# Classical extended charge subjected to linear forces and Rayleigh-Jeans radiation

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We study a rigid classical extended charge in the nonrelativistic approximation, first subjected to a linear force, and second immersed in an electromagnetic radiation with a Rayleigh-Jeans (RJ) spectrum. A Yukawa distribution is considered for the charge, when necessary, to get explicit results. A comparison with the Abraham-Lorentz (AL) model is made. Our results show that the AL model is a good approximation for the extended charge only if the external forces do not contain high frequencies. However, if we consider RJ radiation big discrepancies appear. We also find that the linear system follows the Maxwell-Boltzmann law only for large enough values of the radius.

## I. INTRODUCTION

The problem of the dynamics of charged particles, i.e., electrons, in the framework of classical electrodynamics, has never ceased to interest physicists, even after the development of the theory initiated by Dirac[1] in 1938, essentially completed by Haag in 1955, and afterwards by Rohrlich[2] and Teitelboim.[3] The reason for this lies in the well-known anomalous effects (runaway solutions, preacceleration, infinite mass, etc.) displayed by the Lorentz-Dirac (LD) equation, and also in some features related to the derivation of such an equation. Because of all that, many authors[4] have tried to elaborate alternative theories for the electron in order to avoid those difficulties. An interesting review and analysis of the several models can be found in Ref. 5. One of these theories, already developed by Abraham in the beginning of the century, considers the electron as an extended charge. Most of the studies devoted to this problem deal with a spherically symmetric rigid charge in the nonrelativistic approximation.[6–11] When the radius of the charge tends to zero, this model converges to the Abraham-Lorentz (AL) one. However, the validity of this limiting process is not clear because of the appearance of infinities. The AL equation rather could be considered as a good approximation for a real extended charge. But, it is difficult to estimate the magnitude of the terms neglected in this approximation that contain derivatives of the acceleration of order larger than one. One of the aims of this paper is to profoundly analyze this problem by studying the exact solutions for an extended charge, first subjected to a linear force, and second immersed in addition in an electromagnetic radiation with Rayleigh-Jeans (RJ) spectrum. We also study this interaction of a harmonic oscillator with RJ radiation to get some insight of the problem of the radiative equilibrium for an extended charge. In order to obtain explicit results we consider a Yukawa distribution for the charge, although some of them are, as it will be seen, general. The extension of our results to arbitrary distributions will be considered in future work.

As concerns the model for the extended charge different approaches have been considered. Some of them make use of the Lorentz self-force in order to account for the radiation reaction (see, for instance, Refs. 6, 7, and 10). However, it turns out that the resulting equation does not possess Lorentz invariance. This has led some authors[8,9] to perform covariant derivations in order to obtain a Lorentz-invariant equation of motion. (For a deeper analysis of this point, see Refs. 11 and 12.)

Nevertheless, the problem is not closed. The reason is that one of the main problems of the extended-charge models, namely, the problem of the "cohesive forces," is not solved but rather eluded in all mentioned papers. Obviously only a model including these forces can be fully satisfactory. However, if one assumes that the "cohesive forces" do not affect some global motions of the charge, as, for instance, the motion of its "center" (suitably defined), the equation governing those motions and including only the electromagnetic forces must be Lorentz invariant.

In this context, the model we are going to deal with is precisely the nonrelativistic limit of the Lorentz-invariant relativistic model devised in Refs. 8 and 9. Some general features of this model are studied in this paper. An important property is the existence of a critical radius $r_e^0$ such that for radius $r_e$ larger than $r_e^0$ the solution is unique, whereas when $r_e < r_e^0$ the solution is not determined and it can display preacceleration and/or runaway behavior. Assuming that the extended-charge model with $r_e > r_e^0$ is exact, we study the "validity" of the AL equation by comparing the solutions obtained from both theories.

We show that the AL model is a good approximation only if the external forces do not contain high frequencies. On the contrary, if we consider external forces with high frequencies, i.e., RJ radiation, a strong interaction appears which leads to appreciable changes in the trajectories at least as concerns the acceleration and its derivatives. We also find that the linear system follows the Maxwell-Boltzmann law only for values of $r_e$ much larger than $r_e^0$. These results could be very important for a future study of the radiative equilibrium for an extended charge.

## II. THE CHARGE MODEL

The equation of motion for a nonrelativistic rigid charge with spherical symmetry has been derived both in a fully nonrelativistic way[6,7,10,11] and from the corresponding relativistic covariant equation.[8,9] As we have said in the Introduction we follow the second approach.

The equation is, for the one-dimensional case,

$$m\ddot{x}=F-m_1\int_{-\infty}^{t}\gamma(t-t')[\ddot{x}(t')-\ddot{x}(t)]dt' \tag{2.1a}$$

or alternatively

$$m_1\ddot{x}=F-m_1\int_{-\infty}^{t}\gamma(t-t')\ddot{x}(t')dt' , \tag{2.1b}$$

where $\gamma(t)$ accounts for the retarded effect of the charge on itself and is given by

$$\gamma(t)=\frac{8\pi e^2}{3m_1}t\int d^3r'\rho(r')\rho(|\mathbf{r}+\mathbf{r}'|) \tag{2.2a}$$

$$=\frac{32\pi^2e^2}{3m_1c^3}\int_0^\infty d\omega\,\hat{\rho}^2(\omega)\omega\sin\omega t, \quad |\mathbf{r}|=ct , \tag{2.2b}$$

$\hat{\rho}$ being the Fourier transform of the distribution:

$$\hat{\rho}(\omega)=\frac{1}{(2\pi)^{3/2}}\int d^3r\,\rho(r)e^{i\mathbf{k}\cdot\mathbf{r}}, \quad \omega=ck . \tag{2.3}$$

$F$ in Eqs. (2.1a) and (2.1b) accounts for the effect of the external force $F_{\rm ext}$ and is given by the $x$ component of

$$\mathbf{F}(\mathbf{r},t)=\int d^3\xi\,\mathbf{F}_{\rm ext}(\mathbf{r}+\xi,t)\rho(\xi) . \tag{2.4}$$

As concerns $m_1$, if we denote the mechanical and electromagnetic masses by $m_0$ and $m_e$, respectively, we have

$$m_1=m_0-\tfrac{1}{3}m_e . \tag{2.5}$$

Now, if we call

$$\epsilon=\int_0^\infty dt\,\gamma(t), \quad \tau=\frac{2e^2}{3mc^3} , \tag{2.6}$$

the following relations hold:

$$m=m_0+m_e=m_1(1+\epsilon) , \tag{2.7a}$$

$$\int_0^\infty dt\,\gamma(t)t=(1+\epsilon)\tau . \tag{2.7b}$$

As we have said in the Introduction, we consider, when no general results are immediate, a Yukawa-type distribution:

$$\rho(r)=\frac{1}{4\pi r_e^{\,2}}\frac{e^{-r/r_e}}{r} . \tag{2.8}$$

In this case the former expressions read

$$\hat{\rho}(\omega)=\frac{\sigma^2}{(2\pi)^{3/2}}\frac{1}{(\sigma^2+\omega^2)} , \tag{2.9}$$

$$\gamma(t)=\sigma^2\epsilon te^{-\sigma t} \quad (t\geq 0) , \tag{2.10}$$

$$m_e=\frac{e^2\sigma}{4c^3}, \quad \epsilon=\frac{e^2\sigma}{3m_1c^3} , \tag{2.11}$$

where

$$\sigma=\frac{c}{r_e}\equiv\frac{1}{\tau_e} . \tag{2.12}$$

Using Eqs. (2.7b) and (2.10) we get

$$\frac{\epsilon}{1+\epsilon}=\tfrac{1}{2}\sigma\tau \tag{2.13}$$

from which one can see that there exists a crucial value for the radius

$$r_e^0=\tfrac{1}{2}c\tau=\frac{e^2}{3mc^2} \tag{2.14}$$

corresponding to $\epsilon=\infty$, $m_1=0$, and $m_1\epsilon=m$, and such that, when $r_e>(<)\ r_e^0$ we have $m_1>(<)\ 0$. As we shall see this fact is decisive in the behavior of the solutions.

Obviously the value of the radius cannot be arbitrarily large. This imposes restrictions on the reasonable values of $\epsilon$. If we require the radius to be much smaller than the Bohr radius, that is, $r_e \ll 5\times 10^4 F$, we obtain, for $\epsilon$,

$$\epsilon \gg 2\times 10^{-5} . \tag{2.15}$$

Finally, a remark concerning the point-particle limit has to be made. If we perform this limit by taking $\rho(\mathbf{r})=\delta^{(3)}(\mathbf{r})$, we recover the AL equation from (2.1a). However, this is a formal procedure and it is necessary to analyze the behavior of the solutions in this limit. This will be made in subsequent sections.

## III. SOME FEATURES OF THE MODEL

In order to see whether this model lacks the inconvenience of the AL one, it is usual to study the existence of runaway solutions and preacceleration, as is made, for instance, in Refs. 6–11. However our point of view about some of these topics differs from the one shown in those papers. Moreover, a few of the points we are interested in either do not appear or are incompletely studied in the mentioned works. Our aim in this section is to study the problem posed above by showing the exact behavior for the case of the Yukawa-type charge. We also clarify some of the results existing in the literature.[6–11]

First of all we study the possible solutions of the homogeneous equation and then consider the behavior under a time-dependent force.

### A. Free particle

Strictly speaking this problem has only been studied in Ref. 10 for a general distribution. The result appearing in this work can be slightly generalized in the following way.

If $\rho$ has a constant sign, $0<\epsilon<1$ and the acceleration is bounded at $t\to-\infty$, the unique solution of the homogeneous equation is the trivial one: $\ddot{x}=0$.

In Ref. 10 the boundedness of $\ddot{x}$ at $t\to-\infty$ is substituted by a stronger condition, namely, that $\ddot{x}\to 0$ when $t\to-\infty$.

The proof of the statement is as follows. The equation of motion is written

$$\ddot{x}=-\int_{-\infty}^{t}dt'\gamma(t-t')\ddot{x}(t') . \tag{3.1}$$

If $\ddot{x}$ is bounded when $t\to-\infty$, there exists $t_0$ such that $|\ddot{x}(t_0)|=M$, and $|\ddot{x}(t)|\leq M\ \forall t\leq t_0$. Then we have, from Eq. (3.1),

$$M=|\ddot{x}(t_0)|=\left|\int_{-\infty}^{t_0}dt'\gamma(t_0-t')\ddot{x}(t')\right|$$
$$\leq\int_{-\infty}^{t_0}dt'\,|\gamma(t_0-t')|\,M=M\,|\epsilon|\ ,$$

where we have taken into account that according to (2.2a) $\gamma(t)$ has a definite sign. Now, it is clear that for $|\epsilon|<1$ only $\ddot{x}(t)=0$ can be a solution of (3.1). Furthermore, since for $-1<\epsilon<0$, $m<0$, the condition $|\epsilon|<1$ reduces to $0<\epsilon<1$ which ends the proof.

As we have indicated this result is valid for a general distribution. When $|\epsilon|\geq 1$ there can exist in general nontrivial solutions. We are now going to analyze the situation for a Yukawa distribution.

If we restrict ourselves to solutions that are bounded at $t\to-\infty$, we can use the Laplace transform of $\alpha(t)=\ddot{x}(-t)$ to get, when $\epsilon>0$,

$$\alpha(t)=\tilde{\alpha}'(\sigma)\sigma^2\epsilon e^{\sigma t}\cos(\sigma\sqrt{\epsilon}t)$$
$$+\tilde{\alpha}(\sigma)\sigma\sqrt{\epsilon}e^{\sigma t}\sin(\sigma\sqrt{\epsilon}t)\ , \tag{3.2}$$

which is not bounded at $t=\infty$ unless $\tilde{\alpha}(\sigma)=\tilde{\alpha}'(\sigma)=0$, that is $\ddot{x}(t)=\alpha(-t)=0$.

On the contrary, for $\epsilon<0$ we obtain

$$\alpha(t)=\tilde{\alpha}'(\sigma)\sigma^2\epsilon e^{\sigma t}\cosh(\sigma\sqrt{|\epsilon|}\,t)$$
$$+\tilde{\alpha}(\sigma)\sigma\sqrt{|\epsilon|}\,e^{\sigma t}\sinh(\sigma\sqrt{|\epsilon|}\,t)\ , \tag{3.3}$$

which gives the following solution bounded when $t\to\infty$:

$$\alpha(t)=\text{const}\times\exp[-\sigma(\sqrt{|\epsilon|}-1)t]\ . \tag{3.4}$$

Consequently, for the Yukawa distribution nontrivial solutions of the homogeneous equation (3.1) appear if and only if $\epsilon<0$ (in fact $\epsilon<-1$), and such solutions are of the form given by

$$\ddot{x}(t)\equiv\alpha(-t)=\text{const}\times\exp[\sigma(\sqrt{|\epsilon|}-1)t]\ , \tag{3.5}$$

which clearly shows a runaway behavior. No self-oscillating solutions occur when a Yukawa distribution is considered (see Ref. 7).

### B. Time-dependent external force

In this section we consider a time-dependent external force that is connected at $t=0$. In this case we have again two different behaviors depending on the sign of $\epsilon$. However, as we shall see, the only cause for this difference lies on the features of the homogeneous equation indicated above.

The first important point concerns the uniqueness of solutions of the equation of motion (2.1b) which can now be written as

$$\ddot{x}(t)=\phi(t)-\int_0^t dt'\gamma(t-t')\ddot{x}(t') \tag{3.6}$$

with

$$\phi(t)=\frac{F(t)}{m_1}-\int_{-\infty}^{0}dt'\gamma(t-t')\ddot{x}(t')\ . \tag{3.7}$$

Note that we admit $\ddot{x}(t)\neq 0$ for $t<0$. For a Yukawa distribution it is easy to prove the following: If we know the motion for $t<0$ the solution of (3.6) is unique.

To prove this assertion we introduce the new variable

$$\xi(t)=\int_0^t dt'\gamma(t-t')\ddot{x}(t')\ . \tag{3.8}$$

With very simple calculations one can prove that Eq. (3.6) is equivalent to the following system of ordinary differential equations:

$$\ddot{x}=\phi(t)-\xi(t)\ , \tag{3.9a}$$

$$\ddot{\xi}+2\sigma\dot{\xi}+\sigma^2\xi=\sigma^2\epsilon\ddot{x}\ , \tag{3.9b}$$

plus the conditions

$$\xi(0)=\dot{\xi}(0)=0\ . \tag{3.9c}$$

The uniqueness is now immediate, and we look for the solution using the Laplace transform.

For $\epsilon>0$, $\ddot{x}(t)=0$ if $t<0$ as was seen in the preceding section, and the solution of (3.6) results in (see Ref. 11)

$$\ddot{x}(t)=\frac{F(t)}{m_1}-\int_0^t dt'\chi(t-t')\frac{F(t')}{m_1} \tag{3.10}$$

with

$$\chi(t)=\sigma\sqrt{\epsilon}e^{-\sigma t}\sin(\sigma\sqrt{\epsilon}t)\ . \tag{3.11}$$

When $F(t)$ is bounded, it is clear from Eqs. (3.10) and (3.11) that $\ddot{x}$ is also bounded. Consequently, for $\epsilon>0$, neither runaway nor preacceleration phenomena occur. Furthermore, if $\epsilon\ll 1$, $\ddot{x}\simeq F(t)/m$ and the integral term in (3.10) can be considered as perturbative.

In the case $\epsilon<0$ there can be nontrivial solutions for $t<0$, which are of the form given by (3.5). The solution for $t>0$ can be found writting $\phi(t)$ instead of $F(t)/m_1$ and

$$\bar{\chi}(t)=\sigma\sqrt{|\epsilon|}\,e^{-\sigma t}\sinh(\sigma\sqrt{|\epsilon|}\,t) \tag{3.12}$$

instead of $\chi(t)$, in (3.10). Thus we obtain, for the general solution,

$$\ddot{x}(t)=\begin{cases}A\exp[\sigma(\sqrt{|\epsilon|}-1)t],\quad t<0\ , & (3.13\text{a})\\[4pt] \dfrac{F(t)}{m_1}-\displaystyle\int_0^t dt'\bar{\chi}(t-t')\frac{F(t')}{m_1}\\ \qquad +A\exp[\sigma(\sqrt{|\epsilon|}-1)t],\quad t>0\ , & (3.13\text{b})\end{cases}$$

where the constant $A$ is completely arbitrary. Consequently, a nonvanishing acceleration can appear before the force is switched on. However, this has nothing to do with the preacceleration phenomenon occurring in the AL model that is strictly caused by the force. On the contrary, in our model the acceleration may or may not exist at time $t<0$ whether or not the force is switched on. This is why we do not consider this phenomenon as a preacceleration. The reason for it is the existence of solutions of the homogeneous equation (in fact an infinity of solutions). It can be seen that in Ref. 9 the term that is claimed to violate causality has exactly the form given by (3.5) with a constant depending on the force. This corresponds to a particular choice of the constant $A$ in (3.13) but does not give the general solution.

As can be seen in (3.13) the system shows in general a runaway behavior. However it is possible in some cases to eliminate this behavior with a proper choice of $A$, but this leads to the existence of preacceleration. This choice corresponds to the one used in the AL model. However, in the extended model all values of $A$ are allowed.

Another strange feature of the model for $\epsilon<0$ must be pointed out. If we consider that $\ddot{x}=0$ for $t<0$, we see in Eq. (2.1b) that the initial inertia shown by the system is given by $m_1$ which is negative. We conclude that the behavior of the system for $\epsilon<0$ escapes from physical intuition and consequently a deeper study of this situation is necessary.

To end this section we want to make some comments concerning previous papers. As we have indicated above, the equation of motion obtained in some of these works[6,7,10] differs from ours in that an amount of $\frac{1}{3}m_e\ddot{\mathbf{r}}$ is missing in the expression of the radiation reaction (for details see Refs. 9 and 11). In these papers the equation of motion is written like (2.1b), but with $m_0$ instead of $m_1$. As a consequence the anomalous effects would appear only for $m_0<0$ which is physically unacceptable. (See Ref. 6 for a discussion of this point.) However, in our case, we see from (2.5) that it is possible to have $m_0>0$ and $m_1<0$ and so anomalous effects can appear.

## IV. STUDY OF A HARMONIC OSCILLATOR

When the extended charge is submitted to an external force, it is usually assumed[8,9,11] that this force remains essentially unchanged within the dimensions of the charge. In this way the force is substituted by its value at the center of the charge. For a linear force $F=-m\omega_0^2x$ that means that

$$\Delta F\sim m\omega_0^2 r_e \ll F\sim m\omega_0^2 x \tag{4.1}$$

and then the radius must be small compared with the dimensions of the trajectory. In fact, very small trajectories would require more detailed models for the charge structure. Now, if we combine (4.1) with the nonrelativistic approximation, i.e., $\omega_0 x\ll c$, we get $\omega_0 r_e\ll c$, and using $\sigma=r_e/c$,

$$\omega_0/\sigma\ll 1 \ . \tag{4.2}$$

In the following we shall assume that (4.2) is always satisfied. Under these approximations we study in this section an extended charge subjected to a linear force that sets in at $t=0$. Taking into account the results obtained in Sec. III, we consider $r_e>r_e^0$, that is, $\epsilon>0$ $(m_1>0)$. Then, we have $\ddot{x}(t)=0$ for $t<0$, and from (2.1b)

$$\ddot{x}(t)=-\omega_1^2x(t)-\int_0^t dt'\gamma(t-t')\ddot{x}(t'),\quad t>0\ , \tag{4.3}$$

where [see (2.7a)]

$$\omega_1^2\equiv\frac{m}{m_1}\omega_0^2=(1+\epsilon)\omega_0^2\ . \tag{4.4}$$

In the same way as for (3.6) it is easy to show that the initial conditions $x_0, v_0$ determine the solution of (4.3). Using the Laplace transform we obtain the expressions

$$x(t)=x_0\chi_1(t)+v_0\chi_0(t)\ , \tag{4.5a}$$

$$v(t)=v_0\chi_1(t)-x_0\omega_0^2(1+\epsilon)\chi_2(t)\ , \tag{4.5b}$$

where

$$\chi_0(t)=\int_0^t du\,\chi_1(u)\ , \tag{4.6a}$$

$$\chi_1(t)=1-\omega_0^2(1+\epsilon)\int_0^t du\,\chi_2(u)\ , \tag{4.6b}$$

$$\chi_2(0)=0,\quad \dot{\chi}_2(0)=1\ , \tag{4.6c}$$

$$\tilde{\chi}_2(s)=\frac{(s+\sigma)^2}{[\omega_0^2(1+\epsilon)+s^2](s+\sigma)^2+s^2\sigma^2\epsilon}\ . \tag{4.6d}$$

A straightforward but lengthy calculation permits us to write explicitly the functions $\chi_i(t)$ $(\epsilon>0)$:

$$\chi_i(t)=(A_i\cos\bar{\omega}t+B_i\sin\bar{\omega}t)e^{-t/\tau_r}+(C_i\cos\sigma_v t+D_i\sin\sigma_v t)e^{-\sigma_r t}\ , \tag{4.7}$$

where

$$\bar{\omega}=\omega_0[1+O((\omega_0/\sigma)^2)]\ , \tag{4.8a}$$

$$1/\tau_r=\omega_0[\tfrac{1}{2}\omega_0\tau+O((\omega_0/\sigma)^3)]\ , \tag{4.8b}$$

$$\sigma_r=\sigma\left[-1+\frac{1}{2}\frac{\omega_0^2\tau}{\sigma}+O((\omega_0/\sigma)^3)\right]\ , \tag{4.8c}$$

$$\sigma_v=\sigma\sqrt{\epsilon}\left[1-\frac{\omega_0^2(1-\epsilon)}{2\sigma^2(1+\epsilon)}+O((\omega_0/\sigma)^3)\right]\ , \tag{4.8d}$$

and the coefficients are, to the lowest order in $\omega_0/\sigma$,

$$\begin{aligned}&A_0=\frac{4\epsilon(\epsilon-1)}{(1+\epsilon)^3}\frac{\omega_0^2}{\sigma^3},\quad B_0=1/\omega_0\ ,\\ &C_0=\frac{4\epsilon(1-\epsilon)}{(1+\epsilon)^3}\frac{\omega_0^2}{\sigma^3},\quad D_0=\frac{\sqrt{\epsilon}(1+\epsilon^2-6\epsilon)}{(1+\epsilon)^3}\frac{\omega_0^2}{\sigma^3}\ ;\end{aligned} \tag{4.9a}$$

$$\begin{aligned}&A_1=1,\quad B_1=-\frac{\epsilon}{1+\epsilon}\frac{\omega_0}{\sigma}=-\tfrac{1}{2}\omega_0\tau\ ,\\ &C_1=\frac{\epsilon(\epsilon-3)\omega_0^2}{(1+\epsilon)^2\sigma^2},\quad D_1=\frac{\sqrt{\epsilon}(3\epsilon-1)}{(1+\epsilon)^2}\frac{\omega_0^2}{\sigma^2}\ ;\end{aligned} \tag{4.9b}$$

$$\begin{aligned}&A_2=\frac{2\epsilon}{\sigma(1+\epsilon)^2},\quad B_2=\frac{1}{\omega_0(1+\epsilon)}\ ,\\ &C_2=\frac{-2\epsilon}{\sigma(1+\epsilon)^2},\quad D_2=\frac{\sqrt{\epsilon}(\epsilon-1)}{\sigma(1+\epsilon)^2}\ .\end{aligned} \tag{4.9c}$$

We also consider the expressions for $\dot{\chi}_2$ and $\ddot{\chi}_2$ that will be useful later. Obviously they have the same form as $\chi_i$ with other coefficients. Using the subscripts 3 and 4, respectively, for $\dot{\chi}_2$ and $\ddot{\chi}_2$, we get

$$\begin{aligned}&A_3=\frac{1}{1+\epsilon},\quad B_3=\frac{-3\epsilon\omega_0}{(1+\epsilon)^2\sigma}\ ,\\ &C_3=\frac{\epsilon}{1+\epsilon},\quad D_3=\frac{\sqrt{\epsilon}}{1+\epsilon}\ ;\end{aligned} \tag{4.9d}$$

$$\begin{aligned}&A_4=\frac{-4\epsilon\omega_0^2}{(1+\epsilon)^2\sigma},\quad B_4=\frac{-\omega_0}{1+\epsilon}\ ,\\ &C_4=\frac{4\epsilon\omega_0^2}{(1+\epsilon)^2\sigma},\quad D_4=-\sigma\sqrt{\epsilon}\ .\end{aligned} \tag{4.9e}$$

From (4.7)–(4.9) it is easy to analyze the behavior of the solutions. The two first terms oscillate with a frequency very close to $\omega_0$, and they decay rather slowly. (In a period its value remains essentially unchanged since $\omega_0\tau/2 < \omega_0/\sigma \ll 1$.) On the contrary, the two last terms are highly oscillating with a frequency $\sigma\sqrt{\epsilon}$, which will most generally be much larger than $\omega_0$. [If we consider that $\epsilon > 10^{-3}$ (see Sec. II), we have, up to frequencies $\omega_0 \simeq 10^{18}$ sec$^{-1}$, $(\omega_0/\sigma)^2 < 10^{-4} \ll \epsilon$.] These terms decay very quickly with a characteristic time given by $\sigma^{-1} \ll T$.

It is clear from the above expressions that the magnitude of the highly oscillating terms with frequency $\sigma_v$ increase when differentiating. But they decay very quickly, and then they are not important at large times. However, as we shall see later, in the presence of RJ radiation this situation changes and these terms become very important.

Once we have obtained the solutions for the extended charge oscillator, we compare them with the ones corresponding to the uncharged oscillator and to the AL equation. In this way we show the perturbative character of the damping, and we analyze to what extent the AL model is an approximation to the extended one.

### A. Comparison with the uncharged oscillator

It is easy to conclude from (4.9a)–(4.9c) that, neglecting terms of order $\omega_0/\sigma$, the trajectories in phase space for the uncharged oscillator and our model [see (4.5)] coincide during many periods before the decay of the charge to the origin, $x = v = 0$, becomes appreciable. This damping effect is important at times of order $\tau_r \gg T$. We can say that, in this sense, the damping is perturbative. However, because of the highly oscillating terms, the acceleration is very different when $\epsilon$ is not small. But, this only happens for very short times not greater that $\sigma^{-1} \ll T$, and then we can still consider perturbative the effect of the "damping." However, as we shall see later this kind of effect will be important for a charge immersed in a RJ radiation.

### B. Comparison with the Abraham-Lorentz solution

The AL equation

$$\ddot{x} = -\omega_0^2 x + \tau \dddot{x} \tag{4.10}$$

has three independent solutions, one of which goes as $e^{t/\tau}$. If we impose the additional condition[2]

$$\ddot{x} \underset{t\to\infty}{\longrightarrow} 0 ,$$

we keep the other two solutions, since they decay for $t \to \infty$. The general solution is a combination of exponentials whose coefficients can be obtained in powers of the parameter $\delta = \omega_0\tau$ that satisfies

$$\delta = \omega_0\tau = \frac{2\epsilon}{1+\epsilon}\frac{\omega_0}{\sigma} < \frac{2\omega_0}{\sigma} \ll 1 , \tag{4.11}$$

where we have used (2.13) and (4.2). The solution is given up to order $\delta$ by

$$x(t) = e^{-\omega_0\delta t/2}\left[x_0\left[\cos\omega_0 t + \frac{\delta}{2}\sin\omega_0 t\right] + \frac{v_0}{\omega_0}\sin\omega_0 t\right] , \tag{4.12a}$$

$$v(t) = e^{-\omega_0\delta t/2}[v_0\cos\omega_0 t - (x_0\omega_0 + v_0\delta/2)\sin\omega_0 t] . \tag{4.12b}$$

We observe that highly oscillating terms do not appear in (4.12). As concerns the other two terms appearing in $\chi_i$ ($i = 0,1,2$) the frequency and the decay time coincide with the ones of the extended charge model up to order $\omega_0/\sigma$ [see (4.8) and (4.11)]. This result holds for any model, since (2.7b) is verified for any $\rho$. [Note that the poles of (4.6d) can be obtained developing $\tilde{\gamma}(z)$ around $z = 0$.] As concerns the coefficients, they coincide only at zero order in $\omega_0/\sigma$, as it can be seen from (4.9) and (4.12). We conclude then that the AL equation approximates the extended charge model if we do not consider the runaway solutions.

It is also straightforward to show that the damping integral term of Eq. (2.1a) can be approached, for the exact solution (4.5), by $m\tau\dddot{x}$, corresponding to Eq. (4.10), whenever we do not consider too short times. This is obvious, since if the force sets in instantaneously at $t = 0$, the acceleration changes suddenly at this time. Then, it is not possible to develop in a Taylor series until a time $t \gg \sigma^{-1}$ elapses, since then the behavior at times $t < 0$ has been forgotten. Finally, we note that many derivations, in classical[12] and quantum[6(a),13] frameworks, of the equation of motion for the extended charge are based on a development of the integral damping using a Taylor series of the acceleration. It is then interesting to analyze the convergence of this damping expansion. It is shown in the Appendix that the development is absolutely convergent for the oscillator. However, as we shall see in the next section, it is not always so.

## V. STUDY OF AN EXTENDED CHARGE IN A RAYLEIGH-JEANS RADIATION

The interest of studying this problem lies on the fluctuation-dissipation property verified by our extended charge model in a RJ radiation. This fact leads us to conjecture that the extended charge enclosed in a box attains equilibrium with the RJ radiation. Then, the classical results for the blackbody would be reproduced. Nevertheless, it is obvious that nonlinear systems must be studied to get a definite answer for this conjecture. For the time being we just analyze some revealing aspects of the extended charge RJ radiation interaction.

We divide this section into three parts. We define first the model we shall use in the following for the radiation field. Afterwards we study the interaction of this field with the free charge, and finally with a harmonic oscillator. Once we obtain the general solution for both cases, we shall devote the rest of the sections to analyze two different aspects of the model.

On the one hand, it is usually claimed that the effect of the RJ radiation and the damping is a perturbation upon the motion due to the conservative force. In order to clarify this point we shall study two effects in the case of the oscillator; namely, (a) the perturbation of the radiation over the phase-space variables, $x$ and $v$, and (b) the effect over the acceleration, in connection with the interchange of energy between the charge and the radiation. In fact we shall prove that (a) the perturbation in phase-space is important at times of the order of the period unless $\epsilon \ll 1$, and (b) the acceleration is predominantly given by the corresponding to the radiation field, and consequently the charge and the radiation interchange a great amount of energy.

On the other hand, we want to clarify to what extent the AL model gives a good approximation to our model. We shall prove concretely that in the presence of RJ radiation the integral damping of Eq. (2.1a) cannot be approximated by the damping of the AL equation, $m\tau\dddot{x}$, which makes the latter clearly invalid. We also show that a Taylor-series expansion of the acceleration in the integrand of the damping, which is sometimes claimed to justify the AL equation, is not valid.

### A. The model for the radiation field

If the radiation is produced by many charges, the only way to treat it is as a stochastic process. The model we shall use is based on the "natural radiation," an idea that was introduced by Planck[14] in his research on the black-body radiation. He considered the field as a linear combination of plane waves with phases and amplitudes randomly distributed. Taking into account that the field is produced by a huge number of charges, and imposing homogeneity and isotropy, the radiation is modeled by considering the coefficients of the plane waves as independent Gaussian random variables with zero mean value.[15] The radiation field turns out to be a Gaussian process with zero mean and its spectrum (the Fourier transform of the correlation function) is related to the energy density of the field, $u(\omega)$, in the form[16]

$$S(\omega)=\frac{4\pi^2}{3}u(\omega)=\frac{4\pi^2\omega^2}{3c^3}\mathscr{Y}^2(\omega) . \tag{5.1}$$

The quantity $\mathscr{Y}^2(\omega)$ characterizes the radiation field. For the RJ radiation it is constant: $\mathscr{Y}^2=k_BT/\pi^2$.

Now, we analyze the interaction of the radiation field with the extended charge. In the nonrelativistic approximation we can neglect the magnetic field. Then, the force $F^{\rm st}$ on the charge due to the field is given by (2.4), where $F_{\rm ext}$ is replaced by $eE$. In this case we cannot approach this expression by its value at the center of the charge, because the radiation contains indefinitely high frequencies $\omega$ with an intensity proportional to $\omega^2$, and then the field changes appreciably within the dimensions of the charge. Therefore, the force $F^{\rm st}$ is a Gaussian random process with zero mean, and its spectrum is given in the dipole approximation by[17]

$$S^{\rm eff}(\omega)=e^2S(\omega)8\pi^3\hat{\rho}^2(\omega) . \tag{5.2}$$

For the Yukawa distribution and the RJ radiation we get from (2.9) and (5.2), the effective spectral density

$$S^{\rm eff}(\omega)=\text{const}\times\frac{\omega^2}{(\sigma^2+\omega^2)^2} , \tag{5.3}$$

that is integrable. This is a characteristic feature of the extended charge. The reason for the cutoff in $S^{\rm eff}(\omega)$ is that the components of the radiation field with very short wavelength are not "seen" by the charge. The force acting on the charge has, unlike the pointlike case, a well-defined correlation function. This allows a rigorous treatment of the problem.

We end this section by noting that the following relation holds for the RJ radiation:

$$\begin{aligned} B(t)&=\langle F_i^{\rm st}(t_1+t)F_i^{\rm st}(t_1)\rangle \\ &=k_BTm_1\dot{\gamma}(|t|) \end{aligned} \tag{5.4}$$

[see (2.2) and (5.2)]. This is a fluctuation-dissipation property.[18] Note that the integral term of Eq. (2.1b) can be written by integrating by parts as

$$\int_{-\infty}^{t}dt'\gamma(t-t')\ddot{x}(t')=-\gamma(t)\dot{x}_0+\int_0^t dt'\dot{\gamma}(t-t')\dot{x}(t') , \tag{5.5}$$

where we have assumed that $a(t)=0$ for $t<0$.

### B. Free charge immersed in a RJ radiation field

In the following we assume that $\epsilon>0$, and the fields set in at $t=0$. The equation of motion is (3.6), where $\phi(t)$ must be replaced by $F^{\rm st}(t)$, the stochastic process defined above. The solution is given by (3.10) and (3.11). From (3.10) we obtain the velocity

$$v(t)=v_0+\frac{1}{m_1}\int_0^t dt'\varphi(t-t')F^{\rm st}(t') , \tag{5.6}$$

where

$$\varphi(t)=\frac{1}{1+\epsilon}[1+e^{-\sigma t}(\epsilon\cos\sigma\sqrt{\epsilon}t+\sqrt{\epsilon}\sin\sigma\sqrt{\epsilon}t)] . \tag{5.7}$$

Because of the properties of $F^{\rm st}$ and the linearity of the equation of motion, $v(t)$ is a Gaussian process with zero mean. Its mean quadratic velocity can be obtained from (5.4), (5.6), and (5.7). A little algebra yields

$$\langle v^2(t)\rangle=v_0{}^2+\frac{k_BT}{m_1}[1-\varphi^2(t)] . \tag{5.8}$$

The probability distribution for $v$ is then given by

$$P(v,t)=\text{const}\times\exp(-(v-v_0)^2/\{2k_BTm_1{}^{-1}[1-\varphi^2(t)]\}) . \tag{5.9}$$

In the limit $t\to\infty$ we get

$$P_{\rm st}(v)=\text{const}\times\exp[-(v-v_0)^2/(\mu k_BTm_1{}^{-1})] , \tag{5.10}$$

where

$$\mu\equiv1-\varphi^2(\infty)=\frac{\epsilon^2+2\epsilon}{(1+\epsilon)^2} . \tag{5.11}$$

Note that these results are valid for any charge distribution.

We get then steady state for $v$; on the contrary the position increases indefinitely. Other properties of the free particle are essentially equivalent to the ones obtained for the oscillator, which is analyzed in the next section.

### C. Oscillator in a RJ radiation field

#### *1. Stationary state*

Using the notation of Sec. IV, keeping the condition (4.2) and $\epsilon>0$, the equation of motion is now

$$\ddot{x}=-\omega_1{}^2x-\int_0^t dt'\gamma(t-t')\ddot{x}(t')+\frac{F^{\rm st}(t)}{m_1}\ . \tag{5.12}$$

It can be shown, by using $\xi$ given by (3.8), that the solution of (5.12) is again determined by the initial conditions $x_0,v_0$. Using the Laplace transform and the functions $X_i$ $(i=0,1,2)$ [see (4.6d) and (4.5)], we get

$$x(t)=x_0X_1(t)+v_0X_0(t)+\int_0^t dt'X_2(t-t')\frac{F^{\rm st}(t')}{m_1}\ , \tag{5.13a}$$

$$v(t)=v_0X_1(t)-x_0\omega_0^2(1+\epsilon)X_2(t)+\int_0^t dt'\dot{X}_2(t-t')\frac{F^{\rm st}(t')}{m_1}\ . \tag{5.13b}$$

The position and velocity are then Gaussian processes whose mean values are given by the solutions without field (4.5). The covariance matrix is obtained[19] from (5.13):

$$C_{xx}(t)\equiv\langle x^2(t)\rangle-\langle x(t)\rangle^2=\frac{k_BT}{m_1}\left[\frac{1-X_1{}^2(t)}{\omega_1{}^2}-X_2{}^2(t)\right], \tag{5.14a}$$

$$C_{vv}(t)\equiv\langle v^2(t)\rangle-\langle v(t)\rangle^2=\frac{k_BT}{m_1}[1-\omega_1{}^2X_2{}^2(t)-\dot{X}_2{}^2(t)]\ , \tag{5.14b}$$

$$C_{xv}(t)=\langle x(t)v(t)\rangle-\langle x(t)\rangle\langle v(t)\rangle=\frac{k_BT}{m_1}X_2(t)[X_1(t)-\dot{X}_2(t)]\ . \tag{5.14c}$$

In the steady state $t\to\infty$, the mean values (4.5), and $X_1,X_2,\dot{X}_2$, decay to zero. Therefore, the stationary probability density is

$$P_{\rm ST}(x,v)={\rm const}\times\exp[-(\tfrac{1}{2}m_1v^2+\tfrac{1}{2}m\omega_0{}^2x^2)/(k_BT)]\ , \tag{5.15}$$

and the mean energy takes the value

$$\langle\mathscr{E}\rangle_{\rm ST}=k_BT\left[1+\frac{\epsilon}{2}\right]. \tag{5.16}$$

Note that for the nonrelativistic approximation to be valid, we must have $\langle\mathscr{E}\rangle_{\rm ST}\ll mc^2$. Then, for $r_e$ close to its minimum value, $r_e^0$ $(\epsilon\gg1)$, (5.16) leads to very restrictive values of the temperature of the system $T$. We now make some comments on (5.15). We first note that it is only valid for $\epsilon>0$. If $\epsilon<0$ the mean values of $x$ and $v$ diverge, due to the runaway solutions, and there is no stationary state. Second, when $\epsilon>0$ the result (5.15) is valid for any charge distribution, because only general properties of $X_i$ $(i=0,1,2)$ have been used.

We finally note that if $\epsilon\ll1$, then $m_1\simeq m$, and (5.15) coincides with the Maxwell-Boltzmann (MB) law. Therefore RJ and MB are compatible only if $\epsilon\ll1$, which is a reasonable value (see Sec. II). This is crucial as concerns statistical mechanics and the study of the blackdody radiation law. We find then that for some models there are discrepancies with the usual result of classical physics. (It must be kept in mind that the implication RJ$\Longrightarrow$MB is based on the AL or LD equations.) It remains an open problem to obtain the spectrum compatible with MB law. However, since the universal character of (5.15) is due to the fluctuation-dissipation property (5.4), it seems that with another spectrum the result will depend on the charge distribution. Consequently, in order to get the MB law, the spectrum should also be charge distribution dependent.

#### *2. Analysis of the interaction with the RJ radiation field*

*a. Study of the phase-space trajectories perturbation.* In order to analyze to what extent the damping and radiation terms of Eq. (5.12) can be considered as perturbations, we first compare the trajectories given by (5.13) with the ones corresponding to the uncharged oscillator for times larger than the period but shorter than $\tau_r$ [see (4.8b)]. To do this we calculate the quantities

$$\Delta_x=\frac{\langle(x-x_c)^2\rangle}{x_M{}^2},\quad \Delta_v=\frac{\langle(v-v_c)^2\rangle}{v_M{}^2}\ , \tag{5.17}$$

where $x_c,v_c$ are the solutions for the uncharged oscillator, and $x_M$ $(v_M)$ the maximum value of $x$ $(v)$.

According to the discussion in Sec. IV A, $x_c$ and $v_c$ coincide, for the times we are considering, and neglecting terms of order $\omega_0/\sigma$, with (4.5), i.e., with the mean values of $x$ and $v$. Since we may estimate $x_M$ and $v_M$ by $\omega_0x_M\sim v_M\sim(2\mathscr{E}_0/m)^{1/2}$, we get

$$\Delta_x\sim\frac{C_{xx}m\omega_0{}^2}{2\mathscr{E}_0},\quad \Delta_v\sim\frac{C_{vv}m}{2\mathscr{E}_0}\ , \tag{5.18}$$

where $\mathscr{E}_0$ is the initial energy and $C_{xx}$ and $C_{vv}$ are given by (5.14a) and (5.14b). Expressions (5.18) give an estimation of the mean deviation of the trajectory with respect to the "neutral" (uncharged) one for times $t\ll\tau_r$. It is obvious that if $\mathscr{E}_0$ is very small these two trajectories are very different, even for short times. On the contrary, when $\mathscr{E}_0$ is big enough there will never be appreciable deviation for $t\ll\tau_r$. Anyhow, what is important is the behavior of the trajectories with initial energies $\mathscr{E}_0$ similar to the mean energy (5.16) that give the most important contribution to the stationary state. Taking then $\mathscr{E}_0\sim\langle\mathscr{E}\rangle_{\rm ST}$, and using Eqs. (4.9b)–(4.9d) we get, for times $t\ll\tau_r$,

$$\Delta_x = \frac{\epsilon}{(1+\epsilon)(2+\epsilon)}\sin^2\omega_0 t + O\left[\frac{\omega_0}{\sigma}\right], \quad (5.19a)$$

$$\Delta_v = \frac{\epsilon^2+2\epsilon-\epsilon\sin^2\omega_0 t + N(t)}{(1+\epsilon)(2+\epsilon)} + O\left[\frac{\omega_0}{\sigma}\right], \quad (5.19b)$$

where $N(t)$ contains terms coming from $\dot{\chi}_2{}^2$, that decay as $e^{-\sigma_r t}$. Then, they are important only when $t \lesssim \sigma_r{}^{-1} \ll T$. It is clear from (5.19) that these quantities are small only if $\epsilon \ll 1$. When $\epsilon$ is not small, the trajectories are very different from the neutral ones, and the radiation field cannot be considered as a perturbation.

*b. Study of the acceleration perturbation.* It is also interesting to study the deviation for the acceleration:

$$\Delta_a = \frac{\langle (a-\langle a\rangle)^2\rangle}{\omega_0{}^4 x_0{}^2 + \omega_0{}^2 v_0{}^2} . \quad (5.20)$$

From (5.13b), we obtain after a straightforward calculation

$$\langle (a(t)-\langle a(t)\rangle)^2\rangle = \frac{B(0)}{m_1{}^2} + \frac{k_B T}{m_1}\{[1-\dot{\chi}_2{}^2(t)]\omega_1{}^2 - \ddot{\chi}_2{}^2(t)\}. \quad (5.21)$$

Using Eqs. (4.9d) and (4.9e), we get, for times $t \gtrsim 2\pi/\omega_0 \gg \sigma^{-1}$ but $t \ll \tau_r$ and $\mathcal{E}_0 \sim \langle \mathcal{E}\rangle_{\mathrm{ST}}$,

$$\Delta_a \simeq \frac{1+\epsilon}{2+\epsilon}\frac{\sigma^2\epsilon}{\omega_0{}^2} \quad (5.22)$$

that is never small, since $\sigma^2\epsilon/\omega_0{}^2 \ll 1$ would require an enormous radius for the electron. For instance, if $\omega_0/\sigma \simeq 10^{-2}$ and $\sigma^2\epsilon/\omega_0{}^2 \lesssim 10^{-1}$, we need $\epsilon \lesssim 10^{-5}$ that corresponds to $r_e$ bigger than the radius of the first Bohr orbit (see Sec. II).

*c. Analysis of the interchange of energy.* The results obtained in the foregoing sections lead us to think that the emission of energy by the charge, clearly related to $a$, is very intense, and then the interaction between the charge and the radiation is very strong. To see this, we calculate the emitted and absorbed power in the stationary state. Since the mean energy is conserved in the stationary state, we have $P_{a(\mathrm{ST})}+P_{e(\mathrm{ST})}=0$, and it is enough to obtain $P_{a(\mathrm{ST})}$, that is given from (5.13b) by

$$P_{a(\mathrm{ST})} = \lim_{t\to\infty}\langle v(t)F^{\mathrm{st}}(t)\rangle = \frac{1}{m_1}\int_0^\infty du\,\dot{\chi}_2(\omega)B(\omega) . \quad (5.23)$$

Using (2.10) and (5.4), we obtain

$$P_{a(\mathrm{ST})} = k_B T\sigma^2\epsilon\left[\frac{\partial}{\partial s}s^2\tilde{\chi}_2(s)\right]_{s=\sigma} = k_B T\sigma\frac{4\epsilon(\epsilon+8\lambda^2)}{(4+\epsilon)^2[1+4\lambda^2/(4+\epsilon)]^2} , \quad (5.24)$$

where $\lambda = (\omega_0/\sigma)(1+\epsilon)^{1/2}$. The expression (5.24) is exact. Since $\omega_0/\sigma \ll 1$, we develop $P_{a(\mathrm{ST})}$ in powers of $\lambda$:

$$P_{a(\mathrm{ST})} = k_B T\sigma\frac{4\epsilon}{(4+\epsilon)^2}\left[\epsilon + \frac{32\lambda^2}{4+\epsilon} + O(\lambda^4)\right] . \quad (5.25)$$

Note that the limit $\lambda\to 0$ corresponds to the free-particle case. From (5.25) we see that after a time

$$\tau_a \simeq \frac{(4+\epsilon)^2(1+\epsilon/2)}{4\epsilon^2}\sigma^{-1} \equiv \Omega\sigma^{-1} , \quad (5.26)$$

the absorbed energy turns out to be of the same order as the mean energy (5.16). $\Omega$ has a minimum for $\epsilon=6.5$ given by $\Omega_{\min}=2.8$. Then for many systems $\tau_a$ is of the same order as $\sigma^{-1} \equiv r_e/c$. This strong interaction is due to the high frequencies of order $\sigma$. The contribution of low frequencies to the absorbed power (5.23) is given by

$$P_a^L \sim k_B T\frac{2\epsilon}{1+\epsilon}\omega_0\left[\left[\frac{\omega_0}{\sigma}\right] + O\left[\left[\frac{\omega_0}{\sigma}\right]^2\right]\right] . \quad (5.27)$$

The energy exchange in a period, $P_a^L 2\pi/\omega_0$, is then a fraction of order $(\omega_0/\sigma) \ll 1$ of the mean energy. Therefore, as concerns low frequencies the interaction is weak.

### 3. Comparison with AL model

To end the analysis of the oscillator case we shall show that the damping cannot be approached by the AL expression, $m\tau\dddot{x}$. To do this, we calculate $m\tau\langle\dddot{x}(t)v(t)\rangle_{\mathrm{ST}}$ and compare it with the emitted power, $P_{e(\mathrm{ST})} = -P_{a(\mathrm{ST})}$ [cf. (5.24)]. From (5.21) we get

$$\langle m\tau\dddot{x}v\rangle_{\mathrm{ST}} = -m\tau\langle a^2\rangle_{\mathrm{ST}} = -2k_B T\sigma\epsilon(\epsilon+\lambda^2) . \quad (5.28)$$

It is clear that (5.28) never coincides with $P_{e(\mathrm{ST})}$ obtained from (5.24) ($P_{e(\mathrm{ST})} = -P_{a(\mathrm{ST})}$).

The emitted power given by the AL equation, $\langle m\tau\dddot{x}v\rangle$, can be considered as the first term of an expansion, obtained by using a Taylor series of the acceleration of the integral term. We have shown that this term does not approach the emitted power. As concerns the other terms of this expansion, it can be shown that the odd ones are divergent. Therefore, unlike the case without radiation field (see Sec. IV B and the Appendix), the expansion of the integral term does not make sense. To show those divergences we consider here the third term of this expansion:

$$-m_1\langle v(t)\dddot{a}(t)\rangle\frac{1}{3!}\int_0^t dt'\gamma(t-t')(t-t')^3 . \quad (5.29)$$

We use the relation

$$\langle v(t)\dddot{a}(t)\rangle = \frac{1}{2}\frac{d^4}{dt^4}\langle v^2(t)\rangle - 2\frac{d^2}{dt^2}\langle a^2(t)\rangle + \langle\dot{a}^2(t)\rangle , \quad (5.30)$$

where the two first terms are bounded and go to zero when $t\to\infty$ [see (5.13b), (5.14b), (5.21), and (4.7)]. Differentiating (5.13b), we see that $\langle\dot{a}^2\rangle$ contains a divergent term given by [see (5.1)–(5.3)]

$$\langle\dot{F}^{\mathrm{st}2}(t)\rangle/m_1{}^2 = \frac{32\pi^2 e^2}{3m_1{}^2 c^3}k_B T\int_0^\infty d\omega\,\omega^4\hat{\rho}^2(\omega) = \infty . \quad (5.31)$$

## VI. CONCLUSIONS AND DISCUSSION

We have shown that the extended charge model we have studied lacks the inconsistencies of the punctual model, but not for all the values of the radius $r_e$. This last point allows us to understand why runaway solutions and preacceleration appear in the AL equation. When $r_e < r_e^0$ the solutions become divergent and moreover there is no uniqueness. The first feature is obviously responsible for the runaway solutions of the AL equations, whereas the second one leads to a correct understanding of the phenomenon called "preacceleration." If we consider a given force $F$, the initial position and velocity do not determine the solution, and another condition is needed. Depending on this extra condition the solution can or cannot display this phenomenon. This indetermination, that leads to the existence of "preacceleration," and the runaway solutions clearly make unsatisfactory a model with $r_e < r_e^0$. We note that this analysis can be also applied to the LD equation, if we do not consider the additional condition, $a^\mu \to 0$ when $t \to \infty$.

The punctual model can approach in some cases the extended model with $r_e > r_e^0$. In this context we have shown that, at least for forces without very high frequencies, this is correct as concerns all the relevant quantities (coefficients, frequencies, relaxation times, etc.) for the phase-space coordinates. However, if we consider forces including high frequencies there are big discrepancies between the extended and punctual models. This is due to the order, $\tau_e^{-1} = \sigma$, of the characteristic frequency of the extended charge that leads to highly oscillating terms which can be excited by a force with high enough frequency. Then, an extended charged oscillator immersed in a RJ radiation field has a very strong interaction with the radiation, due to the high frequencies of order $\sigma$. This strong interaction with high frequencies is also basically responsible for the expansion in Taylor series of the acceleration inside the integral term to be incorrect. This expansion is again only valid for forces without high frequencies. Therefore, it is not possible to ground any theoretical model on this kind of development. An important result of our study of an extended charge in a RJ radiation is that if the electromagnetic mass of the charge is comparable to the mass $m$, i.e., $\epsilon \gtrsim 1$, neither the effect of the field on the phase-space trajectory can be considered as a perturbation, nor does the stationary distribution follow the MB law. These two facts, which seem to be related could be important in the study of the radiative equilibrium between radiation and matter.

We finally want to remark that most of the results we have obtained in this paper have not to be considered as intrinsic features of the nonrelativistic rigid spherically symmetric extended charge. In fact our results differ in some respects from the ones obtained in Refs. 6, 7, and 10, where another starting point has been considered for the derivation of the equation of motion. Both models are not fully satisfactory, and a theory including "cohesive forces" appears to be necessary. But even within the scheme of our model, it is obvious that different charge distributions will give different behaviors. A study of our model for a wide class of charge distributions is the aim of a different paper.


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## APPENDIX

In this appendix we show, by using the exact solution for the oscillator, that the "damping" term of Eq. (2.1a) can be developed by means of a Taylor series of the acceleration:

$$F^d = -m_1 \int_0^t dt' \gamma(t-t')[a(t') - a(t)]$$
$$= m_1 \sum_{n=1}^{\infty} \frac{a^{(n)}(t)}{n!} (-)^{n+1} \gamma_n \; , \tag{A1}$$

where [cf. (2.10)]

$$\gamma_n = \int_0^t du\, \gamma(u) u^n$$
$$= \sigma^2 \epsilon (-)^{n+1} \frac{\partial^{n+1}}{\partial \sigma^{n+1}} \int_0^t du\, e^{-\sigma u}$$
$$= \sigma^2 \epsilon A_{n+1}(\sigma, t) \; . \tag{A2}$$

$A_n$ can be written

$$A_n = \frac{n!}{\sigma^{n+1}} - e^{-\sigma t} \frac{P_n(\sigma t)}{\sigma^{n+1}} \; , \tag{A3}$$

where $(\sigma t = x)$

$$P_n(x) = (-)^n x^{n+1} e^x \frac{\partial^n}{\partial x^n} \left[ \frac{e^{-x}}{x} \right] . \tag{A4}$$

Using the following recurrence relation for $P_n(x)$

$$P_{n+1}(x) = (n+1) P_n(x) + x P_n(x) - x P_n'(x) \; , \tag{A5}$$

we get, by induction with $P_0 = 1$ and $P_1 = 1 + x$,

$$P_n(x) = n! \sum_{k=0}^{n} \frac{x^k}{k!} \; . \tag{A6}$$

Then, (A1) is given by

$$F^d = m_1 \epsilon \sum_{n=1}^{\infty} a^{(n)}(t) \frac{n+1}{\sigma^n} \left[ 1 - e^{-\sigma t} \sum_{k=0}^{n+1} \frac{(\sigma t)^k}{k!} \right]$$
$$= m_1 \epsilon e^{-\sigma t} \sum_{n=1}^{\infty} \frac{a^{(n)}(t)(n+1)}{\sigma^n} \sum_{k=n+2}^{\infty} \frac{(\sigma t)^k}{k!} \; . \tag{A7}$$

In order to show that this series is absolutely convergent, we consider two parts in $a(t)$, which is given from (4.5b) and (4.6b) by

$$a(t)=-\omega_0^2(1+\epsilon)[x_0\dot{\chi}_2(t)+v_0\chi_2(t)] \ . \tag{A8}$$

The first part $a_1(t)$ includes the terms with frequency $\bar{\omega}$ close to $\omega_0$ [see (4.8a)], and the second one, $a_2(t)$, the highly oscillating terms. $a_1(t)$ can be written

$$a_1(t)=-\omega_0^2(\xi_1\cos\bar{\omega}t+\eta_1\sin\bar{\omega}t)e^{-t/\tau_r} \ , \tag{A9}$$

where $\tau_r^{-1}$ is given by Eq. (4.8b) and $\xi_1 \simeq x_0$, $\eta_1 \simeq v_0/\omega_0$ [cf. (4.9c) and (4.9d)]. Using $\xi_1+i\eta_1=|\Omega_1|e^{i\varphi_1}$ and $\tau_r^{-1}+i\bar{\omega}=|\Lambda_1|e^{i\theta_1}$, we get, from (A9),

$$a_1^{(n)}(t)=(-)^{n+1}\omega_0^2|\Omega_1|\,|\Lambda_1|^n e^{-t/\tau_r} \times \mathrm{Re}[\exp(-i\bar{\omega}t+i\varphi_1+in\theta_1)] \ . \tag{A10}$$

Substituting (A10) in (A7) we get

$$F_1^d=m_1\epsilon e^{-\sigma t}\omega_0^2|\Omega_1|e^{-t/\tau_r}\,\mathrm{Re}\left[e^{i(\varphi_1-\bar{\omega}t)}\left[\sum_{n=1}^{\infty}\frac{(n+1)|\Lambda_1|^n(-)^{n+1}}{\sigma^n}e^{in\theta_1}\sum_{k=n+2}^{\infty}\frac{(\sigma t)^k}{k!}\right]\right] \ . \tag{A11}$$

As

$$\sum_{k=n+2}^{\infty}\frac{(\sigma t)^k}{k!}\le\exp(\sigma t) \ ,$$

we have

$$\left|\frac{|\Lambda_1|^n(-)^{n+1}e^{in\theta_1}(n+1)}{\sigma^n}\sum_{k=n+2}^{\infty}\frac{(\sigma t)^k}{k!}\right|\le\left|\frac{\Lambda_1}{\sigma}\right|^n(n+1)e^{\sigma t} \ . \tag{A12}$$

But

$$|\Lambda_1|=(\tau_r^{-2}+\bar{\omega}^2)^{1/2}\simeq\omega_0\left[1+\frac{\omega_0^2\epsilon^2}{\sigma^2(1+\epsilon)^2}\right]^{1/2}\ll\sigma$$

[see (4.2), (4.8a), and (4.8b)], and therefore the series in (A11) is absolutely convergent.

We consider now $a_2(t)$. In the same way as for $a_1(t)$ we have

$$a_2(t)=-\omega_0^2(\xi_2\cos\sigma_v t+\eta_2\sin\sigma_v t)e^{-\sigma_r t} \ , \tag{A13}$$

where $\sigma_r\simeq\sigma$, $\sigma_v\simeq\sigma\sqrt{\epsilon}$ [cf. (4.8c) and (4.8d)], and $\xi_2\simeq x_0\epsilon$, $\eta_2\simeq x_0\sqrt{\epsilon}$ [cf. (4.9c) and (4.9d)]. Using $\xi_2+i\eta_2=|\Omega_2|e^{i\varphi_2}$ and $\sigma_r+i\sigma_v=|\Lambda_2|e^{i\theta_2}$, we have

$$a_2^{(n)}(t)=(-)^{n+1}\omega_0^2|\Omega_2|\,|\Lambda_2|^n\sigma^n e^{-\sigma_r t}\mathrm{Re}[\exp(-i\sigma_v t+i\varphi_2+in\theta_2)] \ . \tag{A14}$$

Substituting (A14) in (A7) we get

$$F_2^d=m_1\epsilon e^{-\sigma t}\omega_0^2|\Omega_e|e^{-\sigma_r t}\mathrm{Re}\left[e^{i(\varphi_2-\sigma_v t)}\left[\sum_{n=1}^{\infty}(-)^{n+1}|\Lambda_2|^n e^{in\theta_2}(n+1)\sum_{k=n+2}^{\infty}\frac{(\sigma t)^k}{k!}\right]\right] \ . \tag{A15}$$

Using the property

$$\begin{bmatrix} m+n \\ n \end{bmatrix}=\frac{(m+n)!}{m!n!}\ge 1 \tag{A16}$$

we have

$$\sum_{k=n+2}^{\infty}\frac{x^k}{k!}=\sum_{m=0}^{\infty}\frac{x^m x^{n+2}}{(m+n+2)!}\le\frac{x^{n+2}}{(n+2)!}\sum_{m=0}^{\infty}\frac{x^m}{m!}=\frac{x^{n+2}}{(n+2)!}e^x \ , \tag{A17}$$

and then

$$\left||\Lambda_2|^n e^{in\theta_2}(n+1)\sum_{k=n+2}^{\infty}\frac{(\sigma t)^k}{k!}\right|\le\frac{|\Lambda_2|^n(\sigma t)^n}{n!}e^{\sigma t}(\sigma t)^2 \ . \tag{A18}$$

Therefore the series in (A15) is absolutely convergent.

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