!\!\!/) + 2i\epsilon^{\mu\nu}A_\nu\partial_\mu + i(\partial\!\!\!/\tilde G)\gamma_5 - (\partial\!\!\!/\tilde F) - 2\tilde F\partial\!\!\!/ \\ &- 2\epsilon^{\mu\nu}V_\mu A_\nu + 2i\tilde F V\!\!\!/ + 2i\tilde F A\!\!\!/\gamma_5 + 2i\tilde F\tilde G\gamma_5 \,,\end{aligned}$$

and here $\mathscr{D}_\mu = i\partial_\mu + V_\mu - \epsilon_{\mu\nu}A^\nu + i\gamma_\mu F$, so

$$\begin{aligned}\mathscr{D}^2 = &-\partial^2 + V^2 - A^2 - 2\tilde F^2 + i(\partial\cdot V) + 2iV\cdot\partial \\ &- i\epsilon_{\mu\nu}(\partial^\mu A^\nu) - 2i\epsilon_{\mu\nu}A^\nu\partial^\mu - (\partial\!\!\!/\tilde F) - 2\tilde F\partial\!\!\!/ \\ &- 2\epsilon_{\mu\nu}V^\mu A^\nu + 2i\tilde F V\!\!\!/ - 2i\epsilon_{\mu\nu}\gamma^\mu A^\nu\tilde F \,.\end{aligned}$$

Thus we find

$$\begin{aligned}X = &-i\epsilon_{\mu\nu}(\partial^\mu V^\nu)\gamma_5 + i\gamma_5(\partial\cdot A) + i(\partial\!\!\!/\tilde G)\gamma_5 + 2i\tilde F A\!\!\!/\gamma_5 \\ &+ 2i\epsilon_{\mu\nu}\gamma^\mu A^\nu\tilde F - 2G^2 - 2F^2 - 4iG_{\mu\nu}F^{\mu\nu}\gamma_5 \,,\end{aligned}$$

where we have used Eqs. (A1) and (A2) to simplify the expression. Carrying out the trace, expression (4) reduces to

$$(\epsilon_{\mu\nu}\partial^\mu V^\nu - \partial\cdot A + 4F_{\mu\nu}G^{\mu\nu})/\pi \,.$$

Hence for our two-dimensional theory we can now write the chiral Ward identity as

$$\left[i\partial_\mu\frac{\delta}{\delta A_\mu} + 2iG^{\mu\nu}\frac{\delta}{\delta F^{\mu\nu}} - 2iF^{\mu\nu}\frac{\delta}{\delta G^{\mu\nu}}\right] S = \frac{1}{\pi}\epsilon^{\kappa\lambda}(\partial_\kappa V_\lambda) - \frac{1}{\pi}(\partial\cdot A) + \frac{4}{\pi}F_{\kappa\lambda}G^{\kappa\lambda} \,.$$

We may account for the last two terms on the right-hand side by redefining the action

$$S_{\rm eff} = S + (i/2\pi)\int(\alpha F^2 - \beta G^2 - A^2) \quad \text{with } \alpha+\beta = 2$$

for which the chiral Ward identity simplifies to

$$\left[i\partial_\mu\frac{\delta}{\delta A_\mu} + 2iG^{\mu\nu}\frac{\delta}{\delta F^{\mu\nu}} - 2iF^{\mu\nu}\frac{\delta}{\delta G^{\mu\nu}}\right] S_{\rm eff} = \frac{1}{\pi}\epsilon^{\kappa\lambda}(\partial_\kappa V_\lambda) \,;$$

i.e., the minimal anomaly is just the usual one due to vector coupling.

IV. FOUR-DIMENSIONAL THEORIES

The four-dimensional chiral anomaly is given in the standard way by

$$-i\,\mathrm{tr}\{\gamma_5(X^2/2 + Y_{\mu\nu}Y^{\mu\nu}/12 + [\mathscr{D}^\mu,[\mathscr{D}_\mu,X]])\} \,.$$

Owing to the identity (I):

$$\gamma_5\sigma^{\mu\nu} = i\epsilon^{\mu\nu\rho\sigma}\sigma_{\rho\sigma} \,,$$

valid only in four dimensions, the two types of Pauli couplings, namely $G\cdot\sigma$ and $F\cdot\sigma\gamma_5$, are virtually the same but with

$$2G_{\mu\nu} \leftrightarrow i\tilde F_{\mu\nu}$$

and

$$\tilde G_{\mu\nu} \leftrightarrow -2iF_{\mu\nu} \,.$$

Let us look at two instances.

A. Example of Clark and Love (Ref. 6)

This has the action

$$S = \int d^4x\, \bar\psi(i\partial\!\!\!/ + A\!\!\!/\gamma_5 + G\cdot\sigma)\psi$$

leading to

$$\mathscr{D}_\mu = i\partial_\mu - i\gamma_5\sigma_{\mu\nu}A^\nu + \gamma_5\gamma^\nu\tilde G_{\mu\nu}$$

and

$$X = i\gamma_5\partial\cdot A + 2A^2 - 2\partial^\mu G_{\mu\nu}\gamma^\nu - 2G^2 + i\gamma_5 G\cdot\tilde G \,.$$

Then

$$\mathrm{tr}(\gamma_5 X^2/2) = 8i(A^2 - G^2)(\partial\cdot A + G\cdot\tilde G) \,.$$

Also

$$\begin{aligned}Y_{\mu\nu} = &\,2\gamma_5\sigma_{\kappa[\mu}\partial_{\nu]}A^\kappa + 2iA^2\sigma_{\mu\nu} + 4iA^\kappa\sigma_{\kappa[\mu}A_{\nu]} \\ &+ 2i\tilde G_{\mu\kappa}\tilde G_{\nu\lambda}\sigma^{\kappa\lambda} + 2i\gamma_5\gamma^\lambda\partial_{[\mu}\tilde G_{\nu]\lambda} \\ &+ 4i\gamma_5\gamma^\rho\epsilon_{\kappa\lambda\rho[\mu}\tilde G_{\nu]}{}^\lambda A^\kappa \,,\end{aligned}$$

using the definition $[\mu\nu] = (\mu\nu - \nu\mu)/2$. After some work this yields ($A_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$)

$$\begin{aligned}\tfrac{1}{12}\mathrm{tr}\gamma_5 Y_{\mu\nu}Y^{\mu\nu} = &\,\tfrac{1}{6}iA_{\mu\nu}\tilde A^{\mu\nu} - 8iA^2(\partial\cdot A) + \tfrac{8}{3}i\partial^\mu(A_\mu A^2) + \tfrac{16}{3}iG^2\partial\cdot A - \tfrac{32}{3}iG^{\alpha\nu}G_\alpha{}^\mu\partial_\mu A_\nu - \tfrac{32}{3}iG\tilde G A^2 \\ &+ \tfrac{64}{3}i\tilde G^{\nu\lambda}G_{\rho\lambda}A^\rho A_\nu - \tfrac{64}{3}iG_{\alpha\kappa}G^\alpha{}_\rho G_{\beta\lambda}G^\beta{}_\sigma\epsilon^{\kappa\lambda\rho\sigma} \,,\end{aligned}$$

where we have made abundant use of matrix algebra and trace properties. We also find

$$\tfrac{1}{6}\mathrm{tr}\gamma_5[\mathscr{D}^\mu,[\mathscr{D}_\mu,X]]=-\tfrac{2}{3}i\partial^2(\partial\cdot A)-\tfrac{2}{3}i\partial^2(G\widetilde{G})-\tfrac{8}{3}i\widetilde{G}^{\mu\nu}\partial_\mu\partial^\kappa G_{\kappa\nu}-\tfrac{8}{3}i\partial^\mu(\widetilde{G}_{\mu\nu}\partial_\kappa G^{\kappa\nu})+\tfrac{32}{3}iG^2(\partial\cdot A)$$
$$-\tfrac{64}{3}iG^{\nu\lambda}A_\nu\partial^\kappa G_{\kappa\lambda}+\tfrac{32}{3}iG^2G\widetilde{G}\ .$$

Putting all this together and noting that the $A^2\partial\cdot A$ term vanishes (as it should, because it violates the current-algebra consistency relations,[6] we get the chiral anomaly to be the sum of five terms which we label by L_i ($i=1,\ldots,5$) and define by

$$L_1=\tfrac{1}{6}iA_{\mu\nu}\widetilde{A}^{\mu\nu}\ ,$$
$$L_2=\tfrac{8}{3}i\partial^\mu(A_\mu A^2)-\tfrac{2}{3}i\partial^2(\partial\cdot A)+\tfrac{8}{3}iG^2G\widetilde{G}$$
$$-\tfrac{64}{3}iG_{\alpha\kappa}G^\alpha{}_\rho G_{\beta\lambda}G^\beta{}_\sigma\epsilon^{\kappa\lambda\rho\sigma}-\tfrac{8}{3}i\widetilde{G}^{\mu\lambda}\partial_\mu\partial^\kappa G_{\kappa\lambda}\ ,$$
$$L_3=-\tfrac{2}{3}i\partial^2(G\widetilde{G})-\tfrac{8}{3}i\partial^\mu(\widetilde{G}_{\mu\nu}\partial_\kappa G^{\kappa\nu})\ ,$$
$$L_4=8iG^2(\partial\cdot A)-\tfrac{32}{3}iG^{\alpha\nu}G_\alpha{}^\mu\partial_\mu A_\nu-\tfrac{64}{3}iG^{\alpha\nu}A_\nu\partial^\mu G_{\alpha\mu}\ ,$$
$$L_5=\tfrac{8}{3}iG\widetilde{G}A^2\ .$$

Then the relevant Ward identity reads

$$\left[i\partial^\mu\frac{\delta}{\delta A^\mu}-i\epsilon^{\mu\nu\kappa\lambda}G_{\mu\nu}\frac{\delta}{\delta G^{\kappa\lambda}}\right]S=-\frac{i}{4\pi^2}\sum_{j=1}^{5}L_j$$

using the four-dimensional identity (I) stated earlier.

We recognize L_1 as the normal chiral anomaly for an axial vector and no further comment is needed. As for L_2, we can compensate exactly for it by adding to the action,

$$(i/12\pi^2)\int[2A^2A^2+(\partial\cdot A)^2-2G^2G^2$$
$$+16G_{\alpha\mu}G^{\beta\nu}G^\alpha{}_\nu G^{\beta\mu}+4G^{\mu\lambda}\partial_\mu\partial^\kappa G_{\kappa\lambda}]\ .$$

Compensating for the remaining three terms is trickier as no simple counterterm gives any one of them exactly. However, by adding

$$(-i/6\pi^2)\int(2A^2G^2+4A_\mu G^{\alpha\mu}\partial^\beta\widetilde{G}_{\alpha\beta}-G\cdot\widetilde{G}\partial\cdot A)$$

we may eliminate L_4 and L_5, generating only

$$(-i/6\pi^2)[\partial^2(G\cdot\widetilde{G})+4\partial_\mu(G^{\alpha\mu}\partial^\beta\widetilde{G}_{\alpha\beta})]\ .$$

The first part cancels a term from L_3 and all that is left (in addition to L_1) is

$$(2/3\pi^2)\partial_\mu(G^{\nu\mu}\partial^\kappa\widetilde{G}_{\nu\kappa}-\widetilde{G}^{\mu\nu}\partial^\kappa G_{\kappa\nu})$$

which we can account for by adding to S:

$$(-i/4\pi^2)\int(\widetilde{G}^{\alpha\mu}\partial_\mu\partial^\beta\widetilde{G}_{\alpha\beta}/3-4G_{\mu\nu}\partial^\mu\partial_\kappa G^{\kappa\nu}/3)\ .$$

To summarize, we have redefined the action to be

$$S_{\mathrm{eff}}=S+\frac{i}{12\pi^2}\int[2A^2A^2+(\partial\cdot A)^2-2G^2G^2+16G_{\alpha\mu}G^{\beta\nu}G^\alpha{}_\nu G^{\beta\mu}+8G^{\mu\lambda}\partial_\mu\partial^\kappa G_{\kappa\lambda}$$
$$-\widetilde{G}^{\alpha\mu}\partial_\mu\partial^\beta\widetilde{G}_{\alpha\beta}-4A^2G^2-8A_\mu G^{\alpha\mu}\partial^\beta\widetilde{G}_{\alpha\beta}+2G\widetilde{G}(\partial\cdot A)]$$

and have found the chiral U(1) Ward identity

$$\left[i\partial^\mu\frac{\delta}{\delta A^\mu}-i\epsilon^{\mu\nu\kappa\lambda}G_{\mu\nu}\frac{\delta}{\delta G^{\kappa\lambda}}\right]S_{\mathrm{eff}}=\frac{1}{24\pi^2}A_{\mu\nu}\widetilde{A}^{\mu\nu}\ .$$

Hence we have shown explicitly what Clark and Love did implicitly; namely, that with a proper redefinition of the action, the nonminimal coupling gives no new anomalies.

B. An example from Kaluza-Klein theory

Let us consider the case

$$S=\int d^4x\,\bar{\psi}(i\partial\!\!\!/+V\!\!\!\!/+iF\cdot\sigma\gamma_5+im\gamma_5)\psi\ .$$

This is the action one obtains when reducing the conventional five-dimensional Kaluza-Klein action to four dimensions, keeping the first excited mode. The treatment, however, is quite general and we will discuss the effect of including all the modes subsequently.

For this action one must take

$$\mathscr{D}_\mu=i\partial_\mu+V_\mu-2\gamma_5\gamma^\nu F_{\nu\mu}\ ,$$

whereupon

$$X=\tfrac{1}{2}V_{\mu\nu}\sigma^{\mu\nu}+2F^2-i\gamma_5F\widetilde{F}+(\partial_\mu\widetilde{F}^{\mu\rho})\gamma_\rho-2mF\cdot\sigma-m^2$$

and

$$Y_{\mu\nu}=iV_{\mu\nu}-4i\partial_{[\mu}F_{\nu]\lambda}\gamma^\lambda\gamma_5+8iF_{\mu\kappa}F_{\nu\lambda}\sigma^{\kappa\lambda}\ .$$

One then discovers

$$\mathrm{tr}\gamma_5\tfrac{1}{2}X^2=\tfrac{1}{2}iV_{\mu\nu}\widetilde{V}^{\mu\nu}-8iF^2F\widetilde{F}-4imV_{\mu\nu}\widetilde{F}^{\mu\nu}+4im^2F\widetilde{F}\ ,$$
$$\mathrm{tr}\gamma_5\tfrac{1}{12}Y_{\mu\nu}Y^{\mu\nu}=-\tfrac{64}{3}i\epsilon^{\kappa\lambda\rho\sigma}F_{\mu\kappa}F_{\nu\lambda}F^\mu{}_\rho F^\nu{}_\sigma\ ,$$
$$\mathrm{tr}\gamma_5\tfrac{1}{6}[\mathscr{D}^\mu,[\mathscr{D}_\mu,X]]=\tfrac{2}{3}i\partial^2(F\widetilde{F})+\tfrac{8}{3}iF_{\mu\sigma}\partial^\mu\partial_\rho\widetilde{F}^{\rho\sigma}$$
$$+\tfrac{8}{3}i\partial^\mu(F_{\mu\sigma}\partial_\rho\widetilde{F}^{\rho\sigma})+\tfrac{32}{3}iF^2F\widetilde{F}\ .$$

In accordance with our observations about the relationship between the two types of Pauli terms, note that the pure F and derivative pieces are very similar to those of example A. In particular, they have the same numerical coefficients.

After redefining our action to be

$$S_{\rm eff}=S-\frac{i}{4\pi^2}\int\left(\tfrac{2}{3}F^2F^2-4mV_{\mu\nu}F^{\mu\nu}+2m^2F^2\right.$$
$$-\tfrac{16}{3}F_{\mu\kappa}F_{\nu\lambda}F^{\mu\lambda}F^{\nu\kappa}-\tfrac{1}{6}\tilde{F}^{\mu\nu}\partial^2\tilde{F}_{\mu\nu}$$
$$\left.-\tfrac{2}{3}\tilde{F}^{\mu\nu}\partial_\mu\partial^\kappa\tilde{F}_{\kappa\nu}\right),$$

we can write the chiral U(1) Ward identity for this theory as

$$\langle\partial_\mu(\bar\psi\gamma^\mu\gamma_5\psi)\rangle-2m\langle\bar\psi\psi\rangle-\left[i\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}\frac{\delta}{\delta F^{\mu\nu}}\right]S_{\rm eff}$$
$$=\frac{i}{8\pi^2}V_{\mu\nu}\tilde{V}^{\mu\nu}+\frac{1}{3\pi^2}[(\partial_\mu F_{\kappa\lambda})(\partial^\mu\tilde{F}^{\kappa\lambda})$$
$$+2(\partial^\mu F_{\mu\nu})(\partial_\kappa\tilde{F}^{\kappa\nu})]\ . \quad (5)$$

The following question arises: Can one eliminate the last two terms on the right of (5) by adding a (vector) gauge-invariant polynomial contribution to the axial-vector current and by including a counterterm in the action? The answer is yes. Let us modify the axial-vector current by the extra piece

$$\delta j_{\mu5}=-\frac{1}{3\pi^2}(\tilde{F}_{\kappa\lambda}\partial^\mu F^{\kappa\lambda}+2\tilde{F}_{\mu\nu}\partial_\kappa F^{\kappa\nu})\ ,$$

and supplement the effective action by the counterterm

$$\delta S_{\rm eff}=-\frac{i}{6\pi^2}\int(F_{\mu\nu}\partial^2F^{\mu\nu}+2F^{\mu\nu}\partial_\mu\partial^\kappa F_{\kappa\nu})\ .$$

Then (5) will simplify to the conventional anomaly, with the last two terms absent.

V. DIAGRAMMATIC ANALYSIS

Given the extra $\partial F\cdot\partial F$ contribution to the axial anomaly (that is, before its elimination at the end of Sec. IV), it is worthwhile confirming the existence of this term by a diagrammatic, perturbative argument, similar to Adler's original work. In particular we seek to corroborate the coefficient $-1/3\pi^2$ occurring on the right-hand side of Eq. (5). The most convenient and transparent way is by extending the classical equation of motion

$$-\partial(\bar\psi\gamma_\mu\gamma_5\psi)=2\bar\psi(m+F\cdot\sigma)\psi$$

so as to include a Pauli-Villars regulator field ψ with a heavy mass M and wrong spin statistics. In keeping with the Kaluza-Klein structure, it will have a propagator

$$S(p)=(\not p-iM\gamma_5)^{-1}=(\not p-iM\gamma_5)/(p^2-M^2)\ .$$

It will be enough for our purposes to consider the matrix element $\langle 0|\partial(\bar\psi\gamma\gamma_5\psi)|\gamma(k)\gamma(k')\rangle$ with the photons taken on mass shell and $F_{\mu\nu}$ in Eq. (6) interpreted as $\partial_\mu A_\nu-\partial_\nu A_\mu$. There are two one-loop fermion diagrams [Figs. 1(a) and 1(b)] which can contribute to this process in the limit $M\to\infty$, as well as their crossed counterparts

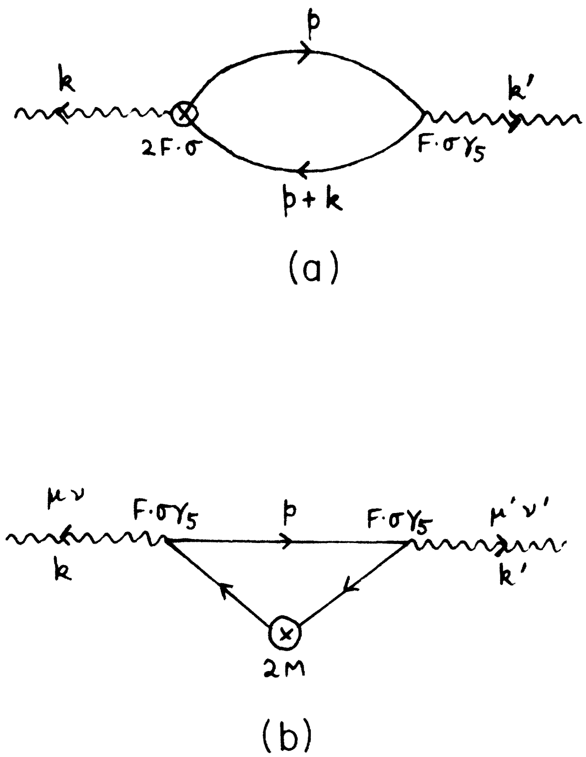

FIG. 1. (a) $2F\cdot\sigma$ regulator contribution to the axial divergence. (b) $2M$ regulator contribution to the axial divergence.

$(k,\mu,\nu\to k',\mu',\nu)$.

After extracting the external factor $(k^\mu\epsilon^\nu-k^\nu\epsilon^\mu)(k'^{\mu'}\epsilon'^{\nu'}-k'^{\nu'}\epsilon'^{\mu'})$, we are left with the Feynman integrals

$$I^{\Sigma}_{\mu\nu\mu'\nu'}=-i\,{\rm Tr}\int d^4p\,S(p+k)\cdot2\sigma_{\mu\nu}\cdot S(p)\cdot\sigma_{\mu'\nu'}\gamma_5\ ,$$
$$I^{\Delta}_{\mu\nu\mu'\nu'}=+i\,{\rm Tr}\int d^4p\,S(p)\cdot\sigma_{\mu\nu}\gamma_5\cdot S(p+k)\cdot2M$$
$$\times S(p-k')\cdot\sigma_{\mu'\nu'}\gamma_5\ .$$

The argument for Pauli coupling is slightly more subtle than the corresponding one for vector coupling. The point is that diagrams 1(a) and 1(b) are each logarithmically divergent (potential quadratic infinities happily disappear thanks to the four-dimensional identity, $\gamma^\lambda\sigma^{\mu\nu}\gamma_\lambda=0$). We have to look forward to a cancellation of these divergences and the absence of possibly damaging terms proportional to M^2. Fortunately this is exactly what happens.

By introducing Feynman parameters as usual and remembering that we are taking $k^2=k'^2=0$ for ease of computation, we can simplify the integrals to

$$I^{\Sigma}_{\mu\nu\mu'\nu'}=2M^2\int_0^1 d\alpha\int d^4p\frac{{\rm Tr}(\sigma_{\mu\nu}\sigma_{\mu'\nu'}\gamma_5)}{(p^2-M^2)^2}\ ,$$
$$I^{\Delta}_{\mu\nu\mu'\nu'}=-4M^2$$
$$\times\int_0^1 d\alpha\int_0^{1-\alpha}d\beta\int d^4p\frac{N_{\mu\nu\mu'\nu'}}{(p^2+2k\cdot k'\alpha\beta-M^2)^3},$$

with

$$N_{\mu\nu\mu'\nu'}=\mathrm{Tr}\{(p^2-M^2)\sigma_{\mu\nu}\sigma_{\mu'\nu'}\gamma_5+(-\not{k}\alpha+\not{k}'\beta)\sigma_{\mu\nu}\gamma_5[\not{k}(1-\alpha)+\not{k}'\beta]\sigma_{\mu'\nu'}$$
$$+\sigma_{\mu\nu}\gamma_5[\not{k}(1-\alpha)+\not{k}'\beta][-\not{k}'(1-\beta)-\not{k}\alpha]\sigma_{\mu'\nu'}+(-\not{k}\alpha+\not{k}'\beta)\sigma_{\mu\nu}[-\not{k}'(1-\beta)-\not{k}\alpha]\sigma_{\mu'\nu'}\gamma_5\}\ .$$

To show the cancellation of the infinities, we shall extend our momentum integrals to $2l$ dimensions, rather than apply a momentum cutoff, and go to the limit $l\to 2$ at the end. Then

$$I^{\Delta}+I^{\Sigma}=\mathrm{Tr}(\sigma_{\mu\nu}\sigma_{\mu'\nu'}\gamma_5)\lim_{l\to 2}\frac{-i}{(4\pi)^l}\left[-\frac{2M^2\Gamma(2-l)}{(M^2)^{2-l}}+2M^2\int d\alpha\, d\beta\left[\frac{l\Gamma(2-l)}{(M^2-2k\cdot k'\alpha\beta)^{2-l}}-\frac{\Gamma(3-l)[k\cdot k'(-1+2\alpha+2\beta-6\alpha\beta)-M^2]}{(M^2-2k\cdot k'\alpha\beta)^{3-l}}\right]\right]$$

$$=\frac{-i\,\mathrm{Tr}(\sigma_{\mu\nu}\sigma_{\mu'\nu'}\gamma_5)}{(4\pi)^2}\cdot 2M^2\int d\alpha\, d\beta\left[-2\ln\left[1-\frac{2k\cdot k'\alpha\beta}{M^2}\right]-1+\frac{[M^2-k\cdot k'(1-2\alpha-2\beta+6\alpha\beta)]}{M^2-2k\cdot k'\alpha\beta}\right]$$

which is perfectly finite. Also, the M^2 terms go away. Thus there is no difficulty in passing to the limit $M\to\infty$ for the regulator so as to extract the anomaly

$$(I^{\Sigma}+I^{\Delta})_{\mu\nu\mu'\nu'}=-\frac{ik\cdot k'}{8\pi^2}\mathrm{Tr}(\sigma_{\mu\nu}\sigma_{\mu'\nu'}\gamma_5)\times\int d\alpha\, d\beta(1-4\alpha-4\beta+12\alpha\beta)$$
$$=\frac{1}{3\pi^2}k\cdot k'\epsilon_{\mu\nu\mu'\nu'}\ ,$$

which precisely corresponds to the extra term

$$(1/3\pi^2)\partial_\lambda F_{\mu\nu}\partial^\lambda\tilde{F}^{\mu\nu}$$

as determined through the heat-kernel method in Eq. (5). There is little doubt that other approaches to regularization (dimensional, path integration, ζ function, etc.) will yield the same answer.

VI. DISCUSSION

If we now identify $F_{\mu\nu}$ of the general result in Sec. IV with $\kappa V_{\mu\nu}$ as in Kaluza-Klein theory, then the very last term on the right-hand side of Eq. (5) vanishes identically by virtue of the Bianchi identity for electromagnetism, $\partial_\mu\tilde{V}^{\mu\nu}=0$ so no counterterm is required for it. However, the previous terms $\partial^\mu V\cdot\partial_\mu\tilde{V}$ persist. This means that we disagree with the formal results of Duff and Toms[1] who claim that under a chiral transformation the change in the measure of the fermion fields for the Kaluza-Klein model is simply the same as conventional vector theory. Rather, we have demonstrated that there are many new terms in the anomaly, associated with derivative couplings, *before* it is minimalized and reduced to the conventional value.

At this point let us take into account all the fermion modes and sum them over. We now show that it is unnecessary to modify the classical axial-vector current or to add counterterms to the action. Reinstate the mode dependence of the charge in front of $V\tilde{V}$ and consider the sum

$$\sum_{n=-\infty}^{\infty}\left[\frac{e^2n^2}{8\pi^2}V_{\mu\nu}\tilde{V}^{\mu\nu}-\frac{\kappa^2}{3\pi^2}(\partial_\mu V_{\alpha\beta})(\partial^\mu\tilde{V}^{\alpha\beta}\right]+\cdots$$
$$=2\zeta(-2)\frac{e^2}{8\pi^2}V\cdot\tilde{V}$$
$$-[1+2\zeta(0)]\frac{\kappa^2}{3\pi^2}(\partial_\mu V_{\alpha\beta})(\partial^\mu\tilde{V}^{\alpha\beta})+\cdots\ .$$

where $\zeta(z)$ is the analytic continuation of the Riemann ζ function. Two of its properties are[10]

$$\zeta(-2m)=0,\quad m=1,2,\ldots,$$
$$\zeta(0)=-\tfrac{1}{2}\ .$$

Thus the summation gives a zero chiral anomaly overall, consistent with the knowledge that there are no chiral anomalies in an odd-dimensional space and agreeing with the end result of Duff and Toms; the sum over modes not only nullifies the counterterms but also the ordinary anomaly.

As a final point, we note that, following a suggestion of Delbourgo and Jarvis,[11] if we use the five-dimensional Christoffel symbol $\Gamma_{\mu 5\nu}=-\kappa V_{\mu\nu}/2$, we may express the term on the right-hand side of (5) as

$$\frac{1}{2\pi^2\kappa^2}\epsilon^{\mu\nu\rho\sigma}\Gamma_{\mu 5\nu}\Gamma_{\rho 5\sigma}-\frac{4}{3\pi^2}\epsilon^{\mu\nu\rho\sigma}(\partial_\alpha\Gamma_{\mu 5\nu})(\partial^\alpha\Gamma_{\rho 5\sigma})\ .$$

It must be admitted that this does not give us much insight into the structure and origin of the anomaly.

APPENDIX: ON CONVENTIONS AND IDENTITIES

We follow the notation of Itzykson and Zuber.[12] Additionally, we shall abbreviate $G_{\mu\nu}G^{\mu\nu}$ by G^2 and $G_{\mu\nu}\tilde{G}^{\mu\nu}$ by $G\tilde{G}$, where we define

$$\tilde{G}^{\mu\nu}=\epsilon^{\mu\nu\rho\sigma}G_{\rho\sigma}\ .$$

In two dimensions we use the identities

$$\gamma_\mu\gamma_\nu=-i\epsilon_{\mu\nu}\gamma_5+\eta_{\mu\nu}\ , \qquad \text{(A1)}$$

$$\epsilon^{\mu\nu}\epsilon^{\rho\sigma}=-\eta^{\mu\rho}\eta^{\nu\sigma}+\eta^{\mu\sigma}\eta^{\nu\rho}\ . \qquad \text{(A2)}$$

In four dimensions,

$$\{\sigma^{\mu\nu},\gamma^{\kappa}\}=2\epsilon^{\mu\nu\kappa\rho}\gamma_5\gamma_\rho ,$$

$$[\sigma^{\mu\nu},\gamma^{\kappa}]=2i(\gamma^{\mu}\eta^{\nu\kappa}-\gamma^{\nu}\eta^{\mu\kappa}) ,$$

$$\{\sigma^{\mu\nu},\sigma^{\kappa\lambda}\}=2\eta^{\mu\kappa}\eta^{\nu\lambda}-2\eta^{\mu\lambda}\eta^{\nu\kappa}+2i\epsilon^{\mu\nu\kappa\lambda}\gamma_5 ,$$

and, since for any vector B^{κ},

$$\epsilon^{\mu\nu\rho\sigma}B^{\kappa}=\epsilon^{\kappa\nu\rho\sigma}B^{\mu}+\epsilon^{\mu\kappa\rho\sigma}B^{\nu}+\epsilon^{\mu\nu\kappa\sigma}B^{\rho}+\epsilon^{\mu\nu\rho\kappa}B^{\sigma}$$

we can derive

$$A_\mu B^\mu \widetilde{S}^{\kappa\nu}T_{\kappa\nu}=2A_\mu B^\kappa \widetilde{S}^{\mu\nu}T_{\kappa\nu}+2A_\mu B^\kappa S_{\kappa\nu}\widetilde{T}^{\mu\nu}$$

for S and T antisymmetric tensors. Special cases of this last identity are used frequently and referred to as Eq. (A3); for example,

$$\partial^2 G\widetilde{G}=4\partial_\mu\partial^\kappa(G_{\kappa\nu}\widetilde{G}^{\mu\nu}) ,$$

$$(\partial_\mu F_{\alpha\beta})(\partial^\mu \widetilde{F}^{\alpha\beta})=2(\partial_\rho F_{\mu\sigma})(\partial^\mu \widetilde{F}^{\rho\sigma})+2(\partial^\mu F_{\mu\sigma})(\partial_\rho \widetilde{F}^{\rho\sigma}) .$$

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