Mass perturbation in the Thirring model

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Using the equivalence with a derivative-coupling model, mass perturbation in the Thirring model is investigated. We show that, for $4\pi(2-\sqrt{3})<\beta^2<8\pi$ all ultraviolet divergences cancel. Finite composite operators are constructed in this range. Ward identities and equations of motion are discussed.

I. INTRODUCTION

The usual approach to perturbative studies of models consists in separating the Lagrangian L describing the system into two pieces L_0 and L_{int}:

$$L=L_0+L_{\text{int}} \tag{1.1}$$

with L_0 a free-field Lagrangian and L_{int} containing all relevant interactions. This division is dictated solely by our ignorance and inability to produce solutions for more general field equations. It is certainly desirable to have at one's disposal a perturbative scheme with L_0 already incorporating as many symmetries as possible. Of course, if a project has too broad a scope it is probably untenable. We have, therefore, limited our attention to mass perturbation around scale-invariant theories. Barring the uninteresting and trivial case of perturbation of free-field theories, this brings us immediately to the context of some soluble two-dimensional models. A prominent member of this class is the Thirring model which has contributed so much to the development of ideas in field theory.[1] In particular, the reader should recall the amazing equivalence of this fermionic model with a bosonic theory, the sine-Gordon model.[2] To fix a notation, let k be the Thirring-model coupling constant in Klaiber's definition.[1] Then the sine-Gordon parameter β which appears in the interaction $\cos(\beta\phi)$, is related to k by

$$\frac{k}{\pi}=-\left[\frac{4\pi}{\beta^2}\right]^{1/2}\left[1-\frac{\beta^2}{4\pi}\right]. \tag{1.2}$$

Attractive and repulsive regions correspond to $\beta^2<4\pi$ ($k<0$) and $\beta^2>4\pi$ ($k>0$). $\beta^2=4\pi$ corresponds to a free fermion theory. Mass perturbation played a basic role in demonstrating this equivalence. Indeed, the result identifying the zeroth-charge sectors of both theories was first obtained by Coleman,[2] comparing the formal mass perturbation for the Thirring model with the perturbation in $\cos(\beta\phi)$ for the sine-Gordon theory. It is one of our objectives to strengthen these results by analyzing in detail the ultraviolet behavior of mass perturbation for the Green's functions of the Thirring model. This is not an entirely trivial task since in principle there could be divergences which should be kept under control (this remark extends also to all two-dimensional scalar theories with nonpolynomial self-interaction).

The Thirring model has other interesting connections. It is also equivalent, in a sense to be made precise later, to a derivative-coupling model describing two massless scalar fields ϕ_1 and ϕ_2 interacting with a massless spinor field ψ via the interaction Lagrangian

$$L_{\text{int}}=g_1(\bar\psi\gamma^\mu\psi)\partial_\mu\phi_1+g_2(\bar\psi\gamma^\mu\gamma^5\psi)\partial_\mu\phi_2\,. \tag{1.3}$$

The model (1.3) will be called the derivative-coupling (DC) model. If $g_1=0$ it becomes a model studied by Schroer.[3] The massive model with $g_1=0$ was considered by Rothe and Stamatescu.[4]

To be equivalent to the Thirring model, the couplings g_1 and g_2 cannot be independent, but are related to Klaiber's constant k by

$$g_2{}^2=k\left\{\left[1+\left[\frac{k}{2\pi}\right]^2\right]^{1/2}+\frac{k}{2\pi}\right\}, \tag{1.4}$$

$$\left[1+\frac{g_1{}^2}{\pi}\right]\left[1+\frac{g_2{}^2}{\pi}\right]=1\,. \tag{1.5}$$

Mass perturbation around a massless theory is plagued by severe infrared divergences. In such a situation, one should attempt to make partial resummations to achieve finiteness. But, without a guiding principle, this is a hopeless task. We shall, therefore, adopt an infrared regulator before proceeding. A detailed discussion of the ultraviolet behavior is then done and the following result obtains.

(1) For $\beta^2<4\pi$ the more divergent contributions are precisely those of the unperturbed model. We found that only for $4\pi(2-\sqrt{3})<\beta^2$ the Thirring Green's functions are well defined. Parenthetically, this does not mean that the Thirring model is pathological for below $4\pi(2-\sqrt{3})$; the Wightman functions as given by Klaiber are, for example, well defined for all values of k. The value $\beta^2=4\pi(2-\sqrt{3})$ is the point where the two-point Green's function becomes singular as a distribution. We could still continue analytically beyond this value, decreasing β^2,

but this process will lead to more and more divergent Green's functions. Finally, at $\beta^2=4\pi[4-(15)^{1/2}]$ all Green's functions will become divergent and no continuation to lower values will be possible. We also mention that in the interval $4(2-\sqrt{3})<\beta^2<4\pi$ the only singularities are volume divergences which, as usual, cancel between numerator and denominator in the Gell-Mann–Low formula.

(2) For $8\pi>\beta^2>4\pi$ there are some additional divergences associated with vacuum bubble diagrams. These are again canceled by the denominator of the Gell-Mann–Low formula.

(3) Formal analysis indicates that the energy density is unbounded below if $\beta^2>8\pi$. However, general field-theoretical arguments show that, already for $\beta^2>4\pi$, the field operator $\cos(\beta\phi)$ (or, equivalently, $\bar{\psi}\psi$) is not well defined by just Wick ordering. Concerning this problem, we verify a conjecture by Swieca.[5] We found that a well-defined operator is obtained by doing a subtraction of the vacuum expectation value besides the usual Wick ordering prescription. We also discuss the construction of other composite operators. In particular this is done for the current which appears in the field equation.

For $\beta^2>8\pi$ the theory is unrenormalizable and some drastic change in the approach is necessary.

The paper is organized as follows. In Sec. II the DC model is introduced, first at the classical level. We then show that the fermionic Green's functions of the model are, for certain identification of the coupling constants, equal to those of the massless Thirring model. The section ends with a brief discussion of composite objects as the fermionic current and the mass operator. Section III begins the discussion of mass perturbation by giving the rules to construct the relevant amplitudes. An infrared cutoff is introduced and the degree of superficial divergency of an arbitrary amplitude is established. The UV behavior is extensively analyzed in Sec. IV where we also discuss the modifications, if any, in the case of composite operators. Equations of motion and Ward identities are discussed in Sec. V. Some remarks concerning the elimination of the infrared cutoff are presented in the conclusions. We have also included an appendix, summarizing Klaiber's notation on the Thirring model.

II. A DERIVATIVE-COUPLING MODEL

From a technical point of view the study of mass perturbation in the Thirring model can be greatly simplified if one takes advantage of the equivalence of this theory with the derivative-coupling model specified by

$$L=\frac{i}{2}\bar{\psi}\overleftrightarrow{\partial\!\!\!/}\psi+\tfrac{1}{2}(\partial_\mu\phi_1)^2+\tfrac{1}{2}(\partial_\mu\phi_2)^2 +g_1(\bar{\psi}\gamma^\mu\psi)\partial_\mu\phi_1+g_2(\bar{\psi}\gamma^\mu\gamma^5\psi)\partial_\mu\phi_2 \;. \tag{2.1}$$

At the classical level the equations of motion derived from such a Lagrangian are

$$\partial^2\phi_1=-g_1\partial_\mu(\bar{\psi}\gamma^\mu\psi)\;, \tag{2.2}$$

$$\partial^2\phi_2=-g_2\partial_\mu(\bar{\psi}\gamma^\mu\gamma^5\psi)\;, \tag{2.3}$$

$$i\partial\!\!\!/\psi=-g_1(\partial_\mu\phi_1)\gamma^\mu\psi-g_2(\partial_\mu\phi_2)\gamma^\mu\gamma^5\psi\;. \tag{2.4}$$

Now, as

$$\gamma^\mu\gamma^5=\epsilon^{\mu\nu}\gamma_\nu,\quad \epsilon^{\mu\nu}=\begin{bmatrix}0&1\\-1&0\end{bmatrix}$$

we could use (2.2) and (2.3) to reconstruct the current

$$\bar{\psi}\gamma^\mu\psi=-\frac{1}{g_1}\partial^\mu-\frac{1}{g_2}\tilde{\partial}^\mu\phi_2,\quad \tilde{\partial}^\mu=\epsilon^{\mu\nu}\partial_\nu\;. \tag{2.5}$$

Comparing this expression with the equation of motion of the Thirring model

$$i\partial\!\!\!/\psi=-g(\bar{\psi}\gamma^\mu\psi)\gamma_\mu\psi\;, \tag{2.6}$$

we see that with the choice $g_1{}^2=-g_2{}^2=-g$, the two models have identical fermionic sectors.

The content of the model (2.1) is actually trivial. As both vector and axial-vector currents are conserved, ϕ_1 and ϕ_2 turn out to be free fields. Moreover, from (2.4) one easily gets

$$\psi=\exp(ig_1\phi_1+i\gamma^5g_2\phi_2)\psi_0$$

with ψ_0 a free massless Dirac field.

The next step is the quantization. It is clear that the equivalence will continue to hold if the same quantization prescription is adopted for both models. At this point it may be instructive to stress a very fundamental difference between the classical and the quantum descriptions of a field theory. The classical equation of motion does not specify a model, because quantum fluctuations make the interaction terms undefined. To promote these formal expressions to the status of bona fide quantum operators requires detailed information about the short-distance behavior of a product of fields. In general terms, this implies that field equations and their solutions must be given simultaneously to, self-consistently, characterize the theory. In our case we suppose that ϕ_1 and ϕ_2 will still be free fields. However, since they are massless an infrared regulator is necessary to achieve finiteness. The infrared regulated two-point functions are

$$\langle T\phi_1(x)\phi_1(0)\rangle=\langle T\phi_2(x)\phi_2(0)\rangle =D_F(x)=-(1/4\pi)\ln\mu^2(-x^2+i0)\;, \tag{2.7}$$

where $D_F(x)$ satisfies $\partial^2D_F(x)=-i\delta(x)$. Because of the infrared cutoff, the Hilbert space of the states constructed from the fields ϕ_1 and ϕ_2 does not have a positive-definite norm. In spite of this, exponentiated fields $:\exp[i\alpha\phi(x)]:$ are in a good shape, provided a certain charge-conservation law is obeyed. The precise statement concerning the last remark is that positivity holds in the subspace reconstructed from Wightman's functions satisfying a charge-conservation law:[6]

$$\langle T{:}\exp[i\alpha_1\phi(x_1)]{::}\exp[i\alpha_2\phi(x_2)]{:}\cdots{:}\exp[i\alpha_\eta\phi(x_n)]\rangle=\exp\left[-\sum_{i<j}\alpha_i\alpha_jD_F(x_i-x_j)\right]\delta_{\sum_i\alpha_i,0}\;. \tag{2.8}$$

Thus, at least for small g_1 and g_2, the fermionic sector could be described by the field

$$\psi_T = :\exp(ig_1\phi_1 + ig_2\phi_2):\psi_0 \;. \tag{2.9}$$

Indeed, using (2.8), the N-point Green's function can be computed and then compared with Klaiber's. We have (see the Appendix)

$$\begin{aligned}\langle T\psi_T(x_1)\cdots\psi_T(x_N)\bar\psi_T(y_1)\cdots\bar\psi_T(y_N)\rangle = {}&\exp\left[\sum_{i<j} -({g_1}^2+{g_2}^2\gamma^5_{x_i}\gamma^5_{x_j})D_F(x_i-x_j)\right]\\ &\times\exp\left[\sum_{i<j} -({g_1}^2+{g_2}^2\gamma^5_{y_i}\gamma^6_{y_j})D_F(y_i-y_j)\right]\\ &\times\exp\left[\sum_{i,j} -(-{g_1}^2+{g_2}^2\gamma^5_{x_1}\gamma^5_{y_j})D_F(x_i-y_j)\right]\\ &\times\langle T\psi_0(x_1)\cdots\psi_0(x_N)\bar\psi_0(y_1)\cdots\bar\psi_0(y_N)\rangle \;,\end{aligned} \tag{2.10}$$

in which we should identify

$$\begin{aligned} a &= {g_1}^2,\quad \tilde a = {g_2}^2 \;,\\ a &= k\left\{-\left[1+\left[\frac{k}{2\pi}\right]^2\right]^{1/2}+\frac{k}{2\pi}\right\} \;,\\ \tilde a &= k\left\{\left[1+\left[\frac{k}{2\pi}\right]^2\right]^{1/2}+\frac{k}{2\pi}\right\} \;,\end{aligned} \tag{2.11}$$

where k is the Thirring-model coupling constant as defined by Klaiber. Note that ${g_1}^2{g_2}^2=-k^2$, implying that one of the g's is imaginary.

We are now in a position to write down all the operators appearing in the equation of motion in terms of ϕ_1, ϕ_2, and ϕ_0. The current, for example, can be identified with

$$g_\mu = \frac{1}{k}(g_1\partial_\mu\phi_1 - g_2\tilde\partial_\mu\phi_2) \;. \tag{2.12}$$

At first sight, from the observation at the end of the last paragraph, it seems that this current is not Hermitian. This would be indeed the case if the identification between the two models were at the operator level. We must stress, however, that the identification holds only in a weaker sense, between the Green's functions of the Thirring model and the corresponding functions of the DC model. In that sector, the current matrix elements depends only on ${g_1}^2$ and ${g_2}^2$ and no problem appears—see our formula (5.2) for an explicit verification of this statement.

Using (2.12), the field equations become

$$i\partial\!\!\!/\psi(x) = -\frac{k}{2}\lim_{\epsilon\to 0}[g^\mu(x+\epsilon)\gamma_\mu\psi(x)+\gamma_\mu\psi(x)g^\mu(x-\epsilon)] \;, \tag{2.13a}$$

$$\partial^2\phi_1 = \frac{k}{g_1}\partial_\mu g^\mu \;, \tag{2.13b}$$

$$\partial^2\phi_2 = \frac{k}{g_2}\tilde\partial_\mu g^\mu \;. \tag{2.13c}$$

Composite operators can also be constructed as local limits of products of the basic fields. In particular, Johnson's limiting procedure furnishes the current[1] (examples of other possible regularizations can be found in Klaiber's paper[1])

$$\begin{aligned} j^\mu(x) &= \frac{1}{4}\left[1+\left[\frac{k}{2\pi}\right]^2\right]^{-1/2}\lim_{\epsilon\to 0}\sum_{\epsilon,\tilde\epsilon} Z(\epsilon)[\bar\psi(x+\epsilon)\gamma^\mu\psi(x)-\gamma^\mu\psi(x)\bar\psi(x-\epsilon)] \;,\\ Z(\epsilon) &= \exp[-({g_1}^2+{g_2}^2)D_F(\epsilon)] \;,\end{aligned} \tag{2.14}$$

which, as discussed elsewhere, differs from (2.12) by a factor containing a spurion field, i.e., a field which has no effect on the fermionic sector. For later reference, we also write the mass operator as a limiting process

$$\begin{aligned} N[\bar\psi_T\psi_T](x) &= \lim_{\epsilon\to 0}\exp[-|{g_1}^2-{g_2}^2|D_F(\epsilon)]\\ &\quad\times\bar\psi_T(x+\epsilon)\psi_T(x)\\ &= :\exp(2ig_2\gamma^5\phi_2)::\bar\psi_0\psi_0:(x)\end{aligned} \tag{2.15}$$

with the understanding that the γ^5 matrix acts immediately on the left of the ψ_0 field.

III. MASS PERTURBATION

In Klaiber's operator approach the field solution of the Thirring model is written as

$$\psi_T = :\exp(i\alpha j + i\tilde\alpha\gamma^5\tilde j)\psi_0: \;, \tag{3.1}$$

where j and $\tilde j$ are the potentials of the free vector and free axial-vector currents, respectively. As j, $\tilde{j}$, and ψ_0 are not independent, the study of mass perturbation may become rather cumbersome. In this respect the representation (2.9) employing independent fields is clearly superior and will be adopted from now on.

The formal study of the perturbative series can be done by defining Green's functions via the Gell-Mann—Low formula

$$\langle T\psi(x_1)\cdots\psi(x_N)\bar{\psi}(y_1)\cdots\bar{\psi}(y_N)\rangle = \frac{\left\langle T\psi_1(x_1)\cdots\psi_T(x_N)\bar{\psi}_T(y_1)\cdots\bar{\psi}_T(y_N)\exp\left[i\int L_{\mathrm{int}}d^2x\right]\right\rangle}{\left\langle T\exp\left[i\int L_{\mathrm{int}}d^2x\right]\right\rangle}, \tag{3.2}$$

where $L_{\mathrm{int}}=N[\bar{\psi}_T\psi_T]$ is the mass operator and ψ_T denotes the solution (3.1) of the Thirring model. The Feynman amplitudes are obtained by expanding the exponential and applying Wick's theorem, always keeping in mind the selection rules (2.8). For that it is useful to use the identity

$$:\exp(2ig_2\gamma^5\phi):\bar{\psi}\psi = :\exp(-2ig_2\phi):\bar{\psi}_1\psi_1 + :\exp(2ig_2\phi):\bar{\psi}_2\psi_2 . \tag{3.3}$$

We shall now study the ultraviolet behavior of the integrals so constructed. To simplify the discussion this will be done explicitly in the Euclidean region. A generic amplitude I_G consists of a product of propagators of various types which, in a graphical representation, are associated with the lines of a graph G. The vertices of G are associated either with the interaction Lagrangian or with external fields. In momentum space the possible propagators are (1) fermion propagator $\not{p}/p^2$, (2) exponentiated fields. These are of various types, depending on the contracted fields $\sigma=g_2{}^2/4\pi=\frac{1}{4}(\beta^2/4\pi-1)$, $\rho=g_1{}^2/4\pi=-\sigma/(1+4\sigma)$.

	Propagator	Contraction
a.	$(p^2)^{-4\sigma-1}$	$:\exp(2ig_2\phi_2):$ with $:\exp(2ig_2\phi_2):$ or $:\exp(-2ig_2\phi_2):$ with $:\exp(-2ig_2\phi_2):$
b.	$(p^2)^{4\sigma-1}$	$:\exp(-2ig_2\phi_2):$ with $:\exp(2ig_2\phi_2):$
c.	$(p^2)^{-2\sigma-1}$	$:\exp(2ig_2\phi_2):$ with $:\exp(ig_2\phi_2):$ or $:\exp(-2ig_2\phi_2):$ with $:\exp(-ig_2\phi_2):$
d.	$(p^2)^{2\sigma-1}$	$:\exp(2ig_2\phi_2):$ with $:\exp(-ig_2\phi_2):$ or $:\exp(-2ig_2\phi_2):$ with $:\exp(ig_2\phi_2):$
e.	$(p^2)^{-\sigma-1}$	$:\exp(ig_2\phi_2):$ with $:\exp(ig_2\phi_2):$ or $:\exp(-ig_2\phi_2):$ with $:\exp(-ig_2\phi_2):$
f.	$(p^2)^{\sigma-1}$	$:\exp(ig_2\phi_2):$ with $:\exp(-ig_2\phi_2):$
g.	$(p^2)^{-\rho-1}$	$:\exp(ig_1\phi_1):$ with $:\exp(ig_1\phi_1):$ or $:\exp(-ig_1\phi_1):$ with $:\exp(-ig_i\phi_1):$
h.	$(p^2)^{\rho-1}$	$:\exp(ig_1\phi_1):$ with $:\exp(-ig_1\phi_1):$

At this point we could introduce a graphical notation to represent the above propagators, but this is not essential. In the figures, all these propagators are generically represented by wavy lines. The precise meaning of each of them must be clear from the context. In any case, it is rapidly seen that a regularization is necessary to avoid infrared divergences. To keep changes at a minimum only propagators associated with the ϕ_2 field will be modified (recall that L_{int} does not depend on ϕ_1). Because of charge conservation the vertices of a graph G can be separated into the following two sets. To the first set V_1 belongs the vertices which are connected to the external vertices of G by fermionic lines. The other set V_2 contains the remaining vertices of G. The fermionic lines, connecting the vertices in V_2, form therefore closed loops. The regularization that we will employ can now be described.

(1) If an exponentiated propagator links a vertex of V_1 with a vertex of V_2, we make the replacement

$$\exp[\alpha D_F(x-y)] \rightarrow \exp[\alpha\Delta_F(x-y,m^2)] , \tag{3.4}$$

where $\Delta_F(x,m^2)$ is the free propagator of mass m. The modification does not change the ultraviolet behavior whereas at large distance we have

$$\exp[\alpha\Delta_F(x)] \xrightarrow{r=(x^2)^{1/2}\rightarrow\infty} 1 . \tag{3.5}$$

(2) Otherwise, if both ends of a line are vertices in V_1 (or in V_2) then the momentum-space propagator is changed as

$$(p^2)^\alpha \rightarrow (p^2+m^2)^\alpha . \tag{3.6}$$

This regularization is not equivalent to (3.4). Indeed, the Fourier transform of the right-hand side (RHS) of (3.6) is not an exponential of a massive propagator but the function

$$\frac{2^{\alpha+2}\pi}{\Gamma(-\alpha)}(m^2)^{\alpha+1}\frac{K_{\alpha+1}(mr)}{r^{\alpha+1}} , \tag{3.7}$$

where $K_{\alpha+1}(mr)$ is a modified Bessel function. We observe that the substitution (3.6) gives a better large distance behavior than (3.4), namely, if $r\rightarrow\infty$ then (3.7) tends to zero. The forthcoming discussion will clarify the reasons for adopting two kinds of regulators instead of only one.

Returning to the study of the ultraviolet behavior of the regulated Feynman integrands, we recall the definition of a generalized vertex. This is any subgraph obtained by deleting some of the vertices (and all lines meeting at these vertices) of the original graph. Only proper one-particle-irreducible generalized vertices can generate counterterms.[7] Consider, therefore, a proper generalized vertex γ. We want to calculate the degree of superficial divergence of γ.

Let then p_1, p_2, l_1, and l_2 be the number of vertices of γ associated with the fields $:\exp(-2ig_2\phi_2):\bar{\psi}_{01}\psi_{01}$,

$:\exp(2ig_2\phi_2):\bar{\psi}_{02}\psi_{02}$, $:\exp(ig_1\phi_1-ig_2\phi_2):\psi_{01}$, and $:\exp(ig_1\phi_1+ig_2\phi_2):\psi_{02}$, respectively. Similarly let $\bar{l}_1$ and $\bar{l}_2$ indicate the number of vertices associated with $:\exp(-ig_1\phi_1-ig_2\phi_2):\bar{\psi}_{01}$ and $:\exp(-ig_1\phi_1+ig_2\phi_2):\bar{\psi}_{02}$. With this notation, the degree of superficial divergence of γ will be

$$\delta(\gamma)=2-p-\tfrac{3}{2}l-\frac{N_F}{2}+\frac{g_1{}^2}{4\pi}(l-f)+\frac{g_2{}^2}{4\pi}(l+4p-h)\ , \tag{3.8}$$

where $l=l_1+l_2+\bar{l}_1+\bar{l}_2$, $p=p_1+p_2$, N_F is the number of external fermionic lines,

$$f=(l_1+l_2-\bar{l}_1-\bar{l}_2)^2\ , \tag{3.9}$$

$$h=(2p_1-2p_2+l_1+\bar{l}_1-l_2-\bar{l}_2)^2\ . \tag{3.10}$$

IV. ULTRAVIOLET ANALYSIS

We first consider the diagrams of the unperturbed theory for which $N_F=p=f=h=0$. We then have

$$\delta(\gamma)=2-[\tfrac{3}{2}-(\sigma+\rho)]l=2-\left[\frac{3}{2}-\frac{4\sigma^2}{1+4\sigma}\right]l\ . \tag{4.1}$$

From this formula we see that the n point Green's functions are well defined for

$$-\sqrt{3}+1<\frac{g_2{}^2}{\pi}<\sqrt{3}+1\ . \tag{4.2}$$

Outside this interval the dimension of ψ_T becomes greater than one. In the repulsive region, the point $g_2{}^2=\pi(\sqrt{3}+1)$ is above the point $\beta^2=8\pi$ where, as we will see, the model becomes unrenormalizable.

The *a priori* inexistence of the Green's functions of the unperturbed model is without physical consequences. The cause is that the divergent parts are proportional to the product of δ functions. The arguments of these δ functions are the coordinates' differences of the external fields. Therefore, the divergent parts can be absorbed into a redefinition of the time ordering. By the same reason, the divergences of the full interacting theory associated with graphs with at least two external vertices (i.e., $l\geq 2$) can be eliminated by a mere redefinition of the time ordering. However these procedures cannot be implemented by the addition of counterterms to the Lagrangian.

From the above observations, it is clear that we need to consider only the cases with $l<2$. Within this constraint we examine each possibility.

(1) $N_F=0$, $l=0$. Some illustrative graphs are depicted in Fig. 1. Power counting, Eq. (3.8), gives

$$\delta(\gamma)=2-(1-4\sigma)p\ .$$

Thus, for $\sigma<0$, δ is negative. For $0<\sigma<\frac{1}{4}$, which corresponds the interval $4\pi<\beta^2<8\pi$, δ is less than two. As $N_F=0$ and also because of chiral symmetry $p_1=p_2$. Therefore the reduced vertex $V(\gamma)$, obtained by contracting the graph γ to a point has no lines. Actually, for this to happen it is important to have a regularization like (3.4). Differently, had we uniformly employed the regularization (3.6), no cancellation of the external lines would occur. The divergence is partially removed by combining these graphs with the corresponding (disconnected) diagrams coming from the denominator of the Gell-Mann—Low formula. In Fig. 2 we show a graph which becomes disconnected when the upper bubble is reduced to a point. For $\sigma<\frac{1}{8}$ the divergence is only logarithmic and is entirely removed in this combination of graphs. For $\frac{1}{8}<\sigma<\frac{1}{4}$ the divergence becomes linear but, because of Lorentz covariance, no counterterm is necessary. For $\sigma>\frac{1}{4}$ $(\beta^2>8\pi)$, δ increases with p and the model becomes unrenormalizable. So, from now on we will restrict the analysis to $\sigma<\frac{1}{4}$.

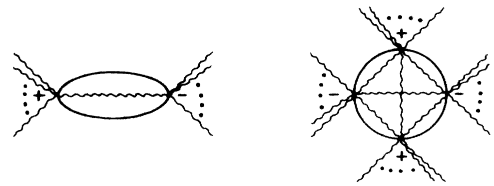

FIG. 1. Divergent graphs without external fermion lines. Solid and wavy lines represent fermion and exponentiated propagators, respectively. The $+$ (or $-$) sign at the vertices indicates the corresponding sign of the exponentiated field.

(2) $N_F=1$, $l=1$. Because of chiral symmetry and charge conservation, $f=h=1$ and therefore

$$\begin{aligned}\delta(\gamma)&=2-(1-4\sigma)p-[\tfrac{3}{2}-(\sigma+\rho)]-\tfrac{1}{2}-\sigma-\rho\\&=-(1-4\sigma)p<0\ .\end{aligned}$$

(3) $N_F=2$. Since in this case l must be even, we have to consider only the possibility $l=0$. There are two subcases.

(a) p is even. We have then $f=h=0$. Thus

$$\delta(\gamma)=1-(1-4\sigma)p\ .$$

For $\sigma<0$ there is no divergence. For $\sigma>0$, δ is less than one. However, as p is even the number of internal fermion lines of γ is odd. Therefore the divergence is absent if symmetric integration is employed.

(b) p is odd. Here $f=0$ but $h=4$. Thus

$$\delta(\gamma)=-(1-4\sigma)(p-1)<0\ .$$

(4) $N_F=3$, $l=1$. We have the following.

(a) p is even. Thus $f=h=1$ so that

$$\begin{aligned}\delta(\gamma)&=2-[(1-4\sigma)p+\tfrac{3}{2}-(\sigma+\rho)]+\tfrac{3}{2}+\sigma+\rho\\&=-1-(1-4\sigma)p<0\ .\end{aligned}$$

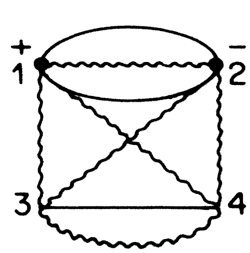

FIG. 2. The lines connecting the vertices 3 to 1 and 2 (and 4 to 1 and 2) cancel, when the bubble is contracted to a point.

(b) p is odd. Here, again, there are two subcases to consider.

(i) $f=h=1$. We get $\delta<0$.

(ii) $f=1$ but $h=9$ [i.e., number of $(\psi_1+\bar{\psi}_1)$ − number of $(\psi_2+\bar{\psi}_2)=3$]. From this results

$$\delta(\gamma)=-1-[(1-4\sigma)p+4\sigma]\leq -3 .$$

(5) Now consider the case with $N_F>4$ arbitrary and $l=0$. We then have $f=0$ and, depending on γ, p_1-p_2 can be equal to $0,1,2,\ldots,N_F/2$. If $p_1=p_2$ then the N_F external fields will consist of equal numbers of ψ_{01}, ψ_{02}, $\bar{\psi}_{01}$, and $\bar{\psi}_{02}$. In the other extreme case, i.e., when $p_1-p_2=N_F/2$, all the external fields will have the same index. Remember now that, because of charge conservation, the fermion lines can end or begin only at the vertices associated to the external fields. Let us treat a generic case in which the fermion lines link the external fields in the following way: x_1 paths connect $x_1\psi_{01}$ external fields to $x_1\bar{\psi}_{01}$ external fields, x_2 connect $x_2\psi_{02}$ to $x_2\bar{\psi}_{02}$, a_1 connect $a_1\psi_{01}$ to $a_1\bar{\psi}_{02}$, and a_2 connect $a_2\psi_{02}$ to $a_2\bar{\psi}_{01}$. Clearly, $x_1+x_2+a_1+a_2=N_F/2$. If this graph is divergent, a typical counterterm will be formed of a certain number of derivatives acting on a field monomial composed of the same ψ_0's as the external fields. The counterterm can be simplified using $\partial_0\psi_2=\partial_1\psi_2$ (and $\partial_0\psi_1=-\partial_1\psi_1$). Indeed, $\partial\!\!\!/\psi=0$ since $\partial\!\!\!/$ cuts a fermion line leaving a result which contains $\exp[\alpha\Delta_F(0)]$ as a factor. This is zero if a convenient ultraviolet regularization (dimensional, for example) is employed. Because of this and Fermi statistics, there is a minimum number of derivatives which should be applied in order to get a nonzero result. For example,

$$\partial_0\psi_{01}\partial_1\psi_{01}\partial_0\bar{\psi}_{01}\partial_1\bar{\psi}_{01}\sim(\partial_0\psi_{01})^2(\partial_0\bar{\psi}_{01})^2=0 .$$

It is not difficult to see that the minimum number of derivatives allowed is

$$D=\tfrac{1}{2}[(x_1+a_1)(x_1+a_1-1)+(x_2+a_2)(x_2+a_2-1)+(x_1+a_2)(x_1+a_2-1)$$
$$+(x_2+a_1)(x_2+a_1-1)]\leq x_1{}^2+x_2{}^2-N_F/2 .$$

On the other hand, the degree of superficial divergence is

$$\delta(\gamma)=2-(1-4\sigma)p-N_F/2-4\sigma(x_1-x_2)^2 .$$

For $\sigma>0$, δ is negative. Also if $-(\sqrt{3}-1)\frac{1}{4}<\sigma<0$, then $D>\delta$ and the divergence will be canceled. The case with N_F arbitrary and $l=1$ can be analyzed analogously giving the same result.

This concludes our discussion of the ultraviolet behavior of the Green's functions. Summing up, we have shown that for $4\pi(2-\sqrt{3})<\beta^2<8\pi$ the only possible divergences are volume divergences which, nonetheless, cancel in the Gell-Mann–Low formula.

Now it is time to justify the use of the two regulators (3.4) and (3.6). The form of the regulator (3.4) enforces the cancellation of "vacuum bubble" diagrams, as explained [case (1)]. Since we want to keep ψ_0 massless, due to (3.5), then we also need the regulator (3.6) to hold infrared divergences away.

A similar discussion can be done for the construction of normal products of the bilinears, $\bar{\psi}\psi$ and $\bar{\psi}\gamma^5\psi$, which are very important for the boson formulation of the model.[2] The graphs contributing to

$$\langle T\mathcal{O}(x)\psi(x_1)\cdots\psi(x_N)\bar{\psi}(y_1)\cdots\bar{\psi}(y_N)\rangle$$

with $\mathcal{O}$ equal either to $\bar{\psi}\psi$ or to $\bar{\psi}\gamma^5\psi$ have a special vertex $V_{\mathcal{O}}$ associated with $\mathcal{O}$. However, since this vertex has the same structure as those coming from the interaction Lagrangian, the power counting will still be the same as before. Therefore, for $4\pi(2-\sqrt{3})<\beta^2<8\pi$ the only new divergences correspond to subgraphs which contain $V_{\mathcal{O}}$, have $p_1=p_2$, and have no external fermion lines. They are of the type (1), discussed previously. From the remarks there, it is clear that the divergent parts, which appear only for $4\pi<\beta^2<8\pi$, can be identified with contributions to the vacuum expectation value of $\mathcal{O}$. Now, because of charge conjugation (or parity) this vacuum expectation value is zero if $\mathcal{O}=\bar{\psi}\gamma^5\psi$. So we get the results that

$$N[\bar{\psi}\psi]=:\bar{\psi}\psi:-\langle:\bar{\psi}\psi:\rangle , \tag{4.3}$$

$$N[\bar{\psi}\gamma^5\psi]=:\bar{\psi}\gamma^5\psi: \tag{4.4}$$

are well-defined operators for $\beta^2<8\pi$ [we stress that the $\langle:\bar{\psi}\psi:\rangle$ in (4.3) is necessary only for $4\pi<\beta^2<8\pi$ where it is divergent]. This agrees with Swieca's conjecture on composite operators of the sine-Gordon model.[5]

V. CURRENT CONSERVATION AND EQUATIONS OF MOTION

For the massive model we can still define a current analogous to (2.12)

$$g^{\mu}=\frac{1}{k}(g_1\partial^{\mu}\phi_1+g_2\tilde{\partial}^{\mu}\phi_2) . \tag{5.1}$$

Indeed, this current is obviously conserved and satisfies $[Z_T=\psi_T(x_2)\cdots\psi_T(x_N)\bar{\psi}_T(y_1)\cdots\bar{\psi}_T(y_N)]$

$$\langle Tg^\mu(x')\psi_T(x)Z_T L_{\rm int}(z_1)\cdots L_{\rm int}(z_q)\rangle = \Bigg[\frac{g_1^2}{k}\partial^\mu\left[D_F(x'-x)+\sum_{i=2}^{N}D_F(x'-x_i)-\sum_{i=1}^{N}D_F(x'-y_i)\right]$$
$$+\frac{g_2^2}{k}\tilde{\partial}^\mu\left[\gamma_x{}^5\Delta_F(x'-x)+\sum_{i=2}^{N}\gamma_{x_i}^5\Delta_F(x'-x_1)+\sum_{i=1}^{N}\gamma_{y_i}^5\Delta_F(x'-y_i)\right]$$
$$+\frac{2g_2^2}{k}\tilde{\partial}^\mu\sum_{i=1}^{q}\gamma_{z_i}^5\Delta_F(x'-z_i)\Bigg]\langle T\psi_T(x)Z_T L_{\rm int}(z_1)\cdots L_{\rm int}(z_q)\rangle\ , \tag{5.2}$$

which shows explicitly the absence of further divergences. However, if we want to define the product $g^\mu(x)\psi(x)$, we should let $x'\to x$. In this situation additional divergences can appear. To see that in detail we have to consider two possibilities. (1) If x and x' are linked by just one line (propagator) we get graphs of the type shown in Fig. 3(a). These divergences are not dangerous since they can be eliminated by Wick ordering. (2) If any path linking x to x' consists of more than one propagator, we obtain graphs as that in Fig. 3(b). Because of (5.2), the graph will contain a line associated with $\partial^\mu D_F(x-w)$ or $\partial^\mu\Delta_F(x-w)$. This factor can be imagined as coming from the differentiation of an exponentiated propagator. In any case, the graph will be more singular, because of the additional momenta factor. Instead of giving an unmotivated definition for its finite part, we first examine the field equations where such a product occurs. We have

$$i\partial\!\!\!/\langle T\psi(x)Z\rangle = k\langle T{:}\gamma_\mu g^\mu\psi{:}(x)Z\rangle + M\{\exp[2g_2{}^2\Delta_F(0)]\}_R\langle T\psi(x)Z\rangle$$
$$+i\sum_{i=1}^{N}(-1)^{i+N}\delta(x-y_i)\{\exp[g_1{}^2D_F(0)+g_2{}^2\Delta_F(0)]\}_R\langle TZ_{y_i}\rangle\ , \tag{5.3}$$

where Z is equal to Z_T with ψ_T replaced by ψ, and Z_{y_i} is the same as Z with $\bar{\psi}(y_i)$ omitted. The index R is to indicate that the quantity in curly brackets is infrared regulated as in (3.5) [or (3.4)]. Note that, because of the factor $\exp[2g_2{}^2\Delta_F(0)]$, the second term in on the RHS of (5.3) is absent if $\sigma<0$. Moreover, for $\sigma>0$ this term is divergent and should be used to compensate a corresponding divergence in the first term. At $\sigma=0$ (5.3) becomes the Dirac equation for a free massive spinor field ψ.

The derivation of (5.3) is standard: In momentum space the graphs contributing to the left-hand side of (5.3) have the structure shown in Fig. 4. Writing $p\!\!\!/=p\!\!\!/+k\!\!\!/-k\!\!\!/$ we get two terms. In the first of these two terms the $p\!\!\!/+k\!\!\!/$ factor is used to cancel a fermion propagator. This produces the second (if the canceled propagator linked x to an interaction vertex) and the third (if the canceled propagator linked x to an external vertex) terms in the right-hand side of (5.3). The remaining term, on the other hand, is easily recognized as a contribution to $\langle T{:}\gamma_\mu g^\mu\psi{:}(x)Z\rangle$.

It is now clear that a useful definition of the finite part of the product of the current with the field is

$$\langle TN[\gamma^\mu g_\mu\psi](x)Z\rangle = \langle T{:}\gamma_\mu g^\mu\psi{:}(x)Z\rangle + \frac{M}{k}\{\exp[2g_2{}^2\Delta_F(0)]-1\}_R\langle T\psi(x)Z\rangle$$
$$+\frac{i}{k}\sum_{i=1}^{N}(-1)^{i+N}\{\exp[g_1{}^2D_F(0)+g_2{}^2\Delta_F(0)]-1\}_R\langle TZ_{y_i}\rangle\ . \tag{5.4}$$

With this definition, the field equation takes the usual form

$$(i\partial\!\!\!/-M)\langle T\psi(x)Z\rangle = k\langle TN[\gamma_\mu g^\mu\psi](x)Z\rangle + i\sum_{i=1}^{N}(-1)^{i+N}\delta(x-y_i)\langle TZ_{y_i}\rangle\ . \tag{5.5}$$

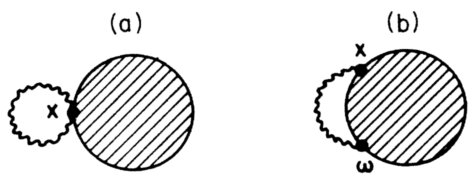

FIG. 3. Graphs contributing to $\lim_{x\to x'}g^\mu(x)\psi(x')$. (a) corresponds to the situation where x and x' were linked just by the indicated wavy line. Any other possibility produces graphs like (b).

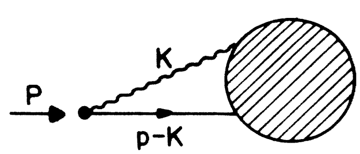

FIG. 4. Graphical structure of the LHS of (5.3).

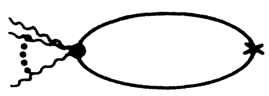

FIG. 5. The only divergent graph in the region $4\pi<\beta^2<8\pi$, after the resummation (5.6). The vertex with the cross corresponds to the additional interaction $:\bar{\psi}_0\psi_0:$ coming from the resummation (5.6).

VI. CONCLUDING REMARKS

In this study of mass perturbation in the Thirring model we have verified that the Green's functions are well defined for $4\pi(2-\sqrt{3})<\beta^2<8\pi$. In this interval the only divergences are those associated with vacuum bubbles which cancel in the Gell-Mann—Low formula. For $\beta^2>8\pi$ the theory is not renormalizable: The degree of superficial divergence increases without bound with the order of perturbation, and our methods are no longer applicable. Besides that, at $\beta^2=8\pi$ the propagator associated with a line linking two interaction vertices develops a nonintegrable singularity.

We have also shown that the mass operator can be made finite in the interval $4\pi(2-\sqrt{3})<\beta^2<8\pi$ by subtracting its vacuum expectation value besides the usual Wick ordering. Up to third order, a similar result has been obtained in Ref. 8.

To avoid infrared divergences, it was necessary to introduce auxiliary mass regulators. The elimination of these regulators requires in principle an infinite resummation of the perturbative series. A possible way to accomplish that could be by writing the interaction as

$$N[\bar{\psi}_T\psi]=:[\exp(2ig_2\gamma^5\phi_2)-1]::\bar{\psi}_0\psi_0:+:\bar{\psi}_0\psi_0: \qquad (6.1)$$

and then transferring the last term to the unperturbed Lagrangian. This would provide a mass to the free fermion propagator and possibly would eliminate the infrared divergences. But more graphs will have to be examined and they could generate additional ultraviolet divergences. The outcome of this analysis depends on the particular value of β^2. For $4\pi<\beta^2<8\pi$ the result is satisfactory since there is only one divergent graph, shown in Fig. 5. Such divergence can be compensated by adding a counterterm $\text{const}\times\cos(2g_2\phi_2)$ to the Lagrangian. The arbitrariness in the finite part can be fixed by imposing a definite value for the mass of the ϕ_2 field.

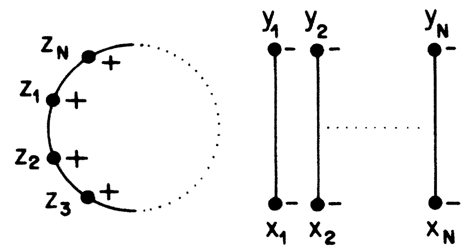

FIG. 6. All vertices of the above graph have exponentiated fields (for simplicity, exponentiated propagators are not explicitly shown). The generalized subgraph made with the vertices on the fermionic loop has a degree of divergence increasing with N, if $\sigma<0$.

Amazingly, the same procedure does not work for $\beta^2<4\pi$. It happens that, in this region σ is negative, which favors the appearance of new divergent graphs. This is illustrated by the graph of Fig. 6 which contains a subgraph divergent for $\sigma<-1/2N$ (the associated counterterm will be a cosine of a higher harmonic of $2g_2\phi_2$). We could say that, in this region, the net effect of the resummation is to replace infrared by ultraviolet divergences. A different resummation procedure, evading this situation would be highly desirable.

ACKNOWLEDGMENTS

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APPENDIX

For the reader's convenience, we collect here some results on the Thirring model. Unless by the following minor technical simplifications, the notation is the same as in Klaiber's paper:[1]

(a) We restrict the (Lorentz) spin to be $\frac{1}{2}$.

(b) We use the current operator in the particular form given by Johnson.[1]

(c) Instead of a non-Lorentz-covariant regularization of the two-point function of the scalar and pseudoscalar fields, we use the regularization (2.7) and (2.8) which guarantee positivity in the fermionic sector.

The model is defined by the set of equations

$$i\partial\!\!\!/\psi(x)=-\frac{k}{2}\gamma^\mu\lim_{\epsilon\to 0}[j_\mu(x+\epsilon)\psi(x)+\psi(x)j_\mu(x-\epsilon)]\ , \qquad (A1)$$

$$j^\mu(x)=\frac{1}{4\left[1+\left[\dfrac{k}{2\pi}\right]^2\right]^{1/2}}\lim_{\epsilon\to 0}\sum_{\epsilon,\tilde{\epsilon}}\exp[-(\alpha-\tilde{\alpha})D^-(\epsilon)][\bar{\psi}(x+\epsilon)\gamma^\mu\psi(x)-\gamma^\mu\psi(x)\bar{\psi}(x-\epsilon)]$$

$$=\frac{\partial^\mu j}{\sqrt{\pi}}=\frac{\tilde{\partial}^\mu\tilde{j}}{\sqrt{\pi}}=:\bar{\psi}_0\gamma^\mu\psi_0:\ , \qquad (A2)$$

$$\psi(x)=\exp\{i[\alpha j^+(x)+\tilde{\alpha}\gamma^5\tilde{j}^+(x)]\}\psi_0(x)\exp\{i[\alpha j^-(x)+\tilde{\alpha}\gamma^5\tilde{j}^-(x)]\}\ , \qquad (A3)$$

where

$$D^{-}(x)=-\frac{1}{4\pi}\ln\mu^2(-x^2+i0x^0)\ , \tag{A4}$$

$$\alpha=\sqrt{\pi}\left\{1+\frac{k}{2\pi}-\left[1+\left[\frac{k}{2\pi}\right]^2\right]^{1/2}\right\}, \tag{A4a}$$

$$\tilde{\alpha}=\sqrt{\pi}\left\{1-\frac{k}{2\pi}-\left[1+\left[\frac{k}{2\pi}\right]^2\right]^{1/2}\right\}, \tag{A4b}$$

$j(x)$, $\tilde{j}(x)$, and $\psi_0(x)$ are massless, free, scalar, pseudoscalar, and Dirac fields, respectively. These are not independent fields. In fact, they are assumed to satisfy the commutation relations

$$[j^{-}(x),\tilde{j}^{\dagger}(y)]=\tilde{D}^{-}(x-y)=-\frac{1}{4\pi}\ln\frac{x^0-y^0-(x^1-y^1)-i0}{x^0-y^0+(x^1-y^1)-i0}\ , \tag{A5a}$$

$$[j^{-}(x),\psi_0(y)]=-i\sqrt{\pi}[D^{-}(x-y)+\gamma^5\tilde{D}^{-}(x-y)]\psi_0(y)\ , \tag{A5b}$$

$$[\tilde{j}^{-}(x),\psi_0(y)]=-i\sqrt{\pi}[\tilde{D}^{-}(x-y)+\gamma^5 D^{-}(x-y)]\psi_0(y)\ . \tag{A5c}$$

By extensively using the commutations relations, the $2N$-point functions can be calculated. The results are

$$\begin{aligned}\langle T\psi(x_1)\cdots\psi(x_N)\bar{\psi}(y_1)\cdots\bar{\psi}(y_N)\rangle=&\exp\left[-\sum_{j<k}(a+\tilde{a}\gamma^5_{x_j}\gamma^5_{x_k})D_F(x_j-x_k)\right]\exp\left[\sum_{j<k}(a+\tilde{a}\gamma^5_{y_j}\gamma^5_{y_k})D_F(y_j-y_k)\right]\\ &\times\exp\left[-\sum_{j,k}(-a+\tilde{a}\gamma^5_{x_j}\gamma^5_{y_k})D_F(x_j-y_k)\right]\\ &\times\langle T\psi_0(x_1)\cdots\psi_0(x_N)\bar{\psi}_0(y_1)\cdots\bar{\psi}_0(y_N)\rangle\ ,\end{aligned} \tag{A6}$$

where

$$a=k\left\{\frac{k}{2\pi}-\left[1+\left[\frac{k}{2\pi}\right]^2\right]^{1/2}\right\}, \tag{A7a}$$

$$\tilde{a}=k\left\{\frac{k}{2\pi}+\left[1+\left[\frac{k}{2\pi}\right]^2\right]^{1/2}\right\}. \tag{A7b}$$

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