Unified dynamics for microscopic and macroscopic systems

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An explicit model allowing a unified description of microscopic and macroscopic systems is exhibited. First, a modified quantum dynamics for the description of macroscopic objects is constructed and it is shown that it forbids the occurrence of linear superpositions of states localized in far-away spatial regions and induces an evolution agreeing with classical mechanics. This dynamics also allows a description of the evolution in terms of trajectories. To set up a unified description of all physical phenomena, a modification of the dynamics, with respect to the standard Hamiltonian one, is then postulated also for microscopic systems. It is shown that one can consistently deduce from it the previously considered dynamics for the center of mass of macroscopic systems. Choosing in an appropriate way the parameters of the so-obtained model one can show that both the standard quantum theory for microscopic objects and the classical behavior for macroscopic objects can all be derived in a consistent way. In the case of a macroscopic system one can obtain, by means of appropriate approximations, a description of the evolution in terms of a phase-space density distribution obeying a Fokker-Planck diffusion equation. The model also provides the basis for a conceptually appealing description of quantum measurement.

I. INTRODUCTION

Despite the success of quantum mechanics in accounting with striking accuracy for a vast variety of physical phenomena, this theory presents crucial conceptual difficulties, about which a lively scientific debate is still going on. Almost all the difficulties can be traced back to the problem of accounting for the behavior of macroscopic objects and for their interactions with microscopic ones, and are strictly related to the occurrence (allowed by the theory) of linear superpositions of macroscopically distinguishable states of a macroscopic system (a typical example being the macroscopically different pointer positions of a measuring apparatus). This very fact, i.e., that the linearity of quantum theory unavoidably leads one to consider such superpositions, constitutes a basic difficulty for all trials of deriving a unified description of the physical reality from microscopic to macroscopic phenomena.[1] In particular, it is a source of difficulties when one tries to describe the dynamics of macroscopic systems in terms of trajectories, which are one of the most immediate data of our experience with these kinds of objects.

The above statements deserve some clarification. Actually, even though most features of the behavior of macroscopic objects are accounted for by quantum mechanics in a natural way, due to the irrelevant spreads of wave packets for macroscopic masses, in those cases in which the evolution of the system leads to a linear superposition of states which are localized in far-away spatial regions (as it happens, e.g., for the final pointer position of a measuring instrument), the description in terms of trajectories becomes problematic. The standard way out is contained in the reduction postulate, which hypothesizes the transition from pure states to statistical mixtures.

The clash between these two descriptions, one in terms of linear superpositions, the other in terms of statistical mixtures, constitutes the basic difficulty of the quantum theory of measurement.

Various solutions for these difficulties have been proposed. They can be fitted into one of the following schemes.

(a) One accepts two principles of evolution yielding a different dynamical behavior for microscopic and macroscopic objects. This introduces a dualism in nature[2] and would require in any case the introduction of a precise criterion allowing the identification of which objects must be considered as macroscopic.

(b) One limits in principle the set of observables for a macroscopic system to an Abelian set. Because of the fact that, as is well known,[3] this assumption implies the equivalence (for what concerns the physical predictions) of pure states and appropriate mixtures, it yields a way out from the difficulties of the quantum theory of measurement.[4] We want to remark that from the literature it is not always clear if this restriction of the class of the observables of a macroscopic object is assumed to hold for all such objects or only for those which can be used as measuring apparatuses. The first attitude amounts again

to accepting two different categories of physical objects. The second alternative does not eliminate the difficulties for those objects which cannot be used as measuring apparatuses. In fact, if one accepts linear superpositions of macroscopically distinguishable states, one still meets the difficulty that the direct perception of these macroscopic differences would "create" the physical properties of such objects.

(c) A solution which is not based on an *a priori* partition of all objects in two classes, but makes appeal to actual experimental limitations, is obtained by pointing out the extreme difficulty of performing experiments which could allow discrimination between linear superpositions and statistical mixtures for a macroscopic object. This point of view makes the definition of what has to be considered as macroscopic dependent on the skillfulness of the experimenter at the present time and, as such, could gradually render the set of macroscopic objects empty. A physically correct attitude within this context is the one taken by those authors who investigate whether it might be possible to observe the effects of quantum-mechanical interference at the macroscopic level and where to look for them.[5]

To conclude this short discussion we want to stress that in our opinion, to keep the standard quantum dynamics and to abandon or to make ineffective the superposition principle for some systems amounts to accepting (at least to a certain extent) a dualistic description of natural phenomena. This means to give up the program of a unified derivation of the behavior of all objects from the basic dynamics of the microscopic world.

We present here an attempt of such a unified description through the discussion of a dynamical model in which linear superpositions of states corresponding to the same macroscopic object being localized in far apart spatial regions are naturally suppressed. This is an essential step in order to give a description of the evolution of a macroscopic system in terms of trajectories. This result, if achieved, would yield a formalism satisfying the general requirements put forward by Ludwig in his basic approach to the description of physical phenomena.[1] Previous important contributions along this line have been given by Barchielli, Lanz, and Prosperi.[6]

To get the above-mentioned result we start by accepting a modification of the dynamics of macroscopic objects with respect to the standard one implied by quantum mechanics, keeping in mind the requirement of suppressing linear combinations of far apart localized states. To yield this suppression the dynamical equation must induce transitions from pure states to statistical mixtures. A way of obtaining this is to add in the dynamical equation a stochastic term corresponding to a localization process. This process is formally identical to an approximate position measurement and actually it was introduced as such in Ref. 6. As we are dealing with macroscopic objects, the idea of modifying their dynamics could be justified by making reference to a remark which has been made by various authors.[7] It consists in the recognition that the very idea of an isolated system loses its meaning for a macroscopic object, since such an object has so closely spaced quantum levels that almost any interaction, no matter how weak, is sufficient to induce transitions among them. A macroscopic object has then to be considered as embedded in some sort of thermal bath. The equation of motion could then be considered as describing the reduced dynamics for such a nonisolated system. Whether the interaction with the rest of the world can be accounted for by processes having the features of a measurement is obviously open to debate.[8] Here we are not interested in discussing this point since we want to take a very different attitude. Our main aim is that of deriving a dynamics for macroscopic objects with the above-mentioned characteristics, and agreeing with classical dynamics, from a postulated basic dynamics for its microscopic constituents. As we shall show, this program can be consistently fulfilled by assuming that all microscopic systems are subjected to localization processes with an appropriate frequency. We do not consider here the problem of the physical origin of these localizations for microscopic systems (see, however, the remarks in Ref. 16), but we simply postulate that they occur. In this sense we say that they are spontaneous. This assumption turns out to be sensible since, as we shall show, one can choose the parameters in the equation in such a way that the dynamics of microscopic systems coincides for all practical purposes with the standard Hamiltonian quantum dynamics. Moreover the dynamics of a macroscopic object can be consistently deduced from that of its microscopic components and turns out to forbid linear superpositions of far-away states and to yield an evolution compatible with classical mechanics.[9] The quantum-mechanical wave-packet reduction with definite final pointer position can also be deduced by the modified quantum dynamics. This last feature of the model will be exhaustively discussed in a forthcoming paper.

In Sec. II we make some preliminary considerations on the requirements which must be taken into account in order to have a satisfactory description of the dynamics of a macroscopic object and we describe some previous important work on this subject. In Sec. III we explicitly introduce a dynamical equation for such objects and we study its consequences on the evolution of a free macroscopic particle in one dimension. In Sec. IV we compare the evolution induced by the dynamical equation with the standard quantum evolution and with the classical one. In Sec. V we show how our dynamical equation can be used to define particle trajectories. Section VI puts forward a change of attitude, i.e., a modified dynamics is postulated also for microscopic systems. It is shown that one can consistently deduce from it the previously discussed dynamics for the center of mass of the macroscopic bodies. Section VII illustrates how by a proper choice of the parameters of the theory one can derive a set of remarkable consequences which can be summarized by stating that standard quantum theory for microscopic objects, quantum-mechanical wave-packet reduction, and classical behavior for macroscopic objects can be all consistently deduced from the basic microscopic dynamics.

In Sec. VIII we show how, taking advantage of the modified dynamics for macroscopic objects, one can give a description of the dynamical evolution of such objects in terms of a phase-space density distribution.

II. DYNAMICS FOR MACROSCOPIC OBJECTS: PRELIMINARY CONSIDERATIONS

If one is interested in understanding the dynamical evolution implied by quantum mechanics for macroscopic systems and in comparing it with the one following from classical mechanics, it is quite natural to look for the possibility of giving some meaning to the concept of trajectories even in the quantum case. As already said, the major obstacle to such a possibility comes from the fact that quantum mechanics allows the occurrence of linear superpositions of far-away states. It should be clear that to reach the above aim some modifications of the purely Hamiltonian quantum dynamics are necessary.

To better understand the problem and the ideas which can be followed in developing such a program, it is useful to discuss first the problem within classical mechanics. We shall limit our considerations to the case of an ensemble of classical particles in one dimension. Such an ensemble is described by means of the density function in phase space $\rho(q,p,t)$ obeying the Liouville equation

$$\frac{d\rho}{dt}=\{H,\rho\}\ , \tag{2.1}$$

where H is the Hamiltonian and the symbol $\{\ ,\ \}$ denotes the Poisson brackets. Let us choose arbitrary continuous functions $q(t)$ and $p(t)$ and, for fixed t, let us define the subset of phase space E_t as

$$E_t=\{q,p\mid |q-q(t)|<\Delta q,\ |p-p(t)|<\Delta p\}\ , \tag{2.2}$$

for two arbitrarily chosen Δq and Δp. When t is varied the family E_t identifies a "tube" in the (q,p,t) space.

We are now interested in the following problem: given the density function ρ at time $t=0$, what is the probability at time t that a particle of the ensemble has followed a trajectory which, for all times up to t, was contained in the tube? A way to tackle this question is the following. Suppose we test at random times, according to a Poisson process with mean frequency λ, whether the members of the ensemble are within the tube or not and we discard those members which are found outside. The ensemble of the surviving systems then becomes poorer as time elapses and will be described by a density function $\tilde{\rho}(q,p,t)$. It is easy to get an equation governing the evolution of $\tilde{\rho}(q,p,t)$. Taking into account that the probability of having no test in the interval dt is $1-\lambda\,dt$ we have

$$\tilde{\rho}(t+dt)=(1-\lambda\,dt)[\tilde{\rho}(t)+\{H,\tilde{\rho}(t)\}dt]+\lambda\,dt\,\chi_{E_t}\tilde{\rho}(t)\ , \tag{2.3}$$

where χ_{E_t} is the characteristic function of the set E_t. From (2.3) we get the differential equation

$$\frac{d\tilde{\rho}}{dt}=\{H,\tilde{\rho}(t)\}-\lambda(1-\chi_{E_t})\tilde{\rho}(t)\ . \tag{2.4}$$

It is easily seen that Eq. (2.4) implies

$$\frac{d}{dt}\int\tilde{\rho}(q,p,t)dq\,dp\le 0\ , \tag{2.5}$$

where the integral is extended to the whole phase space. Equation (2.4) therefore implies a possible loss of probability. If one denotes by Σ_t the map

$$\tilde{\rho}(0)\to\tilde{\rho}(t)=\Sigma_t\tilde{\rho}(0)\ , \tag{2.6}$$

one immediately checks that $\Sigma_{t_1}\Sigma_{t_2}=\Sigma_{t_1+t_2}$ for any $t_1,t_2>0$, showing that the map Σ_t yields a representation of the forward time translation semigroup. Equation (2.4) is the classical analogue of a quantum dynamical semigroup equation. From the derivation it is obvious that the integral of $\tilde{\rho}(q,p,t)$ over the whole phase space gives the probability that a particle of the ensemble is found within the tube in all tests to which it has been subjected in the time interval $0-t$. For λ going to infinity $\tilde{\rho}(q,p,t)$ tends to a density function $\hat{\rho}(q,p,t)$, such that its integral over the whole phase space gives the probability that a trajectory lies within the tube. In view of the arbitrariness of Δq and Δp in Eq. (2.2), this procedure can be used to define the probability of occurrence of a given trajectory with any preassigned accuracy. It is obvious that one could also identify the same probability by testing whether the particles are within the tube or not at fixed, equally spaced times and taking then the limit for vanishing spacing. In Appendix A we discuss a simple case where the semigroup equation (2.4) is explicitly solved.

When trying to identify trajectories in the quantum case, one must first of all take into account that localizations in position and momentum are subjected to the limitations coming from the Heisenberg principle. Moreover, it has to be stressed that, while in the classical case the process of ascertaining whether or not the members of the ensemble lie within a given phase-space region E_t influences the ensemble only by discarding some of its members but does not change the dynamics of the surviving members, in the quantum case the situation is quite different. In fact, any action aimed to ascertain which systems possess some definite properties, induces a drastic change in the statistical operator, so that also the subsequent evolution of the systems which have survived the test is completely different from the one they would have followed in the case of no test. Because of this essential role of the selection mechanism (which cannot be considered simply as a tool to get a formal equation for the identification of the systems whose trajectories lie within the tube), in the quantum case it becomes important to specify precisely the modalities of the measurement process. We will describe in general the changes induced on the statistical operator by this process through the map $\rho\to T[\rho]$. This map accounts for some type of measurement process on the system. It is then useful to make reference to the formalism which has been introduced to generalize the ordinary description of measurement processes. Such a generalized formalism[11] makes use of the ideas of effect-valued measures (EVM) and operation-valued measures (OVM). This framework turns out to be more satisfactory and richer than the standard one, allowing, for instance, the description of approximate measurements and of measurements performed by apparatuses which do not work with efficiency one. For the sake of completeness in Appendix B we have briefly sketched this generalized description of measurement processes.

We now describe briefly the interesting approach to the macroscopic dynamics of Barchielli, Lanz, and Prosperi.[6] These authors are interested in identifying a dynamical

evolution equation for a macroscopic system allowing one to define a functional probability distribution on an appropriate σ algebra of the subsets of the space of the continuous functions of t (trajectories). They deal with a macroscopic particle in one dimension and they are interested in studying the dynamics of such an object when it is subjected to appropriate, obviously approximate, position measurements. If one wants to introduce such processes, one either uses projection operators on definite space intervals (and introduces therefore an arbitrary discretization of space) or resorts to the more general formalism mentioned above, based on the concept of operation valued measures. The authors of Ref. 6 have considered the OVM (Ref. 12):

$$T_I[\rho]=\left[\frac{\alpha}{\pi}\right]^{1/2}\int_I dx\, e^{-(\alpha/2)(\hat{q}-x)^2}\rho\, e^{-(\alpha/2)(\hat{q}-x)^2}, \tag{2.7}$$

where I is a Borel set in $\mathbb{R}$ and $\hat{q}$ is the position operator. $T_I[\rho]$ describes (including its appropriate weight) the subensemble of those systems which in the measurement have been found in I. Accordingly, the probability $P(q\in I\,|\,\rho)$ that in the measurement the position is found within the Borel set I, when the state of the system is ρ, is

$$P(q\in I\,|\,\rho)=\mathrm{tr}T_I[\rho]\ . \tag{2.8}$$

If no selection is performed according to the results of the measurement, the ensemble is represented by the statistical operator

$$T[\rho]=T_{\mathbb{R}}[\rho]=\left[\frac{\alpha}{\pi}\right]^{1/2}\int_{-\infty}^{+\infty}dx\, e^{-(\alpha/2)(\hat{q}-x)^2}\rho\, e^{-(\alpha/2)(\hat{q}-x)^2}\ . \tag{2.9}$$

In the coordinate representation one has

$$\langle q'\,|\,T[\rho]\,|\,q''\rangle=e^{-(\alpha/4)(q'-q'')^2}\langle q'\,|\,\rho\,|\,q''\rangle\ . \tag{2.10}$$

Note that (2.10) is probably the simplest expression having the desired meaning, in spite of the fact that in order to introduce it one has to resort to the generalized formalism involving OVM. We shall make reference to (2.9) as the localization process.

In Ref. 6 the macroscopic system was considered as evolving by pure Hamiltonian dynamics and to be subjected to the localization process at definite, equally spaced instants. The discretization of the time axis was then eliminated by taking in a suitable way the infinite frequency limit. To be more specific, if the time interval between two measurements is denoted by $1/\lambda$, one takes $\lambda\to\infty$, $\alpha\lambda=\mathrm{const}$.

The infinite frequency limit raises some problems; in particular, it forbids the direct use of the process $T_I[\ ]$ to perform selections on the statistical ensemble in order to define the probability distributions on the space of trajectories. This point will become clear in what follows. The way used in Ref. 6 to overcome this difficulty consists in keeping the modified dynamics obtained under the limit as the basic dynamics of the system, and then adding a purely selective process, the selection being performed on the basis of the mean values of the results of the measurements occurring between two selections. In this way one is accepting particles which have been found outside the "tube" which we are considering, provided the mean value of the obtained results fall within the tube.

In the next section, we generalize the dynamical description of a macroscopic object given in Ref. 6.

III. DYNAMICS FOR MACROSCOPIC OBJECTS: EVOLUTION EQUATION AND DETAILED STUDY OF THE FREE MACROSCOPIC PARTICLE

As in Ref. 6 we deal with a macroscopic particle in one dimension. To eliminate the arbitrary discretization of time, we follow a different way: i.e., we assume that the localization process (2.9) occurs at random times.[13] In this way there is no need to take the infinite frequency limit in order to have a continuous evolution of the statistical ensemble and, as we shall see, the dynamical equation itself can be used to perform selections on the ensemble. If no selection on the basis of the result of the measurement is performed and the probability of occurrence of a localization process in the interval dt is $\lambda\,dt$, the evolution for the state of the system is

$$\frac{d}{dt}\rho(t)=-\frac{i}{\hbar}[\hat{H},\rho(t)]-\lambda(\rho(t)-T[\rho(t)])\ . \tag{3.1}$$

Since, owing to (2.10),

$$\langle q\,|\,T[\rho]\,|\,q\rangle=\langle q\,|\,\rho\,|\,q\rangle\ , \tag{3.2}$$

Eq. (3.1) is obviously trace preserving. Moreover, using Eq. (3.1), it can be easily proved that

$$\frac{d}{dt}(\mathrm{tr}\rho^2)<0\ . \tag{3.3}$$

This implies that under the dynamical evolution pure states are necessarily transformed into statistical mixtures.

Let us consider the non-Hamiltonian term $\lambda(\rho-T[\rho])$ in Eq. (3.1). According to (2.10), in the coordinate representation it becomes

$$\lambda\langle q'\,|\,(\rho-T[\rho])\,|\,q''\rangle=\lambda(1-e^{-(\alpha/4)(q'-q'')^2})\langle q'\,|\,\rho\,|\,q''\rangle\ . \tag{3.4}$$

If one takes the limit $\lambda\to\infty$, $\lambda\alpha=\mathrm{const}=\gamma$ the right-hand side of Eq. (3.4) becomes $(\gamma/4)(q'-q'')^2\langle q'\,|\,\rho\,|\,q''\rangle$ which is the matrix element in the coordinate representation of the operator $(\gamma/4)[\hat{q},[\hat{q},\rho]]$. Under the limit Eq. (3.1) becomes

$$\frac{d}{dt}\rho(t)=-\frac{i}{\hbar}[\hat{H},\rho(t)]-\frac{\gamma}{4}[\hat{q},[\hat{q},\rho(t)]]\ , \tag{3.5}$$

which is the basic equation considered in Ref. 6.

Let us now consider Eq. (3.1) in the case in which $\hat{H}$ is the Hamiltonian for a free particle. In the coordinate representation we get

$$\frac{\partial}{\partial t}\langle q' \mid \rho(t) \mid q'' \rangle = \frac{i\hbar}{2m}\left[\frac{\partial^2}{\partial q'^2} - \frac{\partial^2}{\partial q''^2}\right]\langle q' \mid \rho(t) \mid q'' \rangle - \lambda(1 - e^{-(\alpha/4)(q'-q'')^2})\langle q' \mid \rho(t) \mid q'' \rangle \ . \tag{3.6}$$

One can express the solution of the above equation satisfying given initial conditions in terms of the solution $\langle q' \mid \rho_s(t) \mid q'' \rangle$ of the pure Schrödinger equation ($\lambda = 0$) satisfying the same initial conditions, according to

$$\langle q' \mid \rho(t) \mid q'' \rangle = \frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty} dk \int_{-\infty}^{+\infty} dy\, e^{-(i/\hbar)ky} F(k, q'-q'', t)\langle q'+y \mid \rho_s(t) \mid q''+y \rangle \ , \tag{3.7}$$

where

$$F(k,q,t) = \exp\left[-\lambda t\left[1 - \frac{1}{t}\int_0^t d\tau\, e^{-(\alpha/4)(q-k\tau/m)^2}\right]\right] \ . \tag{3.8}$$

The Hermitian symmetry of $\rho(t)$ follows from the property $F(k,q,t) = F(-k,-q,t)$. In Appendix C we have shown how this solution can be obtained.

For $\lambda = 0$ one has $F(k,q,t) = 1$, implying $\langle q' \mid \rho(t) \mid q'' \rangle = \langle q' \mid \rho_s(t) \mid q'' \rangle$, as it must be. Let us list some properties of the function F which will be useful in what follows; one easily finds

$$F(0,0,t) = 1 \ , \tag{3.9a}$$

$$F_k(0,0,t) = 0, \quad F_q(0,0,t) = 0 \ , \tag{3.9b}$$

$$F_{kk}(0,0,t) = -\frac{\alpha\lambda}{6m^2}t^3, \quad F_{kq}(0,0,t) = \frac{\alpha\lambda}{4m}t^2, \quad F_{qq}(0,0,t) = -\frac{\alpha\lambda}{2}t \ , \tag{3.9c}$$

where the indices mean differentiation with respect to the indicated variables.

To understand the dynamical evolution described by Eq. (3.6) we evaluate now, by making use of the explicit solution given by Eqs. (3.7) and (3.8), the mean values and spreads of the position and momentum operators for all times. One has, using Eqs. (3.9),

$$\begin{aligned}\langle \hat{q} \rangle &= \mathrm{tr}[\hat{q}\rho(t)] \\ &= \frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty} dk \int_{-\infty}^{+\infty} dy \int_{-\infty}^{+\infty} dq\,(q-y) e^{-(i/\hbar)ky} F(k,0,t)\langle q \mid \rho_s(t) \mid q \rangle \\ &= \langle \hat{q} \rangle_s F(0,0,t) + i\hbar\,\mathrm{tr}[\rho_s(t)]F_k(0,0,t) = \langle \hat{q} \rangle_s \ .\end{aligned} \tag{3.10}$$

In deriving this result, as well as those which follow in this section, we have assumed appropriate regularity properties of $\langle q' \mid \rho_s(t) \mid q'' \rangle$. In Eq. (3.10) we have denoted by $\langle \hat{q} \rangle_s$ the mean value associated to the pure Schrödinger evolution. In a completely analogous way we get

$$\begin{aligned}\langle \hat{q}^2 \rangle &= \langle \hat{q}^2 \rangle_s F(0,0,t) + 2i\hbar\langle \hat{q} \rangle_s F_k(0,0,t) - \hbar^2\,\mathrm{tr}[\rho_s(t)]F_{kk}(0,0,t) \\ &= \langle \hat{q}^2 \rangle_s + \frac{\alpha\lambda\hbar^2}{6m^2}t^3 \ ,\end{aligned} \tag{3.11}$$

where we have used the fact that the pure Schrödinger evolution is trace preserving. With an analogous procedure one gets for the mean values of $\hat{p}$ and $\hat{p}^2$

$$\langle \hat{p} \rangle = \langle \hat{p} \rangle_s - i\hbar\,\mathrm{tr}[\rho_s(t)]F_q(0,0,t) = \langle \hat{p} \rangle_s \tag{3.12}$$

and

$$\langle \hat{p}^2 \rangle = \langle \hat{p}^2 \rangle_s - 2i\hbar\langle \hat{p} \rangle_s F_q(0,0,t) - \hbar^2\,\mathrm{tr}[\rho_s(t)]F_{qq}(0,0,t) = \langle \hat{p}^2 \rangle_s + \frac{\alpha\lambda\hbar^2}{2}t \ . \tag{3.13}$$

It is also useful to evaluate $\mathrm{tr}[\hat{q}\hat{p}\rho(t)]$. Using Eqs. (3.9) and proceeding as above we get

$$\begin{aligned}\mathrm{tr}[\hat{q}\hat{p}\rho(t)] &= \mathrm{tr}[\hat{q}\hat{p}\rho_s(t)]F(0,0,t) + i\hbar\langle \hat{p} \rangle_s F_k(0,0,t) - i\hbar\langle \hat{q} \rangle_s F_q(0,0,t) + \hbar^2\,\mathrm{tr}[\rho_s(t)]F_{kq}(0,0,t) \\ &= \mathrm{tr}[\hat{q}\hat{p}\rho_s(t)] + \frac{\alpha\lambda\hbar^2}{4m}t^2 \ .\end{aligned} \tag{3.14}$$

Summarizing, in the case of the free particle the evolution induced by Eq. (3.1) gives rise to mean values, spreads, and correlation for the position and momentum variables which are related to those of the pure Schrödinger evolution by

$$\langle \hat{q} \rangle = \langle \hat{q} \rangle_s \ , \tag{3.15a}$$

$$\langle \hat{p} \rangle = \langle \hat{p} \rangle_s \ , \tag{3.15b}$$

$$\{\hat{q}\} \equiv \langle (\hat{q} - \langle \hat{q} \rangle)^2 \rangle = \{\hat{q}\}_s + \frac{\alpha\lambda\hbar^2}{6m^2} t^3 \ , \tag{3.16a}$$

$$\{\hat{q}\hat{p}\} \equiv \langle [(\hat{q} - \langle \hat{q} \rangle)(\hat{p} - \langle \hat{p} \rangle)]_{\rm sym} \rangle = \{\hat{q}\hat{p}\}_s + \frac{\alpha\lambda\hbar^2}{4m} t^2 \ , \tag{3.16b}$$

$$\{\hat{p}\} \equiv \langle (\hat{p} - \langle \hat{p} \rangle)^2 \rangle = \{\hat{p}\}_s + \frac{\alpha\lambda\hbar^2}{2} t \ . \tag{3.16c}$$

In Eq. (3.16b) we have denoted by $[\ ]_{\rm sym}$ the Hermitian part of the quantity in square brackets. The shorthands $\{\hat{q}\}$, $\{\hat{q}\hat{p}\}$, $\{\hat{p}\}$ have been introduced to simplify the notation of the formal developments of the following sections. We note that the mean values are not affected by the non-Hamiltonian term in Eq. (3.1). For what concerns spreads and correlation, the corrections depend only on the combination $\alpha\lambda$ of the parameters α and λ.

IV. DISCUSSION OF THE NON-HAMILTONIAN DYNAMICS AND COMPARISON WITH THE CLASSICAL EVOLUTION

The dynamical equation we have discussed in the previous section has been introduced as referring to a macroscopic system. From this point of view, the first problem which is interesting to discuss is how this dynamics is related to the classical one. We remark that the standard quantum dynamics, in the case of a free particle, induces for the mean values $\langle \hat{q} \rangle_s$ and $\langle \hat{p} \rangle_s$ exactly the classical evolution. Moreover, for any reasonable choice of the initial spreads of the position $\Delta q = (\{\hat{q}\})^{1/2}$ and of the momentum $\Delta p = (\{\hat{p}\})^{1/2}$, the increase of Δq when time elapses, in virtue of the smallness of the Planck constant and of the large value of the mass for a macroscopic object, can be completely disregarded for all interesting times. However, as already discussed, the recognition of this fact does not exhaust the problem of the derivation of the classical behavior of a macroscopic object from quantum principles, since problems remain open when linear superpositions of macroscopically distinguishable states can occur. In such cases a satisfactory classical description would require that the statistical ensemble decomposes into a statistical mixture of macroscopically distinguishable states. Let us discuss the above points within the framework of the non-Hamiltonian dynamics of Sec. III.

First of all we can observe that Ehrenfest's theorem holds true also for the modified dynamics. In fact, for any dynamical variable X which is a function of the operator $\hat{q}$ only it is easily shown that

$$\mathrm{tr}\{X(\hat{q})T[\rho]\} = \mathrm{tr}[X(\hat{q})\rho] \ . \tag{4.1}$$

This in turn implies

$$\frac{d}{dt}\langle X(\hat{q}) \rangle = \mathrm{tr}\left[X(\hat{q}) \frac{d\rho}{dt} \right] = -\frac{i}{\hbar} \mathrm{tr}\{X(\hat{q})[\hat{H},\rho]\} \ , \tag{4.2}$$

as it happens for the Schrödinger evolution. There follows

$$\frac{d}{dt}\langle \hat{q} \rangle = \frac{1}{m}\langle \hat{p} \rangle \ . \tag{4.3}$$

For the operator $\hat{p}$ one finds

$$\mathrm{tr}(\hat{p}T[\rho]) = \mathrm{tr}(\hat{p}\rho) \ . \tag{4.4}$$

Then, if $\hat{H} = \hat{p}^2/2m + V(\hat{q})$, we have

$$\frac{d}{dt}\langle \hat{p} \rangle = -\left\langle \frac{\partial V}{\partial \hat{q}} \right\rangle \ . \tag{4.5}$$

In accordance with this property, Eqs. (3.15) show that in the case of a free particle the mean values of position and momentum are not affected by the non-Hamiltonian terms. On the contrary, in the expression for the spreads additional terms appear. These terms increase with time, so that one can identify a characteristic time interval T during which they remain smaller than those expressing the Schrödinger evolution. T is then of the order of the smaller of the two times T_1 and T_2 given by

$$T_1 = \left[\frac{6m^2(\Delta q_s)^2}{\alpha\lambda\hbar^2} \right]^{1/3} , \quad T_2 = \frac{2(\Delta p_s)^2}{\alpha\lambda\hbar^2} \ . \tag{4.6}$$

For the time interval T the spreads given by Eqs. (3.16) coincide practically with the Schrödinger values, which in turn are negligible for any reasonable choice of their initial values. We shall discuss below the values taken by T when the parameters of the model are appropriately chosen.

The fact that Δq^2 and Δp^2 are very close to the Schrödinger values for an appropriate time interval is strictly related to the small influence of the non-Hamiltonian term on the matrix elements of the statistical operator $\langle q' | \rho | q'' \rangle$ when $|q' - q''| \ll 1/\sqrt{\alpha}$. On the contrary, we shall see that the non-Hamiltonian dynamics has a drastic effect on the off-diagonal elements when $|q' - q''| \gtrsim 1/\sqrt{\alpha}$. This can be easily understood by observing that the properties of the function $F(k,q,t)$ are remarkably different in the two cases $q = 0$ and $q \neq 0$. In fact, when $q = 0$ the integral at the exponent in Eq. (3.8) for sufficiently small t behaves like t, yielding the cancellation of the factor $e^{-\lambda t}$ and making $F(k,0,t)$ very near to 1. Since $F = 1$ implies $\langle q' | \rho(t) | q'' \rangle = \langle q' | \rho_s(t) | q'' \rangle$, this shows that the almost diagonal matrix elements of the statistical operator in the coordinate representation are practically unaffected for an appropriate time interval by the non-Hamiltonian term in the evolution equation. On the contrary, for $q \neq 0$, the integral in Eq. (3.8) cannot cancel, even for small times, the damping factor $e^{-\lambda t}$, so that the off-diagonal elements are rapidly suppressed.

To make these statements more precise we derive two

inequalities for the function $F(k,q,t)$ for the two cases $q=0$ and $q>0$.

(a) $q=0$. Since

$$\frac{1}{t}\int_0^t d\tau\, e^{-\alpha k^2\tau^2/4m^2} \geq e^{-\alpha k^2 t^2/4m^2}, \tag{4.7}$$

it follows that

$$F(k,0,t) > \exp[-\lambda t(1-e^{-\alpha k^2 t^2/4m^2})] \geq 1-\frac{\alpha\lambda k^2 t^3}{4m^2}, \tag{4.8}$$

the last inequality being meaningful for $\alpha\lambda k^2t^2/4m^2<1$. We then have

$$1-F(k,0,t)\leq \alpha\lambda k^2 t^3/4m^2 . \tag{4.9}$$

(b) $q>0$. The function F can be written as

$$F(k,q,t)=\exp\left\{-\lambda t\left[1-h\left[(\sqrt{\alpha}/2)\frac{kt}{m},(\sqrt{\alpha}/2)q\right]\right]\right\}, \tag{4.10}$$

where

$$h(x,y)=\frac{1}{x}\int_{-y}^{x-y} dz\, e^{-z^2} . \tag{4.11}$$

The function $h(x,y)$ is the mean value of e^{-z^2} on the interval $(-y,x-y)$. Clearly one has

$$h(x,y)<h(y,y)=h(2y,y) \tag{4.12}$$

for $x<y$, and

$$h(x,y)<h(2y,y) \tag{4.13}$$

for $x>2y$. For $y<x<2y$ one finds

$$h(x,y)<\frac{1}{x}\int_{-y}^{y} dz\, e^{-z^2} <\frac{1}{y}\int_{-y}^{y} dz\, e^{-z^2} =2h(2y,y) , \tag{4.14}$$

so that, on the whole,

$$h(x,y)<2h(2y,y)=\frac{\sqrt{\pi}}{y}\,\mathrm{erf}(y) . \tag{4.15}$$

In turn the function F obeys the inequality

$$F(k,q,t)<e^{-\lambda\beta t} , \tag{4.16}$$

where

$$\beta=1-\frac{\sqrt{\pi}}{(\sqrt{\alpha}/2)q}\mathrm{erf}[(\sqrt{\alpha}/2)q] . \tag{4.17}$$

Coming back to the discussion of the diagonal elements of the statistical operator, we get from Eq. (3.7)

$$\langle q\,|\,\rho_s(t)\,|\,q\rangle-\langle q\,|\,\rho(t)\,|\,q\rangle=\frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty} dk[1-F(k,0,t)]\int_{-\infty}^{+\infty} dy\, e^{-(i/\hbar)ky}\langle q+y\,|\,\rho_s(t)\,|\,q+y\rangle . \tag{4.18}$$

To illustrate the implications of this equation, we discuss a simple example. Suppose that $\langle q\,|\,\rho_s(t)\,|\,q\rangle$ is a mixture of Gaussian terms whose spreads are Δ_i with minimum Δ_0. Then the Fourier transform appearing in Eq. (4.18) yields terms containing Gaussian factors $e^{-\Delta_i^2k^2/2\hbar^2}$, whose maximum width is $\hbar/\Delta_0$, so that the integral in k is concentrated in a region $|k|<\hbar/\Delta_0$. Inequality (4.9) shows then that the integrand in Eq. (4.18) contains a factor smaller than $(\alpha\lambda\hbar^2/4m\Delta_0^2)t^3$. The condition $t\ll T_1$, the time T_1 being given by Eq. (4.6), implies

$$\langle q\,|\,\rho(t)\,|\,q\rangle\simeq\langle q\,|\,\rho_s(t)\,|\,q\rangle .$$

Obviously this result holds for those matrix elements which are appreciably different from zero.

For the off-diagonal elements we consider the case $q'>q''$. Obviously the same results are valid for $q'<q''$ due to the Hermitian symmetry of $\rho(t)$. Inequality (4.16) gives, for $q'-q''>2\sqrt{\pi/\alpha}$, a significant bound on F independent of k. This shows that the expression (3.7) for $\langle q'\,|\,\rho(t)\,|\,q''\rangle$ contains an exponentially damped factor whose lifetime is $\tau=1/\lambda\beta$, so that in a time interval of the order of τ the linear superpositions of states separated by distances larger than the characteristic localization distance $1/\sqrt{\alpha}$ are transformed into statistical mixtures.[14]

As we shall see, one can choose the parameters λ and α in such a way that the time $T=\min(T_1,T_2)$ is very large and τ extremely small, so that we can conclude that the modified dynamics agrees with the classical description for a macroscopic object and overcomes the problems arising from linear superpositions of states localized in far apart regions.

V. PARTICLE TRAJECTORIES

As extensively discussed in the Introduction, one of the most attractive results which should follow from the suppression of the superposition of far-away localized states would be the possibility of accounting for the dynamics of macroscopic systems in terms of trajectories without encountering the difficulties one has to face when dealing with the same problem within the standard theory. In Sec. II we have pointed out that within the framework of Ref. 6 one cannot use directly the localization process $T[\ ]$ to perform selections on the statistical ensemble in order to define the probability distribution for appropriate sets of trajectories. The reason for this impossibility arises from the fact that, as $\alpha\to 0$, $\mathrm{tr}(T_I[\rho])\to 0$ for any finite I even when the support of $\langle q\,|\,\rho\,|\,q\rangle$ is entirely contained in I.

Within our scheme, since we keep λ finite, we can identify trajectories by using the selective form of Eq. (3.1):

$$\frac{d}{dt}\rho(t)=-\frac{i}{\hbar}[\hat{H},\rho(t)]-\lambda\rho(t)+\lambda(\alpha/\pi)^{1/2}\int_{I(t)}dx\, e^{-(\alpha/2)(\hat{q}-x)^2}\rho(t)e^{-(\alpha/2)(\hat{q}-x)^2}\,, \tag{5.1}$$

where $I(t)$, for any fixed t, is an interval in $\mathbb{R}$. The fraction of the members of the ensemble which up to time t have always been found within $I(\bar{t})$, $0\le\bar{t}\le t$, is the probability associated with the set of trajectories $I(\bar{t})$; in terms of the solution of Eq. (5.1) it is given by $\mathrm{tr}[\rho(t)]$. Since the localization process, as discussed in the previous section, decomposes the statistical ensemble in subensembles corresponding to systems localized on distances of the order $1/\sqrt{\alpha}$, Eq. (5.1) selects in a natural way those subensembles whose members lie within $I(t)$. Obviously, the use of Eq. (5.1) to define trajectories is meaningful only if the amplitude of the interval $I(t)$ is always significantly larger than the localization distance $1/\sqrt{\alpha}$. Moreover, in the time interval in which we are interested, many tests must occur, otherwise the trajectory would not be identified with meaningful accuracy.

Equation (5.1) induces a loss of particles. One can derive from Eq. (5.1) an upper bound for the fraction of particles which survives the localization tests. In fact, taking the trace of Eq. (5.1) we get

$$\frac{d[\mathrm{tr}\rho(t)]}{dt}=-\lambda\,\mathrm{tr}\rho(t)+\lambda\int_{-\infty}^{+\infty}dq\,\rho(q,q,t)\sqrt{\alpha/\pi}\int_{I(t)}dx\, e^{-\alpha(q-x)^2}\,. \tag{5.2}$$

If we call M the maximum extent of the interval $I(t)$ and we observe that

$$\sqrt{\alpha/\pi}\int_{I(t)}dx\, e^{-\alpha(q-x)^2}\le\sqrt{\alpha/\pi}\int_{q-M/2}^{q+M/2}dx\, e^{-\alpha(q-x)^2}=\mathrm{erf}(M\sqrt{\alpha}/2)\,, \tag{5.3}$$

we obtain from Eq. (5.2)

$$\frac{d[\mathrm{tr}\rho(t)]}{dt}\le-\lambda\Gamma\,\mathrm{tr}\rho(t)\,,\qquad \Gamma=1-\mathrm{erf}(M\sqrt{\alpha}/2)\,. \tag{5.4}$$

Taking into account that $\mathrm{tr}\rho(0)=1$ we then get

$$\mathrm{tr}\rho(t)\le e^{-\lambda\Gamma t}\,. \tag{5.5}$$

Since Γ remains finite for $\alpha\to0$, one gets that in the limit $\lambda\to\infty$ (even when $\alpha\lambda$ is kept constant) $\mathrm{Tr}\rho(t)\to0$ for any t, whatever family of intervals $I(t)$ has been chosen, provided $I(t)$ is not the whole real line, which amounts to no selection. This explains why in Ref. 6 one cannot use the process itself to select sets of trajectories. Also in our case, for $t\to\infty$, $\mathrm{tr}\rho(t)\to0$ for any choice of $I(t)$. This gives rise to a disagreement with the classical case, in which the probability of having the particles in a given tube remains constant if the tube is made of possible physical trajectories. The reason for this is that $\mathrm{tr}[T_I(\rho)]<\mathrm{tr}(\rho)$ for any finite I, even when the support of $\langle q\,|\,\rho\,|\,q\rangle$ is entirely contained in I. However, as already remarked, Eq. (5.1) is physically meaningful only if $M\gg1/\sqrt{\alpha}$. This condition shows that the damping factor is ineffective up to times extremely long with respect to $1/\lambda$. For example, if M is chosen in such a way that $M\sqrt{\alpha}\simeq15$, so that $\Gamma\simeq10^{-26}$, the damping factor becomes effective for times of the order of $10^{26}/\lambda$.

VI. MACROSCOPIC DYNAMICS FROM MICROSCOPIC DYNAMICS

Up to now we have introduced a non-purely-Hamiltonian dynamics to describe the motion of a macroscopic particle and we have outlined how this modification can be used to overcome some of the difficulties in the description of such objects. We will make our considerations quantitatively more precise in the following sections. However, macroscopic objects are composite systems and the standard quantum mechanics gives definite prescriptions for their description. It is an important feature of quantum mechanics that, under suitable conditions, the internal and the center-of-mass motions of the composite systems decouple and, moreover, that the equation of motion for the center of mass is formally identical to the equation prescribed by the theory for the description of a single particle. Here we want to investigate whether it is possible to obtain the non-purely-Hamiltonian dynamics for macroscopic particles described in the previous sections from a modification of the standard quantum dynamics for their microscopic constituents.[16] If such a modification leaves practically unaltered the behavior of microscopic systems as accounted for by quantum mechanics we can say we have laid the foundations of a possible unified description able to account for both the quantum and the classical behaviors of microscopic and macroscopic systems, respectively.

In this spirit let us tentatively assume that the localization process $T[\ ]$ occurs individually for each constituent of a many-particle system. We consider a system of N particles in one dimension. Denoting by λ_i the frequency of the process suffered by constituent i and assuming that the accuracy of the localizations is the same for all constituents, the evolution equation for the composite system is

$$\frac{d}{dt}\rho=-\frac{i}{\hbar}[\hat{H},\rho]-\sum_{i=1}^{N}\lambda_i(\rho-T_i[\rho])\,, \tag{6.1}$$

where

$$T_i[\rho]=\sqrt{\alpha/\pi}\int_{-\infty}^{+\infty}dx\, e^{-(\alpha/2)(\hat{q}_i-x)^2}\rho\, e^{-(\alpha/2)(\hat{q}_i-x)^2}\,, \tag{6.2}$$

$\hat{q}_i$ being the position operator for the ith particle of the system.

It is worthwhile to illustrate the physical consequences of the above equation for the important conceptual problem of the possible occurrence of linear superpositions of states corresponding to different localizations of a macroscopic object. Such a situation occurs, for instance, in the

quantum theory of measurement, in connection with possible macroscopically different pointer positions. With reference to such a case we consider the linear superposition $\psi=\psi_1+\psi_2$ of two states corresponding to two different pointer positions. We remark that in the case under discussion there is a macroscopic number $\mathcal{N}$ of particles which are located in macroscopically different positions when the state is ψ_1 or ψ_2 (to be precise, in our model this means located at a distance larger than $1/\sqrt{\alpha}$). If a spontaneous localization process takes place for one of such particles, this particle is found to be either in the spatial region which it occupies when the state is ψ_1, or in the one corresponding to ψ_2. The linear superposition is consequently transformed into a statistical mixture of states ψ_1 and ψ_2. Since the number of differently located particles is $\mathcal{N}$, this means that the phase decorrelation of states ψ_1 and ψ_2 occurs with a frequency which is amplified by a factor $\mathcal{N}$ with respect to the characteristic frequency λ_i of the elementary spontaneous localizations.

The model yields therefore a natural solution to the puzzling situation originating from the occurrence of linear superpositions of differently located states. These considerations, however, do not exhaust the problems to be discussed. In fact, we must still check that the modification of the dynamics for the microscopic constituents does not imply physically unacceptable consequences for the dynamics of the system as a whole. Actually, according to the previous discussions, we would like to have for the macroscopic object a dynamical equation of the type considered in Sec. III. To discuss this point, let us introduce the center of mass and relative motion position operators $\hat{Q}$ and $\hat{r}_j$ $(j=1,2,\ldots,N-1)$, related to the operators $\hat{q}_i$ by

$$\hat{q}_i=\hat{Q}+\sum_{j=1}^{N-1} c_{ij}\hat{r}_j \ . \tag{6.3}$$

Equation (6.1), when the Hamiltonian $\hat{H}$ can be split into the sum of the center-of-mass and internal motion parts $\hat{H}_Q$ and $\hat{H}_r$ acting in the respective state spaces, reads

$$\frac{d}{dt}\rho=-\frac{i}{\hbar}[\hat{H}_Q,\rho]-\frac{i}{\hbar}[\hat{H}_r,\rho]-\sum_i \lambda_i(\rho-T_i[\rho]) \ , \tag{6.4}$$

where the operation $T_i[\rho]$ can now be written as

$$T_i[\rho]=\sqrt{\alpha/\pi}\int_{-\infty}^{+\infty} dx \exp\left[-\frac{\alpha}{2}\left[\hat{Q}+\sum_j c_{ij}\hat{r}_j-x\right]^2\right]\rho\exp\left[-\frac{\alpha}{2}\left[\hat{Q}+\sum_j c_{ij}\hat{r}_j-x\right]^2\right] . \tag{6.5}$$

The dynamical evolution of the center of mass of the system is described by the statistical operator

$$\rho_Q=\mathrm{tr}^{(r)}(\rho) \ , \tag{6.6}$$

obtained by taking the partial trace on the internal degrees of freedom of the statistical operator ρ for the complete N-particle system. Taking the r trace of the operation $T_i[\rho]$ one gets

$$\int dr_1\cdots dr_{N-1}\sqrt{\alpha/\pi}\int_{-\infty}^{+\infty} dx \exp\left[-\frac{\alpha}{2}\left[\hat{Q}+\sum_j c_{ij}r_j-x\right]^2\right]\langle r_1\cdots r_{N-1}|\rho|r_1\cdots r_{N-1}\rangle \times\exp\left[-\frac{\alpha}{2}\left[\hat{Q}+\sum_j c_{ij}r_j-x\right]^2\right] , \tag{6.7}$$

so that, by shifting the integration variable x by the amount $\sum_j c_{ij}r_j$, one finds

$$\mathrm{tr}^{(r)}(T_i[\rho])=T_Q[\mathrm{tr}^{(r)}(\rho)] \ , \tag{6.8}$$

where

$$T_Q[\ ]=\sqrt{\alpha/\pi}\int_{-\infty}^{+\infty} dx\, e^{-(\alpha/2)(\hat{Q}-x)^2}e^{-(\alpha/2)(\hat{Q}-x)^2} \tag{6.9}$$

Taking the r trace of Eq. (6.4) one then gets

$$\frac{d}{dt}\rho_Q=-\frac{i}{\hbar}[\hat{H}_Q,\rho_Q]-\sum_i \lambda_i(\rho_Q-T_Q[\rho_Q]) \ . \tag{6.10}$$

The equation describing the reduced dynamics of the center of mass has exactly the same form of Eq. (3.1), the parameter λ being substituted by the sum of the λ_i's for the individual constituents of the many-body system. This is a direct consequence of the formal property (6.8).

It is worthwhile stressing that the non-Hamiltonian term in Eq. (6.10) is directly generated by the analogous terms of Eq. (6.1) and is not due to the elimination of the internal degrees of freedom. In fact, if one starts with a composite system with Hamiltonian dynamics for which $\hat{H}=\hat{H}_Q+\hat{H}_r$, the reduced dynamics for the center-of-mass motion is necessarily Hamiltonian, and therefore allows the occurrence of linear superpositions of far-away states of the center of mass. To avoid this, one could couple the system to some other system whose dynamics is then eliminated.[19] This, however, gives rise to a chain procedure when larger and larger external parts are included. If one wants to reach a point where linear superpositions of far-away states cannot occur, one has to break this chain in an arbitrary way. In our approach the non-Hamiltonian dynamics for a macroscopic object is induced by a basic non-Hamiltonian dynamics for its micro-

scopic constituents.

From a physical point of view it is particularly simple and interesting the case in which the internal motion Hamiltonian gives rise to a sharp (with respect to $1/\sqrt{\alpha}$) localization of the internal coordinates, as it happens, for an appropriate choice of α, in an insulating solid. In such a case it is evident that localizing with an accuracy $1/\sqrt{\alpha}$ any one of the points of the almost rigid structure of the solid induces a corresponding localization of the center of mass. In this situation something more can be proved, i.e., that the internal and the center-of-mass motion decouple almost exactly and the internal motion is not affected by the non-Hamiltonian terms in Eq. (6.1). To be precise, we assume that the matrix elements $\langle Q',r' \mid \rho \mid Q'',r'' \rangle$ are non-negligible only when the conditions

$$\left| \sum_j c_{ij} r_j' - a_i \right| \ll \frac{1}{\sqrt{\alpha}} , \quad \left| \sum_j c_{ij} r_j'' - a_i \right| \ll \frac{1}{\sqrt{\alpha}} , \quad i = 1, \ldots, N \tag{6.11}$$

are satisfied, a_i being the equilibrium position of constituent i relative to the center of mass. Since conditions (6.11) imply

$$\left| \sum_j c_{ij} (r_j' - r_j'') \right| \ll \frac{1}{\sqrt{\alpha}} , \quad i = 1, \ldots, N , \tag{6.12}$$

$\langle Q',r' \mid \rho \mid Q'',r'' \rangle$ is negligibly small unless condition (6.12) is satisfied. From the definition (6.2) one gets

$$\begin{aligned} &\langle Q',r' \mid T_i[\rho] \mid Q'',r'' \rangle \\ &= \sqrt{\alpha/\pi} \int_{-\infty}^{+\infty} dx \exp\left[-\frac{\alpha}{2} \left[Q' + \sum_j c_{ij} r_j' - x \right]^2 \right] \langle Q',r' \mid \rho \mid Q'',r'' \rangle \exp\left[-\frac{\alpha}{2} \left[Q'' + \sum_j c_{ij} r_j'' - x \right]^2 \right] \\ &= \exp\left[-\frac{\alpha}{4} \left[Q' - Q'' + \sum_j c_{ij} (r_j' - r_j'') \right]^2 \right] \langle Q',r' \mid \rho \mid Q'',r'' \rangle . \end{aligned} \tag{6.13}$$

The exponential factor appearing in the last line of Eq. (6.13) is a Gaussian in the variable $Q'-Q''$ displaced by the amount $\sum_j c_{ij}(r_j''-r_j')$. Because of Eq. (6.12) the displacement of the Gaussian can be neglected with respect to its width, so that in this approximation

$$T_i[\rho] = T_Q[\rho] . \tag{6.14}$$

The physical meaning of Eq. (6.14) is that, as foreseen, a localization of a single constituent of a rigid system is equivalent to a localization of the center of mass. Equation (6.4) shows that, if the initial statistical operator has the form of a direct product $\rho_Q \rho_r$, it remains of the same type, and the statistical operators ρ_r and ρ_Q obey the equations

$$\frac{d}{dt}\rho_r = -\frac{i}{\hbar}[\hat{H}_r, \rho_r] \tag{6.15}$$

and (6.10), respectively. We conclude that in the considered case the internal and the center-of-mass motions decouple, the internal motion of the solid remaining unaffected by the localization process introduced in Eq. (6.1) and the center-of-mass motion being affected by such a process with a characteristic frequency equal to the sum of the frequencies for all single constituents.

A comment on the significance of the reduced statistical operator ρ_Q is appropriate. When the state of a composite system is such that there are correlations between the center-of-mass and the internal states, it may happen that the system has a long-distance coherence and that nevertheless no trace of this coherence remains in $\rho_Q = \mathrm{tr}^r \rho$. As an example one can consider the case in which the whole system is associated to the state $\mid \psi \rangle = \mid \psi_1 \rangle + \mid \psi_2 \rangle$ and the states $\mid \psi_i \rangle$ $(i=1,2)$ are of the factorized form $\psi_i(Q,r_j) = \phi_i(Q) \cdot \varphi_i(r_j)$ between the center-of-mass and relative coordinates, respectively. Suppose also that the two states $\phi_1(Q)$ and $\phi_2(Q)$ are localized in far-away spatial regions. In such a case, if $\langle \varphi_1 \mid \varphi_2 \rangle = 0$ the reduced statistical operator does not contain the terms $\mid \phi_1 \rangle \langle \phi_2 \mid$ and $\mid \phi_2 \rangle \langle \phi_1 \mid$ even though in $\mid \psi \rangle$ there is long-distance coherence. On the other hand, if there are no correlations between the center-of-mass and internal degrees of freedom, ρ_Q describes appropriately the long-distance coherence, when present. As an example one can think of the state $\mid \psi \rangle = (\mid \phi_1 \rangle + \mid \phi_2 \rangle) \otimes \mid \varphi \rangle$, where the states $\mid \phi_i \rangle$ are those previously mentioned and $\mid \varphi \rangle$ is any internal state. In this case ρ_Q contains the terms $\mid \phi_1 \rangle \langle \phi_2 \mid$ and $\mid \phi_2 \rangle \langle \phi_1 \mid$ reflecting the long-distance coherence which is present.

We have shown that Eq. (6.10) holds completely in general. In the general case, however, the correlations between the center-of-mass and internal degrees of freedom are not excluded, so that the suppression of the off-diagonal elements of ρ_Q implied by Eq. (6.10) is not sufficient by itself to ensure the nonoccurrence of coherent superpositions of far-away states. In any case our Eq. (6.1) for the complete statistical operator ρ guarantees that the phase decorrelation takes place with frequency $\mathcal{N}\lambda$, as has been discussed after Eq. (6.2).

The considerations which can be done in the case of an almost rigid body ensure that the density operator keeps the form $\rho = \rho_Q \rho_r$ when it is initially of this form. In this case, therefore, Eq. (6.10), entailing the suppression of the off-diagonal elements of ρ_Q, expresses meaningfully the destruction of the long-distance coherence. The situation we have discussed now can be considered, with some

idealization, typical of the case in which one is dealing with a macroscopic body.

To conclude this section we observe that, if one assumes for simplicity that the localization frequencies λ_i of all microscopic (e.g., atomic) constituents of a macroscopic body are of the same magnitude ($\lambda_i = \lambda_{\text{micro}}$), the center of mass is affected by the same process with a frequency $\lambda_{\text{macro}} = \mathcal{N}\lambda_{\text{micro}}$, where $\mathcal{N}$ is of the order of Avogadro's number. As we shall see, this will allow us to choose the parameters λ_{micro} and α in such a way that standard quantum mechanics holds exactly for extremely long times for microscopic systems, while for a macroscopic body possible linear superpositions of far-away states are rapidly suppressed, the dynamical evolution of the center-of-mass position is the classical one and the internal structure remains unaffected.

VII. POSSIBLE NUMERICAL CHOICE OF THE PARAMETERS OF THE MODEL AND ITS CONSEQUENCES

A crucial feature of the point of view which has been adopted in Sec. VI, i.e., that of considering all constituents of any system as subjected to localizations, consists in the fact that one can choose the parameters of the elementary processes in such a way that (i) the quantum-mechanical predictions for microscopic systems are valid for extremely long times, (ii) the dynamics of a macroscopic object, when it is consistently derived from that of its microscopic constituents, turns out to coincide with the classical one for a sufficiently long time interval, (iii) the suppression of long-distance coherence for macroscopic objects be effective enough to imply that, after a microscopic system has triggered a measuring apparatus, the dynamical evolution leads to the reduction of the wave packet with well-defined pointer positions.

To give orientative indications on the numerical values of the parameters appearing in our model, we start by remarking that, as it is clear from the formulas of Secs. III and IV, all physically significant effects of the modified dynamics for a macroscopic object are governed by the product $\alpha\lambda_{\text{macro}}$. For the choice of the parameter λ_{macro} we have some important criteria which must be followed. First of all we want the mean time $1/\lambda_{\text{macro}}$ elapsing between two successive localizations to be such that the transition to statistical mixtures for states spreading over distances larger than the localization distance $1/\sqrt{\alpha}$ takes place in a very small fraction of a second. A further requirement which has to be taken into account is that, when one is trying to identify particle trajectories for a macroscopic system using the selective form of our equation, the disagreement with the classical predictions which, as shown in Sec. V, unavoidably arises for large times, be unimportant for times which are long with respect to those during which one can keep the macroscopic system isolated. Finally, and more important, we want that the modification of the dynamics for microscopic systems with respect to the standard one be totally irrelevant. The simplest way to obtain this is to assume that the mean frequency $\lambda_{\text{micro}} = \lambda_{\text{macro}}/\mathcal{N}$ of the spontaneous localization processes for a microscopic system be extremely small.

For what concerns the parameter α it is appropriate to choose the localization distance $1/\sqrt{\alpha}$ large with respect to the atomic dimensions and to the mean spreads around the equilibrium positions of the lattice points of a crystal. In this way, even when one of the extremely infrequent localization processes takes place for a constituent of an atomic system, the localization itself does not modify the internal structure of that system and the decoupling of the center of mass and relative motions discussed in Sec. VI still holds. On the other hand $1/\sqrt{\alpha}$ represents the distance after which a linear superposition is transformed into a statistical mixture. This parameter must then be chosen in accordance with the requirement of avoiding the embarassing occurrence of linear superpositions of appreciably different locations of a macroscopic object.

These considerations lead us to discuss the following choice for the order of magnitude of the parameters. For the localization frequency of microscopic systems we choose

$$\lambda_{\text{micro}} \simeq 10^{-16}\ \text{sec}^{-1} .$$

This means that such a system is localized once every 10^8–10^9 yr. For the parameter α we choose

$$1/\sqrt{\alpha} \simeq 10^{-5}\ \text{cm} .$$

The fact that a microscopic system is practically never localized, entails that standard quantum mechanics remains fully valid for this type of systems.

Moreover, for a composite system for which the relative coordinates are confined within a spatial range much smaller than the localization distance $1/\sqrt{\alpha}$, as it happens for atoms and molecules, the process $T[\ ]$ is almost uneffective even when it takes place, a fact that strengthens the above conclusion.

For what concerns macroscopic objects (containing a number of constituents of the order of Avogadro's number), according to the considerations of Sec. VI showing that the individual tests on the constituents add for the center-of-mass dynamics, we get as characteristic test frequency

$$\lambda_{\text{macro}} \simeq 10^{7}\ \text{sec}^{-1} .$$

If we take, for the sake of definiteness, the mass of such an object to be of the order of 1 g, and the initial spread of the position Δq_0 again of the order of 10^{-5} cm, we know that the quantum increase of the spread in the position is negligible for extremely long times ($\sim 10^{10}$ yr), so that the quantum evolution is practically the same as the classical one. In such a case [compare Eqs. (4.6)], the additional term appearing in Δq^2 equals $\Delta q_0{}^2$ at the time T_1, which is of the order of 100 yr. This is a very long time for keeping isolated a macroscopic object. A much longer time T_2 is required in order that the additional term in Δp^2 has an appreciable effect for any reasonably chosen initial spread of the momentum. As far as the occurrence of linear superpositions of far away states is concerned, as we have seen, the off-diagonal elements of the statistical operator are exponentially suppressed with the lifetime $\tau = 1/\lambda\beta$. For $|q - q'| = 4\times 10^{-5}$ cm we have $\tau = 10^{-6}$ sec [see Eq. (4.17)]. Therefore after times of this order

linear superpositions of states separated by distances larger than 10^{-4} cm are transformed into statistical mixtures. For what concerns the identification of particle trajectories through the selective form of our equation, we remark that, as already stressed in Sec. V, if the cross section of the tube defining the trajectory is kept of the order indicated there ($M\sqrt{\alpha}\simeq 15$, which means $M > 10^{-4}$ cm), the damping factor becomes effective after 10^{11} yr. At this time all members of the ensemble have left any preassigned tube of the chosen size. The previous considerations, showing that in about 100 yr we have a doubling of the initial spread, means that for times larger than this value a disagreement with classical predictions about trajectories starts to appear. However, we stress that the time which one has to wait in order that Δq becomes of the order of 1 mm, which is a more realistic value for a macroscopic object of the everyday life, is 10^5 yr. It is also worthwhile to observe that, assuming, in our spirit, that λ is roughly proportional to the mass, T_1 behaves like $m^{1/3}$.

We note that all effects originating from the presence of the non-Hamiltonian term in the evolution equation depend essentially on the product $\alpha\lambda$. Therefore there is a large freedom in the choice of the separate values of these two parameters. However, choosing a value of $1/\sqrt{\alpha}$ smaller than the atomic dimensions would make it impossible to get the decoupling of the internal and center-of-mass motion for a solid body, discussed in Sec. VI. On the other hand $1/\sqrt{\alpha}$ cannot be too large in order that trajectories can be identified according to the procedure sketched in Sec. V.

Considerations of this type are important for the quantum theory of measurement. In fact, at least in the case in which the interaction leading to the triggering of the apparatus takes place in a very short time, we can apply our treatment to the macroscopic parts of the apparatus itself, obtaining in this way a consistent solution of the difficulties related to the quantum theory of measurement for what concerns the wave-packet reduction and the definite final position of the pointer.

It has to be remarked that the basic evolution equation (3.1), due to the appearance of the non-Hamiltonian terms, implies a nonconservation of energy. Let us give an estimate of this effect in the case of the free particle on the basis of the choice for the parameters we have just made. From Eqs. (3.13) we see that, in our case

$$\langle E\rangle=\langle E\rangle_s+\frac{\hbar^2\lambda\alpha}{4m}t\ , \tag{7.1}$$

where $\langle E\rangle_s$ is the conserved energy for free Schrödinger evolution.[20] Energy nonconservation is then expressed by the term

$$\delta E=\frac{\hbar^2\lambda\alpha}{4m}t\ . \tag{7.2}$$

Let us evaluate this term for the case of a microscopic system. Since $\lambda_{\text{micro}}=10^{-16}\ \text{sec}^{-1}$, $m\simeq 10^{-23}$ g,

$$\delta E/t\simeq 10^{-25}\ \text{eV sec}^{-1}\ ,$$

which means that to have an increase of 1 eV it takes a time of 10^{18} yr. In the case of the center-of-mass equation for a macroscopic system, since both the frequency λ and the mass increase proportionally to the number of constituents, the energy nonconservation is of the same amount. However this argument applies only to the increase of the energy of the center of mass. There is also an increase of energy in the internal motion which, as can be easily understood considering a system of free particles, is the same for all constituents. When this fact is taken into account one can conclude that the estimated energy increase for a system of $\mathcal{N}$ atoms is

$$\delta E/t\simeq 10^{-14}\ \text{erg sec}^{-1}\ .$$

Referring to an ideal monoatomic gas the increase in temperature with time is then of the order of 10^{-15} K per year.

We conclude that our model reproduces in a consistent way quantum mechanics for microscopic objects and classical mechanics for macroscopic objects, and provides the basis for a conceptually appealing description of quantum measurement. As we shall see below the model exhibits another important feature: it allows a description of the evolution of macroscopic objects in terms of a classical phase-space density obeying a Markov diffusion process.

VIII. PHASE-SPACE DENSITY

In the foregoing sections we have seen that, at least in the free macroscopic particle case and for suitable choices of the parameters, $\langle q'|\rho(t)|q''\rangle$ undergoes a rapid suppression of the elements not nearly diagonal and $\langle q'|\rho(t)|q'\rangle$ moves in space obeying essentially the classical laws. In this sense we have already obtained for macroscopic particles the classical mechanics from the modified quantum dynamics. However, the comparison between the two descriptions will be more satisfactory from the formal point of view and more precise from the quantitative point of view if, starting from the modified quantum dynamics, through suitable approximations, we are able to introduce a phase-space density and write an evolution equation for it.

To this purpose we again limit ourselves to the free particle case and we consider the normalized state vectors $|\Gamma(A,B)\rangle, A,B\in\mathbb{C}$, $\text{Re}A>0$, whose coordinate representation has the Gaussian form

$$\langle q'|\Gamma(A,B)\rangle=\frac{1}{\mathcal{N}}e^{-(Aq'^2+2Bq')/2}\ . \tag{8.1}$$

Given a wave function of the type (8.1) we indicate by q,p the corresponding values of the means $\langle\hat{q}\rangle_\Gamma,\langle\hat{p}\rangle_\Gamma$ and by Q,P the values of the square spreads $\{q\}_\Gamma,\{p\}_\Gamma$. It is also useful to introduce the symbol R for the q-p correlation $\{qp\}_\Gamma$ defined as in Eq. (3.16b). For a wave function of the type (8.1), R is not an independent quantity but is related to Q and P by the relation

$$R=(QP-\hbar^2/4)^{1/2}\ . \tag{8.2}$$

Obviously, the quantities q,p,Q,P are functions of A and B. They are given in Appendix D. Conversely, A and B are determined by q,p,Q,P provided the additional condition $\text{Im}A\le 0$ is imposed. This choice corresponds to a wave packet of minimum spread ($\text{Im}A=0$) or to a wave

packet whose spread increases by Schrödinger evolution ($\mathrm{Im}\,A<0$). The other choice would correspond to a wave packet which initially contracts, goes through the minimum-spread packet, and then starts to enlarge. The reasons for our choice will become apparent later. A and B as functions of q,p,Q,P are also given in Appendix D.

A wave function of the form (8.1) is transformed into a wave function of the same type by the pure Schrödinger evolution. The values of q,p,Q,P at the time τ are given in terms of the values of the same quantities at the time 0 by

$$q(\tau)=q(0)+p(0)\tau/m\ , \tag{8.3a}$$

$$p(\tau)=p(0)\ , \tag{8.3b}$$

$$Q(\tau)=Q(0)+2R(0)\tau/m+P(0)\tau^2/m^2\ , \tag{8.4a}$$

$$P(\tau)=P(0)\ . \tag{8.4b}$$

One also finds

$$R(\tau)=R(0)+P(0)\tau/m\ . \tag{8.5}$$

When the localization process described by Eq. (2.9) is applied to the pure state (8.1), this is transformed into a mixture of states of the same type. Putting

$$\rho_\Gamma(q,p,Q,P)=|\,\Gamma(q,p,Q,P)\rangle\langle\,\Gamma(q,p,Q,P)\,|\ , \tag{8.6}$$

it is easily shown (see Appendix D) that

$$T[\rho_\Gamma(q_i,p_i,Q_i,P_i)]=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty}d\xi\,e^{-\xi^2}\rho_\Gamma(q_f,p_f,Q_f,P_f)\ , \tag{8.7}$$

where

$$q_f=q_i+\frac{2\alpha Q_i}{[\alpha(1+2\alpha Q_i)]^{1/2}}\xi\ , \tag{8.8a}$$

$$p_f=p_i+\frac{2\alpha R_i}{[\alpha(1+2\alpha Q_i)]^{1/2}}\xi\ , \tag{8.8b}$$

$$Q_f=\frac{Q_i}{1+2\alpha Q_i}\ , \tag{8.9a}$$

$$P_f=\frac{P_i+\hbar^2\alpha(1+\alpha Q_i)}{1+2\alpha Q_i}\ . \tag{8.9b}$$

One also finds

$$R_f=\frac{R_i}{1+2\alpha Q_i}\ . \tag{8.10}$$

On the whole, the process described by the free-particle version of Eq. (3.1) transforms any mixture of states ρ_Γ into a mixture of the same type. We note that Eqs. (8.4) and (8.9) giving the transformations of variables Q,P do not involve variables q,p or the mixing parameter ξ. Therefore one can introduce a distribution with respect to the variables Q,P and study it separately.

In order to understand the properties of the distribution of the variables Q,P we first suppose that the localization process takes place for all systems of the ensemble at the same equally spaced times as in Ref. 6. The time interval between two successive localization processes is $\tau=1/\lambda$. The evolution of the ensemble is a succession of cycles consisting of a localization process followed by a Schrödinger evolution during the time τ. For the variables in which we are interested we write

$$\begin{aligned} Q=Q_i &\xrightarrow[\text{process (2.9)}]{} Q_f=Q(0)\xrightarrow[\text{Schrödinger evolution}]{} Q(\tau)=Q'\ , \\ P=P_i &\xrightarrow[\text{process (2.9)}]{} P_f=P(0)\xrightarrow[\text{Schrödinger evolution}]{} P(\tau)=P'\ . \end{aligned} \tag{8.11}$$

Using Eqs. (8.9), (8.4), and (8.10) we find

$$\begin{aligned} Q'&=\frac{Q}{1+2\alpha Q}+2\frac{(QP-\hbar^2/4)^{1/2}}{1+2\alpha Q}\frac{1}{\lambda m} \\ &\quad+\frac{P+\hbar^2\alpha(1+\alpha Q)}{1+2\alpha Q}\frac{1}{\lambda^2m^2}\ , \\ P'&=\frac{P+\hbar^2\alpha(1+\alpha Q)}{1+2\alpha Q}\ . \end{aligned} \tag{8.12}$$

It is convenient to introduce the dimensionless variables

$$x=\alpha Q,\quad y=\frac{\alpha}{\lambda^2m^2}P\ , \tag{8.13}$$

$$\epsilon=\left[\frac{\alpha\hbar}{\lambda m}\right]^{1/2}\ . \tag{8.14}$$

In terms of such variables the map (8.12) is written as

$$x'=\frac{x}{1+2x}+\frac{2(xy-\epsilon^4/4)^{1/2}}{1+2x}+\frac{y+\epsilon^4(1+x)}{1+2x}\ , \tag{8.15a}$$

$$y'=\frac{y+\epsilon^4(1+x)}{1+2x}\ . \tag{8.15b}$$

For a macroscopic mass and for any reasonable choice of α and λ, ϵ is a small number. In the numerical example of Sec. VII, ϵ is of the order of 10^{-12}.

The first question about the map (8.12) or (8.15) is whether it has a fixed point. Physically such a fixed point corresponds to the existence of a regime condition in the evolution of Q and P. Equation (8.15b) with the condition $y'=y$ gives

$$y=\epsilon^4\frac{1+x}{2x}\ . \tag{8.16}$$

Inserting this value into Eq. (8.15a) with $x'=x$ gives

$$4x^3=2\epsilon^2x\sqrt{1+2x}+\epsilon^4(1+x)(1+2x)\ . \tag{8.17}$$

It is easily seen that this equation has a positive solution given by

$$x_0=\frac{\epsilon}{\sqrt{2}}\left[1+\frac{\epsilon}{\sqrt{2}}+O(\epsilon^2)\right] . \tag{8.18a}$$

In turn, Eq. (8.16) gives

$$y_0=\frac{\epsilon^3}{\sqrt{2}}[1+O(\epsilon^2)] . \tag{8.18b}$$

The corresponding values of the square spreads are

$$Q_0=\frac{1}{\sqrt{2}}\left[\frac{\hbar}{\alpha\lambda m}\right]^{1/2}\left[1+\frac{\epsilon}{\sqrt{2}}+O(\epsilon^2)\right] , \tag{8.19a}$$

$$P_0=\frac{\hbar^2}{\sqrt{2}}\left[\frac{\alpha\lambda m}{\hbar}\right]^{1/2}[1+O(\epsilon^2)] . \tag{8.19b}$$

The next question about (8.12) is whether the regime condition we have found is stable, i.e., whether a small deviation from the values (8.18) or (8.19) tends to vanish when the map is iterated. We consider small deviations δ_x,δ_y defined by

$$x=x_0(1+\delta_x),\quad y=y_0(1+\delta_y) , \tag{8.20}$$

and look for the corresponding deviations δ'_x,δ'_y of x',y'. In the linear approximation, using matrix notation, one writes

$$\underline{\delta}'=M\underline{\delta} . \tag{8.21}$$

To the first order in ϵ, M is found to be

$$M=I+\sqrt{2}\epsilon A , \tag{8.22}$$

where

$$A=\begin{vmatrix}-1 & 1\\ -1 & -1\end{vmatrix} . \tag{8.23}$$

Indicating by $\tilde{A}$ the transpose of A, one has $\tilde{A}+A=-2I$. The square norm of δ' is then

$$\begin{aligned}||\underline{\delta}'||^2&=[\underline{\delta}(I+\sqrt{2}\epsilon\tilde{A})(I+\sqrt{2}\epsilon A)\underline{\delta}]\\ &=(1-2\sqrt{2}\epsilon)||\underline{\delta}||^2 .\end{aligned} \tag{8.24}$$

We conclude that the squared norm of a small enough deviation tends to zero by iteration of the map, so that the regime condition is stable.

Finally, we would like to know what happens when one starts with a pair of Q,P values far from Q_0,P_0. The map (8.12) is complicated enough to prevent giving an analytic answer to such a question. Numerical experiments are being performed. So far we did not find any initial pair which does not converge to the regime values by repeated applications of the map.

The value of P is not changed by the Schrödinger evolution. In the regime condition it remains constant also under the action of the localization process (2.9). On the other hand, the value of Q is decreased by the localization process and enlarged by the Schrödinger evolution. In the regime condition the two effects compensate each other. Equation (8.19a) gives the maximum of Q during the cycle. It is easily found that the minimum is

$$Q_{00}=\frac{1}{\sqrt{2}}\left[\frac{\hbar}{\alpha\lambda m}\right]^{1/2}\left[1-\frac{\epsilon}{\sqrt{2}}+O(\epsilon^2)\right] . \tag{8.25}$$

The relative change of Q due to its breathing during the cycle is of the order of ϵ.

So far the localization process (2.9) was supposed to occur at times equally spaced by the amount $\tau=1/\lambda$. In our treatment, however, the localization process occurs at random times according to a Poisson process. The values of τ are then distributed around the mean value $1/\lambda$ with root-mean-square deviation also equal to $1/\lambda$. This circumstance introduces fluctuations in the values at the end of the cycle of Q and P. We suppose to start a cycle of time length

$$\tau=\frac{1}{\lambda}(1+\delta) , \tag{8.26}$$

where δ is of the order of 1, with the regime values (8.19). One finds at the end of the cycle

$$Q'=Q_0[1+\sqrt{2}\epsilon\delta+O(\epsilon^2)],\quad P'=P_0 . \tag{8.27}$$

In the next cycle the values (8.27) have to be used as initial values instead of Q_0,P_0. This fact introduces at the end of the new cycle a deviation from the values Q_0,P_0 which is again of the order ϵ for Q and is of order ϵ^2 for P. Since the regime condition is stable, these deviations do not increase by iteration of the cycle.

The Poisson process which runs the times of the localization process takes place independently for each member of the statistical ensemble. Therefore, at a given time, each member of the ensemble is in a certain position within a cycle of a certain length inserted in a certain sequence of cycles. As a consequence the values of the Q,P pair mix. According to the foregoing discussion the distribution of the variables Q and P converges to a distribution centered around the mean values

$$Q=\frac{1}{\sqrt{2}}\left[\frac{\hbar}{\alpha\lambda m}\right]^{1/2} , \tag{8.28a}$$

$$P=\frac{\hbar^2}{\sqrt{2}}\left[\frac{\alpha\lambda m}{\hbar}\right]^{1/2} . \tag{8.28b}$$

The ratio of the width in Q of the distribution to the mean value of Q is of the order of ϵ and the ratio of the width in P to the mean value of P is of the order of ϵ^2. The value of R corresponding to (8.28) is

$$R=\hbar/2 . \tag{8.29}$$

It is worth noting that the values of Q and P given by (8.28) are very small on a macroscopic scale. The product $QP=\hbar^2/2$ is twice the minimum allowed by the uncertainty principle. In the numerical example of Sec. VII one finds $\sqrt{Q}\approx 10^{-11}$ cm, $\sqrt{P}\approx 10^{-16}$ g cm sec^{-1}.

The above considerations show that one can reasonably take the approximation of neglecting the widths of the distribution of Q and P, i.e., of assuming

$$\rho(t)=\int dq \int dp\, \sigma(q,p,t)\rho_\Gamma(q,p,Q,P) , \tag{8.30}$$

where Q and P are understood to have the values (8.28). To find the evolution equation for the phase-space density $\sigma(q,p,t)$, we write Eq. (3.1) in the form

$$\frac{d}{dt}\rho(t)=\frac{d^{(s)}}{dt}\rho(t)-\lambda\{\rho(t)-T[\rho(t)]\} , \tag{8.31}$$

where $d^{(s)}/dt$ indicates the time variation rate induced by the Schrödinger evolution of $\rho_\Gamma(q,p,Q,P)$, the values of Q and P being kept fixed. From Eqs. (8.3) one gets

$$\begin{aligned}\frac{d^{(s)}}{dt}\rho(t)&=\int dq \int dp\, \sigma(q,p,t)\left[\frac{\partial}{\partial\tau}\rho_\Gamma\left[q+\frac{p}{m}\tau,p,Q,P\right]\right]_{\tau=0}\\&=\int dq \int dp\, \sigma(q,p,t)\frac{p}{m}\frac{\partial}{\partial q}\rho_\Gamma(q,p,Q,P)\\&=\int dq \int dp\left[-\frac{p}{m}\right]\left[\frac{\partial}{\partial q}\sigma(q,p,t)\right]\rho_\Gamma(q,p,Q,P) .\end{aligned} \tag{8.32}$$

Similarly, Q and P being again kept fixed, Eqs. (8.8) give

$$\begin{aligned}T[\rho(t)]&=\int dq \int dp\, \sigma(q,p,t)T[\rho_\Gamma(q,p,Q,P)]\\&=\int dq \int dp\, \sigma(q,p,t)\frac{1}{\sqrt{\pi}}\int d\xi\, e^{-\xi^2}\rho_\Gamma(q-a\xi,p-b\xi,Q,P)\\&=\int dq \int dp\frac{1}{\sqrt{\pi}}\int d\xi\, e^{-\xi^2}\sigma(q-a\xi,p-b\xi,t)\rho_\Gamma(q,p,Q,P) ,\end{aligned} \tag{8.33}$$

where

$$a=\frac{2\alpha Q}{[\alpha(1+2\alpha Q)]^{1/2}}=\left[\frac{2\hbar}{\lambda m}\right]^{1/2}[1+O(\epsilon)] , \tag{8.34a}$$

$$b=\frac{2\alpha R}{[\alpha(1+2\alpha Q)]^{1/2}}=\sqrt{\alpha}\,\hbar[1+O(\epsilon)] . \tag{8.34b}$$

From Eq. (8.31), taking into account Eqs. (8.32) and (8.33), we find that the phase-space density $\sigma(q,p,t)$ obeys the evolution equation

$$\begin{aligned}\frac{\partial}{\partial t}\sigma(q,p,t)=&-\frac{p}{m}\frac{\partial}{\partial q}\sigma(q,p,t)\\&+\lambda\left[\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty}d\xi\, e^{-\xi^2}\sigma(q-a\xi,p-b\xi,t)\right.\\&\left.-\sigma(q,p,t)\right] .\end{aligned} \tag{8.35}$$

A significant check of the approximation introduced with assumption (8.30) is given in Appendix E.

The first term on the right-hand side of Eq. (8.35), alone, would give rise to pure classical motion. In fact, $-(p/m)\partial/\partial q$ is the classical Liouville operator for the free particle.

The second term on the right-hand side (RHS) of Eq. (8.35) gives rise to diffusion, as one can see by noting that its presence amounts to substituting with mean frequency λ the density function $\sigma(q,p,t)$ with its "diffused" form represented by the integral. The parameters a and b are given by Eqs. (8.34) and are very small. For example, with the numerical choices of Sec. VII, they are of the orders 10^{-17} cm and 10^{-22} g cm sec^{-1}, respectively. It is possible to give the diffusion term a more familiar form which also allows us to better evaluate the size of the diffusion effects. Since the function $\sigma(q,p,t)$ has the meaning of a phase-space density for an ensemble of macroscopic particles, it changes slightly over distances of the order of a and b. Therefore it is a very good approximation replacing $\sigma(q-a\xi,p-b\xi,t)$ by a truncated expansion in powers of $a\xi$ and $b\xi$. The odd terms do not contribute to the integral. Keeping the even terms up to second order we get

$$\begin{aligned}\frac{\partial}{\partial t}\sigma(q,p,t)=&-\frac{p}{m}\frac{\partial}{\partial q}\sigma(q,p,t)\\&+\frac{\lambda}{4}\left[a^2\frac{\partial^2}{\partial q^2}+2ab\frac{\partial^2}{\partial q\partial p}+b^2\frac{\partial^2}{\partial p^2}\right]\sigma(q,p,t) .\end{aligned} \tag{8.36}$$

Equation (8.36) is of the Fokker-Planck type, with two independent diffusion coefficients for position and momentum. The position diffusion coefficient $\lambda a^2/2$, according to Eq. (8.34a), is equal to $\hbar/m$. This is the same value as the diffusion coefficient appearing in Nelson's stochastic mechanics.[21]

There, however, the position diffusion is simply an interpretation of the spread of the wave packet. Here, wave packets do not spread, their width being blocked by the localization process at the value $\sqrt{Q}$ given by Eq. (8.28a), but the quantum spread is substituted by a sound stochastic spread of the same amount. On the other hand, the localization process introduces a stochastic spread of momentum. This effect is represented by the momentum diffusion coefficient $\lambda b^2/2$ which, according to Eq. (8.34b), is equal to $\alpha\lambda\hbar^2/2$.

This value is very small, as shown by the numerical example of Sec. VII, where it is of the order 10^{-37} $(\text{g cm sec}^{-1})^2\text{sec}^{-1}$.

IX. CONCLUSIONS

We have shown that one can consistently introduce a modification of standard quantum mechanics which leaves things unchanged for microscopic objects, while, for macroscopic objects, transforms quantum mechanics into a stochastic mechanics in phase space exhibiting the classical features. Of course, uncertainty is not eliminated, but it is, in a sense, increased. However the amount of stochasticity is quite small and is compatible with our experience of the behavior of macroscopic bodies. The modified dynamics can provide a consistent and conceptually appealing description of the wave-packet reduction of quantum theory.

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APPENDIX A

With reference to Eq. (2.4) let us choose an ensemble of free particles of mass μ in one dimension with an initial uniform distribution over the position interval $[q_1,q_2]$ and with a sharp momentum p_0. For E_t we choose the tube delimited by the lines $q=q_1+(p_0/\mu)t$ and $q=q_M+(p_0/2\mu)t$ with $q_1<q_M<q_2$, so that

$$\chi_{E_t}=\theta\left[q-q_1-\frac{p_0}{\mu}t\right]\theta\left[q_M+\frac{p_0}{2\mu}t-q\right]. \tag{A1}$$

We can disregard the p dependence due to the linear momentum conservation. The solution of Eq. (2.4) in this case and for the stated initial conditions is, as easily checked

$$\begin{aligned}\tilde{\rho}(q,p,t)=\frac{1}{q_2-q_1}\Bigg\{&\theta\left[q-\frac{p_0}{\mu}t-q_1\right]\theta\left[\frac{p_0}{2\mu}t+q_M-q\right]\\&+\theta\left[q-q_M-\frac{p_0}{2\mu}t\right]\theta\left[q_M+\frac{p_0}{\mu}t-q\right]\exp\left[-\frac{\lambda\mu}{p_0}\left[2q-\frac{p_0}{\mu}t-2q_M\right]\right]\\&+\theta\left[q-q_M-\frac{p_0}{\mu}t\right]\theta\left[q_2-q+\frac{p_0}{\mu}t\right]e^{-\lambda t}\Bigg\}\delta(p-p_0)\,.\end{aligned} \tag{A2}$$

We remark that the third term on the RHS of Eq. (A2) describes trajectories which are for all times outside the tube defined by Eq. (A1). The factor $e^{-\lambda t}$ expresses the probability that no selection has been done up to t. The second term describes particles whose trajectories crossed the boundary of the tube before t and therefore they are still present if they have not been tested from the time in which they have left the tube up to t. Since these particles cross the boundary at time $t_1=2\mu[q_M-q+(p_0/\mu)t]/p_0$ the probability of having not been discarded is $e^{-\lambda(t-t_1)}$, which is just the factor attached to the second term. The first term describes particles which are always within the tube up to time t and so have never been discarded. The probability P_{E_t} that the particles have always been within the tube E_t is evaluated, as already stated, by taking the limit of $\tilde{\rho}$ for $\lambda\to\infty$ and then integrating over the whole phase space. The exponential terms vanish and we get

$$P_{E_t}=\frac{q_M-q_1-(p_0/2\mu)t}{q_2-q_1}\theta\left[q_M-q_1-\frac{p_0}{2\mu}t\right], \tag{A3}$$

which is the correct result.

APPENDIX B

The standard description of measurement processes goes as follows. The observables of the system are associated with self-adjoint operators in the Hilbert space $\mathscr{H}$ of the system. The states of the system are described by statistical operators which are a subset of the set $\mathscr{T}^+(\mathscr{H})$ of the non-negative trace-class operators on $\mathscr{H}$. To any observable A one can relate a spectral family of projection operators $E(\lambda),\lambda\in R$ in terms of which one can express the operator $\hat{A}$ associated to A by

$$\hat{A}=\int_{-\infty}^{+\infty}\lambda\,dE(\lambda)\,. \tag{B1}$$

Definition 1. A projection-valued measure (PVM) is a mapping

$$E:\mathscr{B}(\mathbf{R})\to\mathscr{L}(\mathscr{H})$$

from the σ algebra $\mathscr{B}(\mathbf{R})$ of the Borel sets of $\mathbf{R}$ into the set $\mathscr{L}(\mathscr{H})$ of the bounded operators on $\mathscr{H}$, such that (i) $E(T)$ is a projection operator for any Borel set T, (ii) $E(\cup_{i=1}^{\infty}(T_i)=\sum_{i=1}^{\infty}E(T_i)$ for $T_i\cap T_j=0$, $i\neq j$, (iii) $E(\mathbf{R})=1$.

Remark. Given the spectral family $E(\lambda)$ associated to $\hat{A}$ the map

$$E(T)=\int_T dE(\lambda) \tag{B2}$$

is a projection-valued measure.

Denoting by $P(A\in T\mid\rho)$ the probability that in a measurement of $\hat{A}$ a result belonging to the Borel set T is ob-

tained when the state of the system is ρ, one has

$$P(A\in T\,|\,\rho)=\mathrm{tr}[E(T)\rho]\ . \tag{B3}$$

If the measurement ascertaining whether $A\in T$ is used to select the members of the ensemble for which an answer yes has been obtained, the statistical operator after the measurement is

$$\tilde{\rho}=\frac{E(T)\tilde{\rho}E(T)}{\mathrm{tr}[E(T)\rho]}\ . \tag{B4}$$

In the case of a nonselective measurement ascertaining which systems have given the answer $A\in T_k$ ($\bigcup_k T_k=\mathbf{R}$, $T_k\cap T_j=0$) the statistical operator after the measurement is

$$\tilde{\rho}=\sum_k E(T_k)\rho E(T_k)\ . \tag{B5}$$

In virtue of the above considerations, in the standard description one can identify observables with projection-valued measures.

There are several reasons which make it appropriate to enlarge the above scheme. For instance, one would like to incorporate nonsharp measurements (i.e., measurements which do not correspond to yes-no experiments) or measurements for which the efficiency of the apparatus is less than one. Moreover, one would like to include in the description the modifications of the statistical operator of a composite system when a measurement is performed on one of its constituents. It is enlightening to consider the following case. Suppose we subject the system to two successive measurements of two observables A and B at times t_1 and t_2, aimed to ascertain whether $A\in T$ and $B\in S$, respectively. The probability of getting the answer yes for both questions, given the initial state of the system $\rho(0)$, is

$$P[A\in T,t_1;B\in S,t_2\,|\,\rho(0)]=\mathrm{tr}[\epsilon(S)e^{-iH(t_2-t_1)}E(T)e^{-iHt_1}\rho(0)e^{iHt_1}E(T)e^{iH(t_2-t_1)}\epsilon(S)]\ , \tag{B6}$$

where $\epsilon(S)$ is the PVM related to B. If one wants to express this probability in a form similar to (B3) writing

$$P[A\in T,t_1;B\in S,t_2\,|\,\rho(0)]=\mathrm{tr}[F\rho(0)]\ , \tag{B7}$$

one finds that F satisfies $F=F^\dagger$, $0\le F\le 1$.

We can now formulate the generalized description of observables and measurement process.

Definition 2. An effect is a non-negative bounded (and therefore self-adjoint) operator F satisfying $0\le F\le 1$.

Definition 3. An effect-valued measure (EVM) is a map F from $\mathscr{B}(\mathbf{R})$ into $\mathscr{L}(\mathscr{H})$ such that (i) $F(T)$ is an effect, (ii) $F(\bigcup_{i=1}^{\infty}T_i)=\sum_{i=1}^{\infty}F(T_i)$ for $T_i\cap T_j=0$, $i\neq j$, (iii) $F(\mathbf{R})=1$.

Observables are then associated to EVM and the probability $P(A\in T\,|\,\rho)$ is given by

$$P(A\in T\,|\,\rho)=\mathrm{tr}[F(T)\rho]\ . \tag{B8}$$

To generalize Eqs. (B4) and (B5) describing the modifications induced by measurement processes on the statistical operator one proceeds as follows.

Definition 4. An operation is a map $\phi:\mathscr{T}(\mathscr{H})\to\mathscr{T}(\mathscr{H})$ of the set of trace-class operators into itself which is linear, positive, and bounded with respect to the trace norm with bound less or equal to one, i.e.,

$$\text{(i)}\ \phi[\alpha A+\beta B]=\alpha\phi[A]+\beta\phi[B],\quad \forall A,B\in\mathscr{T}(\mathscr{H})\ ,$$

$$\text{(ii)}\ \phi[A]\ge 0,\quad \forall A\ge 0,\quad \forall A\in\mathscr{T}(\mathscr{H})\ ,$$

$$\text{(iii)}\ ||\phi[A]||_{\mathrm{tr}}\le||A||_{\mathrm{tr}},\quad \forall A\in\mathscr{T}(\mathscr{H})\ .$$

Taking advantage of the fact that the dual of the Banach space $\mathscr{T}(\mathscr{H})$ with the trace norm is $\mathscr{L}(\mathscr{H})$, one can define the adjoint ϕ^* of the map ϕ by

$$\mathrm{tr}(X\cdot\phi[A])=\mathrm{tr}(\phi^*[X]\cdot A),\quad \forall X\in\mathscr{L},\quad \forall A\in\mathscr{T}\ . \tag{B9}$$

Remark. Given an operation ϕ, if one defines an operator F by requiring

$$\mathrm{tr}(\phi[A])=\mathrm{tr}(F\cdot A),\quad \forall A\in\mathscr{T}(\mathscr{H})\ , \tag{B10}$$

then F is an effect and from (B9) $F=\phi^*[1]$. We can then associate an effect to any operation by (B10). The converse is not true, since different operations can give rise to the same effect. As an example one can remark that for a given effect F the map $\phi[A]=F^{1/2}AF^{1/2}$ is an operation whose associated effect is F. However, if one writes $F=\sum_k F_k$, $F_k\ge 0$ and considers arbitrary unitary operators U_k, the map $\tilde{\phi}[A]=\sum_k U_kF_k{}^{1/2}AF_k{}^{1/2}U_k^\dagger$ is an operation whose associated effect is F. Physically the fact that different operations can be related to the same effect is interpreted as accounting for the possibility of setting up different experimental procedures to measure the same observable.

Definition 5. An operation-valued measure (OVM) is a map from $\mathscr{B}(\mathbf{R})$ into the family of operations such that

$$\text{(i)}\ \phi\left[\bigcup_{i=1}^{\infty}T_i\right]=\sum_{i=1}^{\infty}\phi(T_i),\quad T_i\cap T_j=0,\quad i\neq j\ ,$$

$$\text{(ii)}\ \mathrm{tr}\{\phi(\mathbf{R})[X]\}=\mathrm{tr}X,\quad \forall X\in\mathscr{T}(\mathscr{H})\ .$$

The generalization of ordinary quantum rules mentioned in Sec. II can now be formulated.

(a) To any experimental procedure for measuring an observable A a OVM $\phi_A(T)$ is associated.

(b) The probability for finding $A\in T$ in the measurement when the state of the system is ρ is given by

$$P(A\in T\,|\,\rho)=\mathrm{tr}\{\phi_A(T)[\rho]\}=\mathrm{tr}[F_A(T)\cdot\rho]\ , \tag{B11}$$

where $F_A(T)$ is the unique effect corresponding to the operation $\phi_A(T)$.

(c) If in the measurement the value of A is found to belong to the set T and the systems giving this answer are selected, their ensemble is described by

$$\tilde{\rho}=\frac{\phi_A(T)[\rho]}{\mathrm{tr}\{\phi_A(T)[\rho]\}} . \tag{B12}$$

Remark. Usually one also requires for the map $\phi(T)$, besides the properties appearing in definition 5, the complete positivity. To clarify this point let us consider the system S_1 we are interested in, together with another system S_2 whose Hilbert space is N dimensional, and which is uncoupled with S_1. Let us consider an operation ϕ on $\mathcal{T}(\mathcal{H})$ and extend it to the statistical operators of $\mathcal{H}\otimes\mathcal{H}^N$ of the form

$$\rho=\rho_1^{\mathcal{H}}\otimes\rho_2^{\mathcal{H}^N} \tag{B13}$$

by putting

$$\phi[\rho]=\phi[\rho_1^{\mathcal{H}}]\otimes\rho_2^{\mathcal{H}^N} . \tag{B14}$$

Equation (B14) induces then a unique extension of ϕ on the whole space $\mathcal{T}(\mathcal{H}\otimes\mathcal{H}^N)$. We say that the original operator ϕ is completely positive if this extension is a positive operation for all N. If the complete positivity requirement is made for operations the following theorem holds.

Theorem. For any completely positive operation ϕ on $\mathcal{T}(\mathcal{H})$ there exist operators A_k on $\mathcal{H}$, with $k\in K$ (where K is a finite or countable set) satisfying $\sum_{k\in K_0}A_k^\dagger A_k\le 1$ for all finite subsets K_0 of K, such that for any $X\in\mathcal{L}(\mathcal{H})$ and $B\in\mathcal{T}(\mathcal{H})$

$$\phi[B]=\sum_{k\in K}A_kBA_k^\dagger , \tag{B15a}$$

$$\phi^*[X]=\sum_{k\in K}A_k^\dagger XA_k . \tag{B15b}$$

Moreover the effect F determined by ϕ is

$$F=\sum_{k\in K}A_k^\dagger A_k . \tag{B15c}$$

The series in (B15a) converges in the trace norm topology, the other two series converge ultraweakly.

APPENDIX C

To solve Eq. (3.6) we set

$$\rho(q,q',t)\equiv\langle q\,|\,\rho(t)\,|\,q'\rangle = e^{-\lambda t}\exp(\lambda t\,e^{-(\alpha/4)(q-q')^2})\tilde{\rho}(q,q',t) , \tag{C1}$$

getting

$$\frac{\partial\tilde{\rho}(q,q',t)}{\partial t}=\frac{i\hbar}{2m}\left[\frac{\partial^2\tilde{\rho}(q,q',t)}{\partial q^2}-\frac{\partial^2\tilde{\rho}(q,q',t)}{\partial q'^2}-\alpha\lambda t(q-q')e^{-(\alpha/4)(q-q')^2}\left[\frac{\partial\tilde{\rho}(q,q',t)}{\partial q}+\frac{\partial\tilde{\rho}(q,q',t)}{\partial q'}\right]\right] . \tag{C2}$$

Under the change of variables

$$\begin{aligned} x&=q-q' ,\\ y&=q+q' , \end{aligned} \tag{C3}$$

and calling $\bar{\rho}(x,y,t)=\tilde{\rho}((x+y)/2,(y-x)/2,t)$, we obtain

$$\frac{\partial\bar{\rho}(x,y,t)}{\partial t}=\frac{i\hbar}{m}\left[2\frac{\partial^2\bar{\rho}(x,y,t)}{\partial x\partial y}-\alpha\lambda tx\,e^{-(\alpha/4)x^2}\frac{\partial\bar{\rho}(x,y,t)}{\partial y}\right] . \tag{C4}$$

This equation can be solved by the method of separation of variables putting

$$\bar{\rho}(x,y,t)=f(x,t)g(y) . \tag{C5}$$

We get

$$\frac{dg}{dy}+i\mu g=0 , \tag{C6}$$

$$\frac{df}{dt}=\frac{\mu\hbar}{m}\left[2\frac{\partial f}{\partial x}-\alpha\lambda tx\,e^{-(\alpha/4)x^2}f(x,t)\right] , \tag{C7}$$

where μ is an arbitrary integration constant. With the change of variables

$$\begin{aligned} z&=-\frac{\mu\hbar}{m}t+\frac{1}{2}x ,\\ w&=\frac{\mu\hbar}{m}t+\frac{1}{2}x , \end{aligned} \tag{C8}$$

one solves Eq. (C7). Using (C6), (C5), and (C3) one finally gets that

$$\tilde{\rho}(q,q',t)=h\left[\frac{\mu\hbar}{m}t+\frac{1}{2}(q-q')\right]e^{-i\mu(q+q')} \times\exp\left\{\frac{\alpha\lambda m}{4\mu\hbar}\int_0^{-(\mu\hbar/m)t+(q-q')/2}dz'\left[\left[\frac{\mu\hbar}{m}t+\tfrac{1}{2}(q-q')\right]^2-z'^2\right]\exp\left[-\frac{\alpha}{4}\left[\frac{\mu\hbar}{m}t+\tfrac{1}{2}(q-q')+z'\right]^2\right]\right\} \tag{C9}$$

is a solution of Eq. (C2) for any choice of the function h having obvious regularity properties. Taking into account the arbitrariness and the linearity of Eq. (3.6) one then has that

$$\rho(q,q',t)=\int_{-\infty}^{+\infty} d\mu\, h\left[\mu,q-q'+\frac{2\mu\hbar}{m}t\right]e^{-i\mu(q+q')}\exp\left[\frac{\lambda m}{2\mu\hbar}(q-q')(1-e^{-(\alpha/4)(q-q')^2})\right]$$
$$\times\exp\left[-\frac{\lambda m\alpha}{4\mu\hbar}\int_0^{q-q'} dy\, y^2 e^{-(\alpha/4)y^2}\right] \tag{C10}$$

is a solution of Eq. (3.6). Setting $t=0$ and using the inverse Fourier transform to express h in terms of $\rho(q,q',0)$ one gets the solution satisfying the given initial conditions. Expressing the solution of Eq. (3.6) with $\lambda=0$ at time t in terms of the initial condition $\rho(q,q',0)$ and inverting it we can make appear $\rho_s(q,q',t)$ in the expression of $\rho(q,q',t)$, obtaining Eq. (3.7) of the text.

APPENDIX D

Let $|\Gamma(A,B)\rangle$, $A,B\in\mathbb{C}$, $\mathrm{Re}A>0$ be the normalized state vector whose coordinate representative has the form

$$\langle q'|\Gamma(A,B)\rangle=\frac{1}{\mathcal{N}}e^{-(Aq'^2+2Bq')/2}\,. \tag{D1}$$

The normalization factor is given by

$$\frac{1}{\mathcal{N}}=\left[\frac{\mathrm{Re}A}{\pi}\right]^{1/4}e^{-(\mathrm{Re})^2B/2\,\mathrm{Re}A}\,. \tag{D2}$$

For the mean values, square spreads, and q-p correlation relative to the particular wave function (D1) we use the symbols q, p, Q, P, and R. We find

$$q\equiv\langle\hat{q}\rangle_\Gamma=-\frac{\mathrm{Re}B}{\mathrm{Re}A}\,, \tag{D3a}$$

$$p\equiv\langle\hat{p}\rangle_\Gamma=\frac{\hbar}{\mathrm{Re}A}(\mathrm{Im}A\,\mathrm{Re}B-\mathrm{Im}B\,\mathrm{Re}A)\,, \tag{D3b}$$

$$Q\equiv\langle(\hat{q}-\langle\hat{q}\rangle_\Gamma)^2\rangle_\Gamma=\frac{1}{2\,\mathrm{Re}A}\,, \tag{D4a}$$

$$P\equiv\langle(\hat{p}-\langle\hat{p}\rangle_\Gamma)\rangle_\Gamma=\frac{\hbar^2|A|^2}{2\,\mathrm{Re}A}\,, \tag{D4b}$$

$$R=\langle[(\hat{q}-\langle\hat{q}\rangle_\Gamma)(\hat{p}-\langle\hat{p}\rangle_\Gamma)]_{\mathrm{sym}}\rangle_\Gamma$$
$$=-\frac{\hbar\,\mathrm{Im}A}{2\,\mathrm{Re}A}\,. \tag{D5}$$

Equations (D3) and (D4) can be solved for A and B giving, with the condition $\mathrm{Im}A<0$,

$$\mathrm{Re}A=\frac{1}{2Q}\,, \tag{D6a}$$

$$\mathrm{Im}A=-\frac{1}{\hbar Q}(QP-\hbar^2/4)^{1/2}\,, \tag{D6b}$$

$$\mathrm{Re}B=-\frac{q}{2Q}\,, \tag{D7a}$$

$$\mathrm{Im}B=\frac{1}{\hbar Q}[q(QP-\hbar^2/4)^{1/2}-pQ]\,. \tag{D7b}$$

Substituting Eqs. (D6) into (D5) one obtains the relation (8.2) between R and Q,P.

The state vector evolved from $|\Gamma(A,B)\rangle$ according to the Schrödinger equation in the time interval $(0,\tau)$, apart from an unsignificant phase factor, is $|\Gamma(A(\tau),B(\tau))\rangle$, where

$$A(\tau)=A/(1+i\hbar A\tau/m)\,,$$
$$B(\tau)=B/(1+i\hbar A\tau/m)\,. \tag{D8}$$

From Eqs. (D3), (D4), (D8), (D6), and (D7) one easily gets Eqs. (8.3) and (8.4).

We now put

$$\rho_\Gamma(A,B)=|\Gamma(A,B)\rangle\langle\Gamma(A,B)|\,. \tag{D9}$$

The operation (2.9) can then be written as

$$\langle q'|T[\rho_\Gamma(A_i,B_i)]|q''\rangle$$
$$=\sqrt{\alpha/\pi}\int dx\, w(x)\langle q'|\rho_\Gamma(A_f,B_f)|q''\rangle\,, \tag{D10}$$

where

$$A_f=A_i+\alpha,\quad B_f=B_i-\alpha x\,, \tag{D11}$$

and

$$w(x)=\frac{\mathcal{N}^2(A_f,B_f)}{\mathcal{N}^2(A_i,B_i)}e^{-\alpha x^2}\,. \tag{D12}$$

Translating these equations from the variables A,B to the variables q,p,Q,P and performing a trivial change of the integration variable, one gets Eqs. (8.7), (8.8), and (8.9).

APPENDIX E

Let us consider the initial density operator

$$\rho(0)=\int dq\int dp\,\sigma(q,p,0)\rho_\Gamma(q,p,Q,P)\,. \tag{E1}$$

We want to compare the values of the means $\langle\hat{q}\rangle$, $\langle\hat{p}\rangle$, of the square spreads $\{q\},\{p\}$ and of the q-p correlation $\{qp\}$ given by the exact density operator at the time t with the values of the corresponding quantities given by the approximate density operator (8.30) where $\sigma(q,p,t)$ has evolved from $\sigma(q,p,0)$ according to Eq. (8.35). We shall label by a superscript a the approximate density and the related quantities.

Consider a density operator of the form

$$\rho=\int dq\int dp\,\sigma(q,p)\rho_\Gamma(f(q,p),p,F(Q,P),P)\,. \tag{E2}$$

For classical dynamical variables $A(q,p),B(q,p)$ we define

$$\bar{A}=\int dq\int dp\,\sigma(q,p)A(q,p)\,,$$
$$\tilde{A}=\int dq\int dp\,\sigma(q,p)[A(q,p)-\bar{A}]^2\,, \tag{E3}$$
$$\widetilde{AB}=\int dq\int dp\,\sigma(q,p)[A(q,p)-\bar{A}][B(q,p)-\bar{B}]\,.$$

One can then show that

$$\langle \hat{q} \rangle = \bar{f} ,$$
$$\langle \hat{p} \rangle = \bar{p} ,$$
$$\{q\} = F(Q,P) + \tilde{f} , \tag{E4}$$
$$\{qp\} = [F(Q,P)\cdot P - \hbar^2/4]^{1/2} + \tilde{fp} ,$$
$$\{p\} = P + \tilde{p} .$$

If f has the form $f(q,p) = \xi q + \eta p$, one has further

$$\tilde{f} = \xi^2 \tilde{q} + 2\xi\eta \tilde{qp} + \eta^2 \tilde{p}, \quad \tilde{fp} = \xi \tilde{qp} + \eta \tilde{p} . \tag{E5}$$

We recall here the exact results (3.15) and (3.16):

$$\langle \hat{q} \rangle = \langle \hat{q} \rangle_s ,$$
$$\langle \hat{p} \rangle = \langle \hat{p} \rangle_s ,$$
$$\{q\} = \{q\}_s + \frac{S}{6}\frac{t^3}{m^3} , \tag{E6}$$
$$\{qp\} = \{qp\}_s + \frac{S}{4}\frac{t^2}{m^2} ,$$
$$\{p\} = \{p\}_s + \frac{S}{2}\frac{t}{m} ,$$

where

$$S = \alpha\lambda m \hbar^2 . \tag{E7}$$

The density operator evolved from (E1) according to the Schrödinger equation can be written as

$$\rho_s(t) = \int dq \int dp\, \sigma(q,p,0) \times \rho_\Gamma \left[q + p\frac{t}{m}, p, Q + 2R\frac{t}{m} + P\frac{t^2}{m^2}, P \right] , \tag{E8}$$

so that we can apply Eqs. (E4) and (E5) with $\sigma(q,p) = \sigma(q,p,0)$, $f = q + pt/m$, $F = Q = 2Rt/m + Pt^2/m^2$ to the evaluation of $\langle q \rangle_s, \{q\}_s, \ldots$. We find then

$$\langle \hat{q} \rangle = \bar{q}(0) + \bar{p}(0)\frac{t}{m} , \tag{E9a}$$

$$\langle \hat{p} \rangle = \bar{p}(0) , \tag{E9b}$$

$$\{q\} = Q + \tilde{q}(0) + 2[R + \tilde{qp}(0)]\frac{t}{m} + [P + \tilde{p}(0)]\frac{t^2}{m^2} + \frac{S}{6}\frac{t^3}{m^3} , \tag{E10a}$$

$$\{qp\} = R + \tilde{qp}(0) + [P + \tilde{p}(0)]\frac{t}{m} + \frac{S}{4}\frac{t^2}{m^2} , \tag{E10b}$$

$$\{p\} = P + \tilde{p}(0) + \frac{S}{2}\frac{t}{m} . \tag{E10c}$$

The approximate density operator $\rho^{(a)}(t)$ has the form

$$\rho^{(a)}(t) = \int dq \int dp\, \sigma(q,p,t)\rho_\Gamma(q,p,Q,P) . \tag{E11}$$

Using again Eqs. (E4) and (E5), with $\sigma(q,p) = \sigma(q,p,t), f = q, F = Q$, one has

$$\langle \hat{q} \rangle^{(a)} = \bar{q}(t) , \tag{E12a}$$
$$\langle \hat{p} \rangle^{(a)} = \bar{p}(t) , \tag{E12b}$$
$$\{q\}^{(a)} = Q + \tilde{q}(t) , \tag{E13a}$$
$$\{qp\}^{(a)} = R + \tilde{qp}(t) , \tag{E13b}$$
$$\{p\}^{(a)} = P + \tilde{p}(t) . \tag{E13c}$$

In turn, one gets from Eq. (8.35) that the quantities $\bar{q}(t), \tilde{q}(t), \ldots$ obey the equations

$$\frac{d}{dt}\bar{q}(t) = \frac{1}{m}\bar{p}(t), \quad \frac{d}{dt}\bar{p}(t) = 0 ; \tag{E14}$$

$$\frac{d}{dt}\tilde{q}(t) = \frac{2}{m}\tilde{qp}(t) + \frac{\lambda a^2}{2} ,$$
$$\frac{d}{dt}\tilde{qp}(t) = \frac{1}{m}\tilde{p}(t) + \frac{\lambda ab}{2} , \tag{E15}$$
$$\frac{d}{dt}\tilde{p}(t) = \frac{\lambda b^2}{2} .$$

System (E14) is easily solved, giving

$$\bar{q}(t) = \bar{q}(0) + \bar{p}(0)\frac{t}{m}, \quad \bar{p}(t) = \bar{p}(0) . \tag{E16}$$

System (E15) is also easily solved, giving

$$\tilde{q}(t) = \tilde{q}(0) + 2\left[\tilde{qp}(0) + \frac{\lambda m a^2}{4}\right]\frac{t}{m} + \left[\tilde{p}(0) + \frac{\lambda mab}{2}\right]\frac{t^2}{m^2} + \frac{\lambda m b^2}{6}\frac{t^3}{m^3} ,$$
$$\tilde{qp}(t) = \tilde{qp}(0) + \left[\tilde{p}(0) + \frac{\lambda mab}{2}\right]\frac{t}{m} + \frac{\lambda m b^2}{4}\frac{t^2}{m^2} , \tag{E17}$$
$$\tilde{p}(t) = \tilde{p}(0) + \frac{\lambda m b^2}{2}\frac{t}{m} .$$

From Eqs. (8.34), recalling Eqs. (8.28), (8.29), and (E7), we get

$$\frac{\lambda m a^2}{4} = \frac{\hbar}{2}[1 + O(\epsilon)] = R[1 + O(\epsilon)] ,$$
$$\frac{\lambda mab}{2} = \frac{\hbar^2}{\sqrt{2}}\left[\frac{\alpha\lambda m}{\hbar}\right]^{1/2}[1 + O(\epsilon)] = P[1 + O(\epsilon)] , \tag{E18}$$
$$\lambda m b^2 = \alpha\lambda m \hbar^2 = S[1 + O(\epsilon)] .$$

Using Eqs. (E16) in (E12) and comparing with (E9), we find

$$\langle \hat{q} \rangle^{(a)} = \langle \hat{q} \rangle, \quad \langle \hat{p} \rangle^{(a)} = \langle \hat{p} \rangle . \tag{E19}$$

Inserting Eqs. (E17) and (E18) into (E13), we get the approximate values

$$\{q\}^{(a)}=Q+\tilde{q}(0)+2\{R[1+O(\epsilon)]+\tilde{q}\tilde{p}(0)\}\frac{t}{m}+\{P[1+O(\epsilon)]+\tilde{p}(0)\}\frac{t^2}{m^2}+\frac{S[1+O(\epsilon)]}{6}\frac{t^3}{m^3},$$

$$\{qp\}^{(a)}=R+\tilde{q}\tilde{p}(0)+\{P[1+O(\epsilon)]+\tilde{p}(0)\}\frac{t}{m}+\frac{S[1+O(\epsilon)]}{4}\frac{t^2}{m^2}, \tag{E20}$$

$$\{p\}^{(a)}=P+\tilde{p}(0)+\frac{S[1+O(\epsilon)]}{2}\frac{t}{m},$$

to be compared with the exact values (E10). The comparison shows precisely the kind of result which was expected on the basis of the arguments which led in Sec. VIII to the introduction of approximation (8.30).

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[1]G. Ludwig, Commun. Math. Phys. **4**, 331 (1967); **9**, 1 (1968); in *Deutung des Begriffs Physikalische Theorie und Axiomatische Grundlegung der Hilbertraumstruktur der Quantenmechanik durch Hauptsätze des Messens* (Lecture Notes in Physics, Vol. 4) (Springer, Berlin, 1970); A. J. Leggett, in *Essays in Theoretical Physics,* edited by W. E. Parry (Pergamon, Oxford, 1984), p. 95.

[2]B. d'Espagnat, *Conceptual Foundations of Quantum Mechanics* (Benjamin, Reading, Mass., 1976). See, for example, the discussion in Chap. 16.

[3]J. M. Jauch, Helv. Phys. Acta **37**, 293 (1964).

[4]As is well known the physical equivalence of pure states and appropriate statistical mixtures occurs also when one considers systems with infinitely many degrees of freedom (e.g., within the C^*-algebraic approach to quantum theory). We shall not discuss this possibility here, because, after all, the measuring apparatuses as well as the human observer are finite systems (see, for instance, Ref. 2, p. 188).

[5]A. J. Leggett, Suppl. Progr. Theor. Phys. **69**, 80 (1980); in *Proceedings of the International Symposium on the Foundations of Quantum Mechanics,* edited by S. Kamefuchi *et al.* (The Physical Society of Japan, Tokyo, 1983), p. 74; R. de Bruyn Ouboter, *ibid.*, p. 83.

[6]A. Barchielli, L. Lanz, and G. M. Prosperi, Nuovo Cimento **72B**, 79 (1982); Found. Phys. **13**, 779 (1983); in *Proceedings of the International Symposium on the Foundations of Quantum Mechanics* (Ref. 5), p. 165.

[7]L. L. Landau and E. M. Lifshitz, *Statistical Physics* (Pergamon, New York, 1959), p. 14; H. D. Zeh, in *Foundations of Quantum Mechanics,* edited by B. d'Espagnat (Academic, New York, 1971); E. Wigner, *ibid.* p. 1.

[8]Some authors [see, e.g., V. Hakim and V. Ambegaokar, Phys. Rev. **32**, 423 (1985) and references therein] have discussed within the standard quantum formalism the effect of the coupling of a macroscopic system to a linearly dissipative environment. Hakim and Ambegaokar have remarked that the environment acts, in some sense, like a position measuring apparatus. Their approach, in spite of some formal similarities with our work, has assumptions and aims which are completely different from ours. In particular, the dynamics of the macroscopic object is derived from an actual coupling of the object as a whole to some external system. Second, the coupling is dissipative so that the mean value of the momentum is exponentially damped. In our case, the modification of the dynamics for a macroscopic object is consistently derived from a modified dynamics of its microscopic constituents and there are no friction effects for mean values. Our basic aim is to give a unified description of all levels of physical reality.

[9]In a different context, i.e., in looking for motivations for experimental search of effects of macroscopic coherence, Leggett (Ref. 10) has considered that "One might imagine that there are corrections to Schrödinger's equation which are totally negligible at the level of one, two, or even 100 particles but play a major role when the number of particles involved becomes macroscopic (say of the order of 10^{23})." Our model exhibits these features and identifies explicitly the corrective terms such that a unified description of microscopic and macroscopic bodies can be consistently derived.

[10]A. J. Leggett, Contemp. Phys. **25**, 583 (1984).

[11]R. Haag and D. Kastler, J. Math. Phys. **5**, 848 (1964); G. Ludwig, Commun. Math. Phys. **4**, 331 (1967); K. Kraus, in *Foundations of Quantum Mechanics and Ordered Linear Spaces,* edited by A. Hartkämper and H. Neumann (Lecture Notes in Physics, Vol. 29) (Springer, Berlin, 1974), p. 206.

[12]In Ref. 6 also the more general case of a simultaneous approximate measurement of momentum and position has been considered. We will not discuss this possibility in this paper.

[13]The first formulation of this idea can be found in A. Rimini, in *Proceedings of the Meeting of Theoretical Physics, Amalfi, 1983,* edited by A. Giovannini *et al.* (ESI, Naples, 1984). An investigation of its consequences together with a preliminary account of the ideas developed in the present paper can be found in G. C. Ghirardi, A. Rimini, and T. Weber, in *Quantum Probability and Applications II,* edited by L. Accardi and W. von Waldenfels (Lecture Notes in Mathematics, Vol. 1136) (Springer, Berlin, 1985), p. 223. See, also, Proceedings of the Workshop on Fundamental Aspects of Quantum Theory, Como, 1985 (unpublished).

[14]Up to this point, since we are dealing with a macroscopic object, one can interpret, as already stated, the transition to statistical mixtures as due to the interaction with the external environment. The treatment given above constitutes then a precisely defined model of the occurrence of environment-induced superselection rules (see Ref. 15) for which the preferred pointer basis corresponds to localized pointer positions. Compare however the basic change of attitude we will take in Sec. VI.

[15]W. H. Zurek, in *Proceedings of the International Symposium on the Foundations of Quantum Mechanics* (Ref. 5), p. 181.

[16]It has been recently argued (Ref. 17) that, when attempting to construct a quantum theory of gravity, one is compelled to introduce a modified set of axioms which entail allowing pure states to evolve into mixed states. According to a general theorem, any equation which is able to account for such a type of transitions and satisfies some natural requirements, must fall within the general class which has been identified by

Lindblad (Ref. 18). Our equation belongs to this class. The fact that completely different kinds of physical arguments require one to consider a modification of the standard quantum mechanics by the addition of non-Hamiltonian terms inducing transitions from pure states to statistical mixtures looks interesting by itself.

[17]S. W. Hawking, Commun. Math. Phys. **87**, 395 (1983); see also T. Banks, L. Susskind, and M. E. Peskin, Nucl. Phys. **B244**, 125 (1984); J. Ellis, J. S. Hagelin, D. V. Nanopoulos, and M. Srednicki, *ibid.* **B241**, 381 (1984).

[18]G. Lindblad, Commun. Math. Phys. **48**, 119 (1976).

[19]K. Kraus, *States, Effects and Operations* (Springer, Berlin, 1983).

[20]It is easy to prove that the relation $d\langle \hat{H} \rangle /dt = \lambda \alpha \hbar^2/4m$, from which (7.1) can be derived, holds in general even when a potential term $V(\hat{q})$ is present in the Hamiltonian.

[21]E. Nelson, Phys. Rev. **150**, 1070 (1966).