Symmetry and variational methods in higher-dimensional theories

Robert T. Jantzen
Department of Mathematical Sciences, Villanova University, Villanova, Pennsylvania 19085
and Department of Physics, University of Rome, 00185 Rome, Italy
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The problem of imposing symmetry on variational principles for higher-dimensional theories is illustrated by considering spatially homogeneous solutions of Kaluza-Klein theories. Various implications of the group-theoretical nature of this specific situation are addressed.

I. INTRODUCTION

With the importance of higher-dimensional theories in the description of the various fundamental forces in a framework incorporating gravity, people have begun to consider special solutions in an attempt to understand some of the implications of these theories.[1–7] These special solutions are usually characterized by the existence of a large symmetry group. However, when variational principles are used to discuss symmetry-restricted classes of solutions of field equations, one must be careful to check how imposition of the symmetry affects the reduced variational principle.

It is well known from studying symmetric solutions of the Einstein equations in the four-dimensional case that the imposition of symmetry does not necessarily commute with the derivation of the field equations. This question first arose in the context of spatially homogeneous cosmological models[8,9] and extended to the spatially self-similar models,[10] both of which have a three-dimensional symmetry group acting simply transitively on spacelike hypersurfaces, except for the multiply transitive case of the Kantowski-Sachs models[11] and their spatially self-similar generalizations. Under certain circumstances this problem is even relevant to the most familiar symmetry used in relativity, namely spherical symmetry.[12] No general criteria exist for deciding when a symmetry-restricted variational principle yields the correct field equations.

II. SPATIALLY HOMOGENEOUS KALUZA-KLEIN MODELS

For the sake of an example, consider those solutions of a Kaluza-Klein theory in $N+1$ dimensions which generalize the simply transitive spatially homogeneous cosmological models in four spacetime dimensions. These seem to have been first considered by Belinsky and Khalatnikov[13] who were interested in the effect of a scalar field on the classical initial cosmological singularity. The spacetime for these models is a product manifold $R\times G$, where G is an N-dimensional Lie group with Lie algebra $\mathfrak{g}$ of left invariant vector fields having a basis $\{e_a\}_{a=1,\ldots,N}$ and a dual basis which may be identified with left invariant one-forms $\{\omega^a\}$ satisfying $\omega^a(e_b)=\delta^a{}_b$

$$[e_a,e_b]=C^c{}_{ab}e_c,\quad d\omega^a=-\tfrac{1}{2}C^c{}_{ab}\omega^a\wedge\omega^b\ . \tag{1}$$

Letting $e_0=\partial/\partial t$ and $\omega^0=dt$, where t parametrizes the family of copies of G in the product manifold and is identified with one of a set of local coordinates $\{t,x^a\}\equiv\{x^\alpha\}_{\alpha=0,1,\ldots,N}$ adapted to the product manifold, chosen so that the t lines are orthogonal to the G slices, the metric can be expressed in the form

$$\begin{aligned}{}^{(N+1)}g&=g_{\alpha\beta}(t)\omega^\alpha\otimes\omega^\beta\\&=-\mathcal{N}(t)^2dt\otimes dt+g_{ab}(t)\omega^a\otimes\omega^b\ ,\end{aligned} \tag{2}$$

where $\mathbf{g}=(g_{ab})$ is a positive-definite matrix in the case to be considered. Let $g=\det\mathbf{g}$ be its determinant. Thus G acting on $R\times G$ by left translation in the natural way is an isometry group of ${}^{(N+1)}g$ acting simply transitively on the family of spacelike hypersurfaces of constant t values. Fields like the metric which are invariant under this action have components in the frame $\{e_\alpha\}$ which depend only on the time and are called spatially homogeneous. This frame itself is a global spatially homogeneous frame on the spacetime with constant structure functions

$$\omega^\alpha([e_\beta,e_\gamma])=C^\alpha{}_{\beta\gamma}=\delta^\alpha{}_aC^a{}_{bc}\delta^b{}_\beta\delta^c{}_\gamma\ . \tag{3}$$

III. FRAME VARIATIONAL DERIVATIVES

The vacuum Einstein equations may be derived from the usual scalar curvature Lagrangian but it is worth studying first how symmetry affects the derivation of the field equations for any Lagrangian functional of geometric object fields on the spacetime. Let $\mathcal{L}(\Phi,\partial_\alpha\Phi,\partial_\alpha\partial_\beta\Phi)$ be a Lagrangian density functional of a geometric object field or collection of such fields denoted by Φ whose frame indices and field labels are suppressed but symbolized by the superscript A when needed. The notation $\partial_\alpha f=e_\alpha f$ is used to denote the frame derivative of a function f and $\dot f=\partial_0 f$ for the time derivative. Using the abbreviation $\omega^{\alpha_1\cdots\alpha_p}=\omega^{\alpha_1}\wedge\cdots\wedge\omega^{\alpha_p}$ for the wedge product of p basis one-forms, the action functional

$$I=\int_C\mathcal{L}\omega^{01\cdots N}\ , \tag{4}$$

where C is an arbitrary compact region of the spacetime, is varied with respect to Φ holding Φ fixed on the boundary ∂C.

If

$$\sigma=(N!)^{-1}\epsilon_{\alpha_1\cdots\alpha_{N+1}}\Sigma^{\alpha_1}\omega^{\alpha_2\cdots\alpha_{N+1}}=\Sigma\rfloor\omega^{01\cdots N} \tag{5}$$

is an N-form on the spacetime, the well-known identity

$$d\sigma=\partial\!\!\!/_\alpha\Sigma^\alpha\omega^{01\cdots N},\quad \partial\!\!\!/_\alpha\equiv\partial_\alpha-C^\beta{}_{\alpha\beta} \tag{6}$$

expresses the divergence $\partial\!\!\!/_\alpha\Sigma^\alpha$ of the vector density Σ. The divergence operator satisfies the identity

$$\partial\!\!\!/_\alpha(f\Sigma^\alpha)=\Sigma^\alpha\partial_\alpha f+f\partial\!\!\!/_\alpha\Sigma^\alpha \tag{7}$$

which is used in the frame version of integration by parts, while Stokes's theorem for the differential form σ states

$$\int_C d(\Sigma\rfloor\omega^{01\cdots N})=\int_{\partial C}\Sigma\rfloor\omega^{01\cdots N}\,. \tag{8}$$

Consider a one-parameter family of fields $\Phi(\lambda)$, where $\Phi(0)=\Phi$ and $d\Phi/d\lambda(0)=\Phi'$ with Φ' and its frame component derivatives vanishing on ∂C, and require that the variation (i.e., λ derivative at $\lambda=0$) of the action vanish for arbitrary values of Φ' in the interior of C and arbitrary compact regions C,

$$\begin{aligned}0=dI/d\lambda(0)&=\int_C D\mathcal{L}(\Phi)\cdot\Phi'\omega^{01\cdots N}\\&=\int_C(\delta\!\!\!/\mathcal{L}/\delta\Phi^A)\Phi'^A\omega^{01\cdots N}\\&\quad+\int_{\partial C}Z\rfloor\omega^{01\cdots N}\,,\end{aligned} \tag{9}$$

where

$$\begin{aligned}D\mathcal{L}(\Phi)\cdot\Phi'&=(\partial\mathcal{L}/\partial\Phi^A)\Phi'^A+[\partial\mathcal{L}/\partial(\partial_\alpha\Phi^A)]\partial_\alpha\Phi'^A\\&\quad+[\partial\mathcal{L}/\partial(\partial_\alpha\partial_\beta\Phi^A)]\partial_\alpha\partial_\beta\Phi'^A\\&=(\delta\!\!\!/\mathcal{L}/\delta\Phi^A)\Phi'^A+\partial\!\!\!/_\alpha Z^\alpha\end{aligned} \tag{10}$$

and

$$\begin{aligned}Z^\alpha&=\{\partial\mathcal{L}/\partial(\partial_\alpha\Phi^A)-\partial\!\!\!/_\beta[\partial\mathcal{L}/\partial(\partial_\beta\partial_\alpha\Phi^A)]\}\Phi'^A\\&\quad+[\partial\mathcal{L}/\partial(\partial_\alpha\partial_\beta\Phi^A)]\partial_\beta\Phi'^A\,,\end{aligned} \tag{11}$$

while

$$\begin{aligned}\delta\!\!\!/\mathcal{L}/\delta\Phi&=\partial\mathcal{L}/\partial\Phi-\partial\!\!\!/_\alpha[\partial\mathcal{L}/\partial(\partial_\alpha\Phi)]\\&\quad+\partial\!\!\!/_\beta\partial\!\!\!/_\alpha[\partial\mathcal{L}/\partial(\partial_\alpha\partial_\beta\Phi)]\end{aligned} \tag{12}$$

defines the frame components of the Lagrange derivative or variational derivative of the Lagrangian density $\mathcal{L}$. The slash is a reminder that it is a frame derivative like the divergence operator (6). Stokes's theorem has been used to transform the divergence term in the integral (9) to an integral over the boundary where $Z\rfloor\omega^{01\cdots N}$ vanishes. Thus the arbitrariness of Φ' in the interior of the region C implies the Lagrangian field equations

$$\delta\!\!\!/\mathcal{L}/\delta\Phi^A=0\,. \tag{13}$$

All of the above formulas are valid for an arbitrary fixed frame $\{e_\alpha\}$ on an $(N+1)$-dimensional manifold with structure functions $C^\alpha{}_{\beta\gamma}$. For a holonomic frame, or equivalently a local coordinate frame, the structure functions vanish and these formulas reduce to the usual ones. For the present case the structure functions are constants and the variational derivative is explicitly

$$\begin{aligned}\delta\!\!\!/\mathcal{L}/\delta\Phi&=\partial\mathcal{L}/\partial\Phi-\partial_\alpha[\partial\mathcal{L}/\partial(\partial_\alpha\Phi)]+\partial_\beta\partial_\alpha[\partial\mathcal{L}/\partial(\partial_\alpha\partial_\beta\Phi)]\\&\quad+C^\beta{}_{\alpha\beta}\partial\mathcal{L}/\partial(\partial_\alpha\Phi)+C^\gamma{}_{\alpha\gamma}C^\delta{}_{\beta\delta}\partial\mathcal{L}/\partial(\partial_\alpha\partial_\beta\Phi)-(C^\gamma{}_{\beta\gamma}\partial_\alpha+C^\gamma{}_{\alpha\gamma}\partial_\beta)\partial\mathcal{L}/\partial(\partial_\alpha\partial_\beta\Phi)\,.\end{aligned} \tag{14}$$

Now suppose one restricts the fields Φ to be spatially homogeneous, i.e., Φ^A depend only on t, and the region C to be a product region $[t_1,t_2]\times C_G$ and one considers a family of spatially homogeneous fields $\Phi(\lambda)$. The spatially homogeneous functions Φ'^A, being functions only of t, cannot be required to vanish on ∂C without vanishing identically in the interior, and the symmetry-restricted variational principle will in general yield incorrect field equations due to unwanted divergence terms.[8]

The problem is perhaps easiest understood by examining the effect of imposing the symmetry on the field equations. Referring to the imposition of the spatially homogeneous symmetry as "spatial homogenization" and denoting the operation on functionals by $O_{\rm Hom}$, then in $O_{\rm Hom}\,\mathcal{L}$ one may set $\partial_\alpha=\delta^0{}_\alpha\partial_0$ when acting on a spatially homogeneous function. Understanding Φ to be a spatially homogeneous field, the homogenized Lagrangian density depends only on Φ and its time derivatives

$$(O_{\rm Hom}\mathcal{L})(\Phi,\dot\Phi,\ddot\Phi)=\mathcal{L}(\Phi,\delta^0{}_\alpha\dot\Phi,\delta^0{}_\alpha\delta^0{}_\beta\ddot\Phi) \tag{15}$$

and acts as a Lagrangian function on the finite-dimensional configuration space of variables Φ^A treated as functions on the real line. Its Lagrange derivative

$$\begin{aligned}\delta\!\!\!/(O_{\rm Hom}\mathcal{L})/\delta\Phi&=\partial(O_{\rm Hom}\mathcal{L})/\partial\Phi-[(O_{\rm Hom}\mathcal{L})/\partial\dot\Phi]^{\cdot}\\&\quad+[\partial(O_{\rm Hom}\mathcal{L})/\partial\ddot\Phi]^{\cdot\cdot}\,,\end{aligned} \tag{16}$$

when set equal to zero, produces the Lagrangian equations of motion for the finite-dimensional Lagrangian system. However, the homogenized Lagrange derivative is

$$\begin{aligned}O_{\rm Hom}(\delta\!\!\!/\mathcal{L}/\delta\Phi)&=\delta\!\!\!/(O_{\rm Hom}\mathcal{L})/\delta\Phi\\&\quad+O_{\rm Hom}[C^b{}_{ab}\partial\mathcal{L}/\partial(\partial_a\Phi)\\&\qquad+C^c{}_{ac}C^d{}_{bd}\partial\mathcal{L}/\partial(\partial_a\partial_b\Phi)]\,.\end{aligned} \tag{17}$$

Setting it equal to zero produces the correct equations of motion which do not agree with the Lagrangian equations of motion for the finite-dimensional system unless the additional terms $O_{\rm Hom}(\delta\!\!\!/\mathcal{L}/\partial\Phi)-\delta\!\!\!/(O_{\rm Hom}\mathcal{L})/\delta\Phi$ vanish. These terms vanish identically when $C^b{}_{ab}=0$ or possibly on submanifolds of the configuration space when $C^b{}_{ab}\neq0$. The same analysis may be applied to the corresponding Hamiltonian formulation.

The condition $C^b{}_{ab}=0$ is the component form of the relation ${\rm Tr\,ad}=0$, where ad is the adjoint representation of the Lie algebra $\mathscr{g}$, i.e., the adjoint representation of $\mathscr{g}$ must be trace-free and therefore the adjoint representation Ad of the Lie group G must be a unimodular representation. For such a unimodular Lie group G, $\omega^{1\cdots N}$ is a bi-invariant N form and the Lagrangian density $O_{\rm Hom}\mathscr{L}$, in general transforming as a density under inner automorphisms of G, is instead invariant under these transformations, making Ad(G) a symmetry group of the finite-dimensional Lagrangian system.

IV. THE EINSTEIN CASE

For the scalar curvature Lagrangian density $\mathscr{L}={}^{(N+1)}R\,|\,{}^{(N+1)}g\,|^{1/2}$ the field equations are Einstein's equations

$$0=\delta\mathscr{L}/\delta g_{\alpha\beta}=-\,|\,{}^{(N+1)}g\,|^{1/2}\,{}^{(N+1)}G^{\alpha\beta}\,. \tag{18}$$

For a metric of the form (2) with $g_{0a}=0$, only the equations of motion for the variables g_{ab} and the super-Hamiltonian constraint may be obtained from the Lagrangian

$$\begin{aligned}0&=\delta\mathscr{L}/\delta g_{ab}=-\mathscr{N}g^{1/2}\,{}^{(N+1)}G^{ab}\,,\\0&=\delta\mathscr{L}/\delta\mathscr{N}=2\mathscr{N}^2g^{1/2}\,{}^{(N+1)}G^{00}\equiv-\mathscr{H}\,,\end{aligned} \tag{19}$$

while the remaining equations, the supermomentum constraints

$$0=\mathscr{H}_a\equiv2\mathscr{N}g^{1/2}\,{}^{(N+1)}G^0{}_a\,, \tag{20}$$

must be imposed as constraints on the system. The "lapse function" $\mathscr{N}$ must be chosen to fix the choice of parametrization of the family of spatially homogeneous hypersurfaces, i.e., to fix the choice of a cosmic time.

However, for such a metric the Arnowitt-Deser-Misner (ADM) approach to the field equations[14,15] is much more natural. This assumes a spacelike slicing of the spacetime leading to at least a local product representation of the spacetime and considers only local product regions C for the variational principle. Conveniently subtracting a divergence term from the Lagrangian density (which does not affect the Lagrangian field equations), leads to the Lagrangian

$$\mathscr{L}_{\rm ADM}=(4\mathscr{N})^{-1}\mathscr{G}^{abcd}\dot g_{ab}\dot g_{cd}-\mathscr{N}U\,, \tag{21}$$

where

$$\mathscr{G}^{abcd}=g^{1/2}(g^{a(c}g^{d)b}-g^{ab}g^{cd}) \tag{22}$$

is the DeWitt metric on the space of N-dimensional Riemannian metrics,[16] $U=-g^{1/2}R$ is the scalar curvature potential and R is the scalar curvature of the "spatial metric." This Lagrangian density has the familiar form of a kinetic energy minus a potential energy. For the spatially homogeneous metrics, $O_{\rm Hom}\mathscr{L}$ is just a familiar Lagrangian over the configuration space $\mathscr{M}$ of positive-definite inner products on R^N on which $\{g_{ab}\}$ may be interpreted in a generalized sense as local coordinates.

The free motion, ignoring the potential U, was thoroughly studied by DeWitt[16] for the case $N=3$ and is just geodesic motion with respect to the DeWitt metric $\mathscr{G}=\mathscr{G}^{abcd}dg_{ab}\otimes dg_{cd}$ on $\mathscr{M}$, required to be null geodesic motion by the super-Hamiltonian constraint that the total energy vanish. The potential function U is a quadratic function of the structure constant tensor components $C^a{}_{bc}$:

$$\begin{aligned}U&=g^{1/2}g^{ab}(C^c{}_{ac}C^d{}_{bd}+\Gamma^c{}_{da}\Gamma^d{}_{cb})\,,\\\Gamma^a{}_{bc}&=\omega^a(\nabla_{e_b}e_c)=\tfrac12C^a{}_{bc}+C_{(b}{}^a{}_{c)}\,,\end{aligned} \tag{23}$$

where $\Gamma^a{}_{bc}$ are the spatial components of the spacetime metric connection, which in this case coincide with the components of the spatial metric connection.

It is very useful to make a conformal splitting of the metric variables

$$g_{ab}=e^{2\alpha}\tilde g_{ab},\quad\det(\tilde g_{ab})=1,\quad g^{1/2}=e^{N\alpha}\,, \tag{24}$$

effectively decomposing $\mathscr{M}$ into the product manifold $R\times\tilde{\mathscr{M}}$, where $\tilde{\mathscr{M}}$ is the unimodular subspace of $\mathscr{M}$, equivalent to the space of conformal inner products. The limit $\alpha\to-\infty$ or $g=0$ defines the "frontier" at which $\mathbf{g}$ is singular. This representation of the variables clearly shows the Lorentz signature of the DeWitt metric

$$\tfrac14\mathscr{G}=e^{N\alpha}[-N(N-1)d\alpha\otimes d\alpha+\tfrac14\tilde g^{ac}\tilde g^{db}d\tilde g_{ab}\otimes d\tilde g_{cd}] \tag{25}$$

since the induced metric on $\tilde{\mathscr{M}}$ is Riemannian. Note that the decomposition of the variables is orthogonal and α is a timelike variable on $\mathscr{M}$. The Lagrangian is then

$$\begin{aligned}\mathscr{L}&=\tilde N^{-1}[-N(N-1)\dot\alpha^2+\tfrac14\tilde g^{ac}\tilde g^{db}\dot{\tilde g}_{ab}\dot{\tilde g}_{cd}]\\&\quad+\tilde{\mathscr{N}}e^{2(N-1)\alpha}\tilde U\,,\\\mathscr{N}&\equiv e^{N\alpha}\tilde{\mathscr{N}},\quad U\equiv e^{(N-2)\alpha}\tilde U\,,\end{aligned} \tag{26}$$

where $\tilde U$ is the scale invariant part of the potential, a function on $\tilde{\mathscr{M}}$. The choice of lapse $\mathscr{N}=2N(N-1)g^{1/2}$, or equivalently $\tilde{\mathscr{N}}=2N(N-1)$, makes the coefficient of $-\dot\alpha^2$ just $\frac12$. This generalizes Misner's supertime time gauge[17] which, neglecting the factor of $2N(N-1)$, dates back to Taub[18] and was used extensively by Belinsky, Khalatnikov, and Lifshitz,[19] in their qualitative analysis of $N=3$ spatially homogeneous dynamics. The time function in this gauge is an affine parameter for the geodesics of the conformally invariant rescaled DeWitt metric $\bar{\mathscr{G}}=g^{-1/2}\mathscr{G}$ in the free motion case.

To obtain the Hamiltonian formulation one must introduce the momentum π^{ab} canonically conjugate to g_{ab} in the standard way. Apart from a conformal factor this corresponds to "lowering the index" of the "velocity" $\dot g_{ab}$ with the DeWitt metric (note that "covariant indices" on $\mathscr{M}$ are symmetric pairs of contravariant tensor indices, so lowering actually raises the index pair position):

$$\begin{aligned}\pi^{ab}&=\partial\mathscr{L}/\partial\dot g_{ab}=(2\mathscr{N})^{-1}\mathscr{G}^{abcd}\dot g_{ab}\,,\\\dot g_{ab}&=2\mathscr{N}\mathscr{G}^{-1}{}_{abcd}\pi^{cd}\,,\end{aligned} \tag{27}$$

where

$$\mathscr{G}^{-1}{}_{abcd}=g^{1/2}[g_{a(c}g_{d)b}-(N-1)^{-1}g_{ab}g_{cd}] \tag{28}$$

are the components of the "contravariant" DeWitt metric on $\mathscr{M}$ satisfying

$$\mathscr{G}^{abfg}\mathscr{G}^{-1}{}_{fgcd}=\delta^a{}_{(b}\delta^b{}_{d)} \ . \quad (29)$$

Thus the Hamiltonian is

$$H=\pi^{ab}\dot{g}_{ab}-\mathscr{L}_{\rm ADM} \\ =\mathscr{N}\mathscr{G}^{-1}{}_{abcd}\pi^{ab}\pi^{cd}+\mathscr{N}U=\mathscr{N}\mathscr{H} \ . \quad (30)$$

However, neither the Lagrangian nor Hamiltonian equations obtained from the spatially homogeneous Lagrangian or Hamiltonian are correct unless the problematic terms in the variational derivatives vanish. These terms involve spatial derivatives so the kinetic energy offers no problems; only the spatial curvature potential may lead to trouble.

For a general nonspatially homogeneous spatial metric one has the well-known variational identity[8] (denoting the spatial metric covariant derivative by a semicolon)

$$(g^{1/2}R)'=-g^{1/2}G^{ab}g'_{ab}+(\mathscr{G}^{abcd}g'_{cd})_{;cd} \quad (31)$$

which implies from the definition of the variational derivative that

$$\delta(-U)/\delta g_{ab}=-g^{1/2}G^{ab} \ . \quad (32)$$

Restricting (31) to spatially homogeneous metrics (for which $g'^a{}_a$ is a constant) yields

$$(g^{1/2}R)'=-g^{1/2}G^{ab}g'_{ab}-Q^{ab}g'_{ab} \ , \quad (33)$$

where

$$Q^{ab}g'_{ab}=-(\mathscr{G}^{abcd}g'_{cd})_{;ab} \\ =-g^{1/2}(g'_{ab}{}^{;ab}-g'^a{}_{a;b}{}^{;b}) \\ =-g^{1/2}g'_{ab}{}^{;ab} \ . \quad (34)$$

This implies the following exterior derivative relation on the finite-dimensional space $\mathscr{M}$

$$-g^{1/2}G^{ab}dg_{ab}=-dU+Q^{ab}dg_{ab} \quad (35)$$

or the variational result

$$\delta(O_{\rm Hom}(-\mathscr{N}U))/\delta g_{ab}=-\mathscr{N}g^{1/2}G^{ab}-\mathscr{N}Q^{ab} \ . \quad (36)$$

Thus the Lagrange derivative of the finite-dimensional potential term produces not only the desired spatial Einstein tensor driving force but an unwanted term as well; the correct field equations are

$$0=O_{\rm Hom}(\delta\mathscr{L}/\delta g_{ab})=\delta(O_{\rm Hom}\mathscr{L})/\delta g_{ab}-\mathscr{N}Q^{ab} \ , \quad (37)$$

or from the Hamiltonian point of view

$$\dot{g}_{ab}=\{g_{ab},H\}, \ \ \dot{\pi}^{ab}=\{\pi^{ab},H\}+\mathscr{N}Q^{ab} \ . \quad (38)$$

This has a simple interpretation.[20] The finite-dimensional Einstein driving force (35) is not conservative, but has a nonpotential component Q which is a one-form on $\mathscr{M}$; the dynamics is described by a Lagrangian or Hamiltonian system with a nonpotential driving force Q, unless Q vanishes. This system is subject to the super-Hamiltonian constraint that the total energy vanish

$$H=0 \ \text{ or } \ \mathscr{H}=0 \quad (39)$$

and the supermomentum constraint

$$0=\mathscr{H}_a\equiv 2\mathscr{N}g^{1/2\,(N+1)}G^0{}_a \\ =-2\pi_{ab}{}^{;b}=-2\pi^b{}_c\delta_a{}^c{}_b \ , \quad (40)$$

where

$$\delta_a{}^c{}_b\equiv C^c{}_{ab}-\delta^c{}_aC^d{}_{bd} \ , \quad (41)$$

defines a set of trace-free matrices $\boldsymbol{\delta}_a=(\delta_a{}^c{}_b)$.

Formulas for spatial divergences follow from standard results using the spatial connection components $\Gamma^a{}_{bc}$ given above. The divergence of a spatially homogeneous vector or vector density is

$$\partial_aX^a=-C^f{}_{af}X^a \ , \quad (42)$$

while (40) gives the formula for any spatially homogeneous symmetric second-rank tensor or tensor density. It immediately follows that the double divergence which defines Q is

$$Q^{ab}=g^{1/2}\delta_c{}^{(ab)}C^{dc}{}_d \ . \quad (43)$$

This vanishes for all unimodular groups as shown above. For the nonunimodular groups, Q is in general nonvanishing, although it may be zero when restricted to some submanifold of the configuration space consistent with the equations of motion.

Special conditions may be imposed on the spatial metric only if the submanifold of $\mathscr{M}$ to which they correspond is associated with additional discrete or continuous spacetime symmetries of ${}^{(N+1)}g$. These additional symmetries[20,21] are in general related to automorphisms of the Lie algebra $\mathfrak{g}$ of the isometry group G. The diagonal submanifold $\mathscr{M}_D$ of $\mathscr{M}$ is such a submanifold only if certain reflections of the basis vectors $\{e_a\}$ of $\mathfrak{g}$ are automorphisms. This requires not only the existence of certain discrete automorphisms for the Lie algebra $\mathfrak{g}$ but that the basis be chosen so that these discrete automorphisms take the form of reflections; i.e., some thought should be given to the choice of a standard form for the structure constant tensor components $C^a{}_{bc}$. In the case $N=3$ the nonunimodular Lie groups do not allow diagonal metrics except in special cases when additional constraints are necessary; the situation is probably similar for higher dimensions.

In those cases where diagonal solutions are possible, one can choose a basis $\{\mathbf{e}_A\}_{A=1,\ldots,N-1}$ of the space of trace-free diagonal matrices satisfying $\mathrm{Tr}(\mathbf{e}_A\mathbf{e}_B)=N(N-1)\delta_{AB}$, where the normalization factor is $N(N-1)=(\mathrm{Tr}\mathbf{1})^2-\mathrm{Tr}\mathbf{1}^2$, and set $\tilde{g}=e^{2\boldsymbol{\beta}}$ where $\boldsymbol{\beta}=\beta^A\mathbf{e}_A$. A useful choice of such a basis which generalizes the basis $\{\sqrt{3}\mathrm{diag}(1,-1,0), \mathrm{diag}(1,1,-2)\}$ originally introduced by Misner[17] for $N=3$ and recently used by Halpern[7] for $N=4$ is

$$\mathbf{e}_A=\left[\frac{N(N-1)}{A(A+1)}\right]^{1/2}\mathrm{diag}(1,\ldots,1,-A,0,\ldots,0) \ , \quad (44)$$

where the first A diagonal components are unity and $A=1,\ldots,N-1$. This choice is adapted to the relative anisotropy of the $(A+1)$th spatial frame vector relative to the first A vectors. It is also convenient to introduce another notation for the trace-free diagonal matrix $\boldsymbol{\beta}$:

$$\boldsymbol{\beta}=\mathrm{diag}(\beta^{(1)},\ldots,\beta^{(N)}) ,$$
$$\beta^{abc}=\beta^{(a)}-\beta^{(b)}-\beta^{(c)} . \quad (45)$$

In the Misner supertime time gauge the Lagrangian (26) and Hamiltonian (30) then become

$$\mathscr{L}=\tfrac{1}{2}\left[-\dot{\alpha}^2+\sum_{A=1}^{N}(\dot{\beta}^A)^2\right]-2N(N-1)e^{2(N-1)\alpha}\tilde{U} ,$$
$$H=\tfrac{1}{2}\left[-p_\alpha{}^2+\sum_{A=1}^{N}p_A{}^2\right]+2N(N-1)e^{2(N-1)\alpha}\tilde{U} , \quad (46)$$

where $\tilde{U}$ depends only on the conformal variables β^A. Neglecting this potential (the Abelian case), the solutions are just geodesics of the conformally invariant rescaled DeWitt metric $[2N(N-1)g^{1/2}]^{-1}\mathscr{G}$ with respect to which $\{\alpha,\beta^A\}$ are orthonormal coordinates on the flat Lorentz subspace $\mathscr{M}_D$. The super-Hamiltonian constraint in the absence of the potential requires the geodesic to be null. (The same statements hold for the full nonflat space $\mathscr{M}$ itself, where the geodesics are also known explicitly.[16]) Considering the potential U as a time-dependent function on the Euclidean space $\mathscr{M}_D$ of the variables $\{\beta^A\}$ due to the conformal dependence on the "time function" α gives a useful way of visualizing the dynamics in terms of the familiar classical mechanical system where α plays the role of the time. Note that α is not the time function on the spacetime in this gauge.

However, diagonalization is intimately connected with the spatial gauge freedom and the supermomentum constraints. The variables (g_{ab},π^{ab}) are canonical coordinates on the momentum phase space (cotangent space) $T^*\mathscr{M}$ with Poisson brackets $\{g_{ab},\pi^{cd}\}=\delta^c{}_{(a}\delta^d{}_{b)}$. For each matrix $\mathbf{A}\in GL(N,R)$ the moment function $P(\mathbf{A})=-2\,\mathrm{Tr}(\boldsymbol{\pi}\mathbf{A})$, $\boldsymbol{\pi}\equiv(\pi^a{}_b)$, generates a canonical action on the phase space corresponding to the natural action on $\mathscr{M}$ by the one-parameter metric group generated by $\mathbf{A}$ which corresponds to transforming the basis $\{e_a\}$ of $\mathscr{g}$

$$\mathbf{B}\cdot(g_{ab},\pi^{ab})=(B^{-1c}{}_aB^{-1d}{}_bg_{cd},B^a{}_cB^b{}_d\pi^{cd}) ,$$
$$\mathbf{B}\equiv\exp(s\mathbf{A}) . \quad (47)$$

Notice that this incorrectly transforms the canonical momentum which should transform as a density; even though the correct transformation is a symmetry of the equations of motion, it is not a canonical symmetry of the Hamiltonian system except when $\mathbf{A}$ is trace-free and its exponential therefore unimodular. This problem arises because the original Lagrangian, treated as a scalar function on the velocity phase space, is a scalar density on spacetime and the transformation properties of the two interpretations are in conflict. The isometry group of the DeWitt metric is in fact $SL(N,R)$ which is the symmetry group of the kinetic energy, but the potential U breaks the symmetry down to the matrix representation $\mathrm{SAut}_e(\mathscr{g})$ of the special automorphism group of the Lie algebra, namely, the subgroup of $SL(N,R)$ which leaves invariant the structure tensor constant components $C^a{}_{bc}$.

The supermomentum constraints

$$0=\mathscr{H}_a=P(\boldsymbol{\delta}_a) \quad (48)$$

require that the canonical generators or moment functions associated with the trace-free vector space span $\{\boldsymbol{\delta}_a\}$ vanish. In general there are N independent such generators. Consider first the unimodular case where the matrices $\boldsymbol{\delta}_a$ reduce to the adjoint matrices $\mathbf{k}_a=(C^c{}_{ab})\in\mathrm{ad}_e\mathscr{g}$ which generate the matrix representation of the adjoint Lie algebra with respect to the given basis. The generators are independent as long as the center of the Lie algebra is trivial so that the adjoint representation is faithful. The Hamiltonian system may then be reduced by N degrees of freedom by exploiting its invariance under the adjoint group $\mathrm{Ad}_e(\mathscr{g})$ generated by these matrices, using a well-known procedure.[22] This still leaves $N(N+1)/2-N=N(N-1)/2$ degrees of freedom, which is greater than the number of diagonal variables N by $N(N-3)/2$. Thus diagonal metrics are not general if $N>3$, but off-diagonal degrees of freedom are necessary. If the automorphism group is larger than the adjoint group, one may reduce the system further but those additional degrees of freedom will have nonzero moment functions. The nonunimodular case is slightly more complicated.

The minimum number of variables necessary to describe the potential U is associated with the orbit space $\mathscr{M}/\mathrm{Aut}_e(\mathscr{g})$, which is the finite-dimensional part of superspace associated with the particular spatially homogeneous symmetry. One needs to find a slice for the action of the automorphism group on $\mathscr{M}$, which will then parametrize the orbit space. Assuming that G is simply connected, this action on $\mathscr{M}$ arises from those diffeomorphisms of the group manifold which preserve the symmetry, namely, the automorphisms and translations of the group into itself; i.e., the action reflects the spatial diffeomorphism freedom.

The super-Hamiltonian constraint also leads to a reduction by one of the number of degrees of freedom. The obvious way of accomplishing this is to assume Misner's Ω-time time gauge[17] where the time function of $\mathscr{M}$, α, or equivalently $\Omega\equiv-\alpha$, is taken as the spacetime time variable by choosing the lapse $\mathscr{N}=\pm 2N(N-1)p_\alpha{}^{-2}$. This eliminates the need to consider an equation of motion for α and eliminates the super-Hamiltonian constraint which may be used to define p_α and hence the lapse. The lapse enables one to find the proper time function τ defined by $d\tau=\mathscr{N}dt$, usually chosen so that $\tau=0$ corresponds to $\Omega\rightarrow\infty$ when an initial singularity exists.

All of the above ideas extend in a natural way from the case $N=3$ if properly understood. The $N=3$ class A and class B categories of symmetry types divide the groups into the unimodular and nonunimodular groups, respectively, which generalize to all values of N. "Essentially closed" equipotential surfaces of the potential function $\tilde{U}$, which are a prerequisite of the Belinsky-Khalatnikov-Lifshitz (BKL) "oscillatory approach to the initial singularity,"[19] should occur for symmetry groups G which are semisimple, while the nonsemisimple groups have nontrivial continuous outer automorphisms which open up the potential in certain directions (an open set of directions as opposed to a set of measure zero since the latter always exist and require the modifier "essentially" above). For $N=3$ the semisimple Bianchi types VIII and IX have such essentially closed contours on the two-dimensional space of variables $\{\beta^+,\beta^-\}$ introduced by Misner.[17] For $N=4$, for example, there are no semisimple Lie groups, so there are no essentially closed potentials.[7]

In the diagonal case in the Misner supertime time gauge, one may calculate the velocities of the equipotential surfaces of individual terms in the potential.[17,23] Evaluating the potential U at $\mathscr{M}_D$ yields the result

$$\begin{aligned}\tilde{U}_{\rm diag}&=\tfrac{1}{4}\sum_{a,b,c}(C^a{}_{bc}e^{\beta^{abc}})^2+\sum_a(\tfrac{1}{2}\gamma_{aa}+C_a{}^2)e^{-2\beta^{(a)}}\\&=\tfrac{1}{2}\sum_{a\neq b\neq c,\ b<c}(C^a{}_{bc}e^{\beta^{abc}})^2\\&\quad+\sum_a[\tfrac{1}{2}\gamma^{(0)}_{aa}+N(N-1)^{-1}C_a{}^2]e^{-2\beta^{(a)}}\ ,\end{aligned}\tag{49}$$

where γ_{ab} and $\gamma^{(0)}_{ab}$ are the components of the Killing inner product and the "trace-free" Killing inner product defined by (A10), while $C_a=C^b{}_{ab}$ is just the trace covector. This expression has terms of two types, each of which depends on $\boldsymbol{\beta}$ through the exponential factor $e^{2\beta^{abc}}$ where $b\neq c$. For the first type (a,b,c) are distinct, while for the second type either $a=b$ or $a=c$ so that it reduces to the factor $e^{-2\beta^{(c)}}$ or $e^{-2\beta^{(b)}}$. The terms to the first type have been called "dangerous terms"[19,24] for reasons soon to become clear. For a compact semisimple group, $C_a=0$ and one can choose the basis $\{e_a\}$ so that $\gamma_{ab}=-\mathscr{R}^2\delta_{ab}$; it is then easy to see that (49) is negative only in an almost compact region about the point $\boldsymbol{\beta}=0$ (this point corresponds to the bi-invariant metrics).

In the Misner supertime time gauge, a term of the second type leads to a term in U which is a constant times $\exp[2(N-1)\alpha-2\beta^{(a)}]$ which for $a=N$ is $\exp[2(N-1)(\alpha+\beta^N)]$, so a fixed equipotential hyperplane of this potential in $\boldsymbol{\beta}$ space moves with unit Ω speed in the positive β^N direction, i.e., has a null velocity from the perspective of the total space $\mathscr{M}_D$. Since the potential increases in this direction for fixed Ω, it is moving outward from the region allowed to the system point by the super-Hamiltonian constraint as Ω increases towards the singularity.

On the other hand a potential term of the first type, say with the exponential factor $\exp(2\beta^{N,N-1,N-2})$, leads to a term in U whose equipotential hyperplanes move with Ω speed:

$$\begin{aligned}d\beta_{\rm equipot}/d\Omega&=(N-1)^{1/2}/(3N-1)^{1/2}\\&\in[\tfrac{1}{2},3^{-1/2}),\ \ N\geq 3\ ,\end{aligned}\tag{50}$$

which for $N=3$ has the familiar value $\frac{1}{2}$. This potential also moves in the direction in which it increases for fixed Ω and therefore away from the instantaneous location of the system point as Ω increases. Potential terms of this type are the so-called dangerous terms[19,24] which scatter an initially approximate free state known as the Kasner state. Since the free state is a null line in $\mathscr{M}_D$ corresponding to unit Ω speed, the system point can always "overtake" such a potential as long as the perpendicular component of its Ω velocity relative to the potential is greater than the Ω speed of the potential. However, the free state has at best zero relative Ω speed in the direction of a potential term of the second kind and so cannot overtake it.

Consider, for example, the case[3,5,13] where the unimodular group G is $R^{N-3}\times SO(3,R)$, which is the direct product of the Abelian group R^{N-3} with the three-dimensional rotation group, the latter of which is known to lead to an oscillatory approach to the initial singularity in the case $N=3$. The potential equipotential hypersurfaces are cylindrical and hence the motion in the Abelian directions (along the cylinder) reduces the magnitude of the non-Abelian component of its free Ω velocity to a value less than unity. In this particular case a unit Ω speed is the maximum speed the system point can have in the non-Abelian direction so that its relative Ω velocity with one exponential potential of the dangerous type is zero and yet the system point does not run into another such potential. Thus when the non-Abelian Ω velocity component is nonzero, certain "vertex directions" are possible in which the system point will chase one such potential with a negative relative Ω velocity and hence will remain in an asymptotically free state as one approaches the singularity. The explanation is simple.

The diagonal Abelian solution (the "free state")

$$\mathbf{g}=e^{\alpha_{(0)}}e^{\beta_{(0)}}{\rm diag}(\tau^{2p_1},\ldots,\tau^{2p_N})\ ,\tag{51}$$

where τ is the proper time ($\mathscr{N}=1$ time gauge) is known as the Kasner solution,[19] and the nonzero Einstein equations

$$\begin{aligned}0&=g^{\ (N+1)}R^a{}_b=\left[1-\sum_{c=1}^{N}p_c\right]{\rm diag}(p_1,\ldots,p_N)\ ,\\0&=g^{1/2}\mathscr{H}=\sum_{c=1}^{N}p_c{}^2-\left[\sum_{c=1}^{N}p_c\right]^2\end{aligned}\tag{52}$$

lead to the famous Kasner conditions

$$\sum_{c=1}^{N}p_c{}^2=\sum_{c=1}^{N}p_c=1\tag{53}$$

on the Kasner exponents $(p_1,\ldots,p_N)$ which parametrize the canonical momentum of the solution. For such an approximate solution in the non-Abelian case, the exponential factor which appears in the diagonal potential has the value $e^{2\beta^{abc}}=e^{2\beta^{abc}_{(0)}}\tau^{2\alpha_{abc}}$, which decreases to zero as $\tau\to 0$ as long as $\alpha_{abc}\equiv p_a-p_b-p_c>0$. This condition is equivalent to the requirement that the orthogonal component of the Ω velocity of the system point relative to this potential be negative so that the effect of the potential becomes more and more negligible as the singularity is approached. As long as this condition is satisfied for each such dangerous term potential, an initial approximate free state will remain in an asymptotically free state as the singularity is approached. That this is possible on an open region of the space of allowed Kasner exponent values for $N\geq 10$ for all α_{abc} is the observation of Demaret, Henneaux, and Spindel.[24] This means asymptotically free states are allowed for an open set of directions for $N\geq 10$ independent of the group G.

Of course all of the semisimple groups admit special solutions which asymptotically approach Kasner states. These solutions correspond to motion along a set of directions in $\mathscr{M}_D$ of measure zero at which the equipotential surfaces corresponding to nonpositive values extend out to infinity (infinite values of $\boldsymbol{\beta}$). In the $N=3$ case these special solutions are the locally rotationally symmetric solutions first discovered by Taub[18] for the compact case of Bianchi type IX; these solutions extend by the Weyl unitary trick[25] to the noncompact semisimple Bianchi type VIII and by Lie-algebra contraction to certain other unimodular nonsemisimple group types. It is interesting to note that the cosmological singularity $g=0$ in these special solutions is only a Killing horizon across which the spacetime may be continued. Analogs of these locally rotationally symmetric exact solutions exist in higher dimensions.[3]

The Ω velocities of the potentials or dangerous term calculations are simply related to the transformation properties of functions on the space $\mathscr{M}_D$ under the action of the scale group $\mathrm{Diag}^{(0)}(N,R)\equiv\exp[\mathrm{diag}(N,R)] \subset GL(N,R)$ generated by the Abelian Lie algebra $\mathrm{diag}(N,R)$ of diagonal matrices. Any function f on $\mathscr{M}_D$ which itself scales under the natural action of the scale group defines two weights q and s:

$$\mathscr{B}=\mathscr{B}^0\mathbf{1}+\mathscr{B}^A\mathbf{e}_A,\quad \delta^{AB}n_An_B=1,\quad s\geq 0\ ,$$
$$e^{\mathscr{B}}\cdot f(\mathbf{g})=f(e^{2\mathscr{B}}\mathbf{g})=e^{q\mathscr{B}^0+sn_A\mathscr{B}^A}f(\mathbf{g})\ . \tag{54}$$

The first dimension q is just the ordinary dimension of f under uniform scale transformations. To interpret s note that the parametrization $\mathbf{g}=\exp[2(\alpha\mathbf{1}+\beta^A\mathbf{e}_A)]$ enables one to identify $\mathscr{M}_D$ with the scale group which in turn may be identified with its matrix Lie algebra by the exponential map. The rescaled DeWitt metric $2N(N-1)g^{-1/2}\mathscr{G}$ corresponds to the Lorentz metric on $\mathrm{diag}(N,R)$ for which $\{\alpha,\beta^A\}$ are orthonormal coordinates. $\{\beta^A\}$ are Cartesian coordinates on the Euclidean subspace $\alpha=0$ corresponding to $\widetilde{\mathscr{M}}_D$ and the induced metric itself corresponds to the usual trace inner product on $\mathrm{gl}(N,R)$ restricted to the trace-free diagonal subspace $\mathrm{sdiag}(N,R)$. The second dimension s is just the "length" of the covector sn_A which determines the anisotropic scaling properties of the function f. One might call s the anisotropy dimension. The Ω speed associated with such a function f is then just $|q|s^{-1}$.

When the potential $\widetilde{U}$ is non-negative, the Ω speed of the system point is necessarily less than or equal to 1,

$$d\beta/d\Omega\equiv\left[\sum_{A=1}^{N-1}(d\beta^A/d\Omega)^2\right]^{1/2}$$
$$=1-4N(N-1)p_0{}^{-2}e^{2(N-1)\alpha}\widetilde{U}\in[0,1]\ , \tag{55}$$

due to the super-Hamiltonian constraint. This means that the motion is timelike in $\mathscr{M}_D$. Negative values of U associated with positive spatial curvature allow spacelike motion and hence recollapse. (These ideas extend in a natural way to $\mathscr{M}$ itself.) In the case $N=3$ only the compact semisimple case of Bianchi type IX permits positive curvature and only in a region of $\mathscr{M}_D$ which is a finite distance from the projection of the bi-invariant metrics (proportional to the metric of S^3) into $\mathscr{M}_D$ in all directions except along the three directions corresponding to the Taub metrics. The bi-invariant metrics have the maximum symmetry possible and represent the intersection of the three submanifolds of locally rotationally symmetric metrics. Discrete symmetries of the potential $\widetilde{U}$ are related to discrete symmetries of the spatial metric, which in turn are usually related to discrete subgroups of the automorphism group. Additional symmetries of the spacetime metric are also associated with discrete symmetries of the potential. Independent of additional symmetries, some special solutions occur when directions exist in $\mathscr{M}_D$ where the total potential $\widetilde{U}$ scales under the action of the unimodular scale group, as occurs for the generalized Taub solutions and the solutions[2] which generalize the $n^a{}_a=0$ solutions of the case $N=3$, the latter of which are characterized by additional discrete symmetry. The $N=3$ family of diagonal solutions of this type[26] includes both the Ellis-MacCallum type VI solution[27] and its Lie-algebra contraction, the Joseph type V solution.[28] All these ideas are relatively well known in the case $N=3$ and invite investigation for higher-dimensional cases.

The existence of a diagonal case requires that the spatial Ricci tensor be diagonal for $\mathbf{g}\in\mathscr{M}_D$. Naturally this depends on the choice of basis $\{e_a\}$. The spatial Ricci tensor is explicitly

$$R_{ab}=-C_fC_{(a}{}^f{}_{b)}-\gamma_{ab}-\tfrac{1}{4}(2C^{fg}{}_aC_{fgb}-C_a{}^{fg}C_{bfg})\ . \tag{56}$$

Using the decomposition (A3) of the structure constant tensor into trace-free and pure trace parts, this expression may itself be decomposed to yield

$$R_{ab}=R_{ab}^{(0)}-C_fC^{(0)}{}_{(a}{}^f{}_{b)}-(N-1)^{-1}C_cC^cg_{ab}\ , \tag{57}$$

where $R_{ab}^{(0)}$ is the curvature due to the trace-free part alone. In the semisimple case one would choose the basis so that the Killing inner product matrix is diagonal; this seems to allow a diagonal case for these groups and their contractions, all examples of unimodular groups. For the nonunimodular groups whose structure constant tensor has a nontrivial trace-free part, one expects additional constraints on the diagonal metric matrix to kill the off-diagonal components of the spatial Ricci tensor, constraints which may not have a solution.

In the nonunimodular case one must also worry about the potential force. If the basis $\{e_a\}$ is chosen so that $C_a=C_N\delta^N{}_a$, then restricting (A12) to $\mathscr{M}_D$ yields the following result for the scale-invariant quantity $\widetilde{Q}\equiv e^{-(N-2)\alpha}Q$:

$$\widetilde{Q}=e^{2(N-1)\beta^N}\left[\sum_a C_NC^{(0)a}{}_{Na}e_A{}^a{}_ad\beta^A+2N(C_N)^2d\beta^N\right]\ . \tag{58}$$

The role of the second term is merely to eliminate the $C_a{}^2$ term of the potential (49) from the equations of motion for the momentum conjugate to β^N:

$$-\frac{\partial}{\partial\beta^N}[N(N-1)^{-1}C_N{}^2e^{2(N-1)\beta^N}]+\widetilde{Q}(\partial/\partial\beta^N)=0\ . \tag{59}$$

The pure trace part of the structure constant tensor alone makes an isotropic contribution to the spatial curvature as

(57) shows, and hence cannot directly affect the equations of motion for the anisotropy variables $\boldsymbol{\beta}$. In Ω-time time gauge where one does not need an equation of motion for α, one may subtract this term from the potential to yield an effective potential, thus eliminating the pure trace part of the nonpotential force. The first term in the diagonal value of the nonpotential force is allowed to be nonzero only for $N>3$.

The spatially homogeneous discussion is also relevant to the question of an asymptotic "general solution" to the field equations as one approaches the classical initial singularity.[19,24] In this case there is no spacetime symmetry but there is a close relationship to the spatially homogeneous dynamics. One assumes an inhomogeneous metric of the form (2) with $\mathcal{N}=1$, $t=\tau$, and $\mathbf{g}=\mathrm{diag}(\tau^{2p_1},\ldots,\tau^{2p_N})$, where p_a and the structure functions $C^a{}_{bc}$ of the time-independent spatial frame are arbitrary functions of the spatial coordinates. One can examines what conditions must be imposed on these functions in order that one obtain a solution of the Einstein equations to leading order as $\tau\to 0$. The curvature formulas for the spatially homogeneous case change only by the addition of the term

$$\Delta\Gamma^c{}_{ab}=(\delta^c{}_a p_{a,b}+\delta^c{}_b p_{b,a}-\delta_{ab}p_{a,d}\delta^{dc})\ln\tau \qquad (60)$$

to the spatial connection coefficients, a term very similar to the one contributed by C_a. Ignoring the spatial curvature, the spatial Einstein equations and the super-Hamiltonian constraint exactly coincide with (52), which impose the Kasner constraints on the inhomogeneous Kasner exponents. The canonical momentum matrix for this metric is

$$\boldsymbol{\pi}=(\pi^a{}_b)=\mathrm{diag}(p_1,\ldots,p_N)-\mathbf{1}\ , \qquad (61)$$

so the supermomentum constraint (40) becomes

$$0=\mathscr{H}_a=-2\partial_b\pi^b{}_a-2\,\mathrm{Tr}(\boldsymbol{\delta}_a\boldsymbol{\pi})+4\ln\tau\,\mathrm{Tr}(\boldsymbol{\pi}\partial_a\boldsymbol{\pi})\ . \qquad (62)$$

The final term $\mathrm{Tr}(\boldsymbol{\pi}\partial_a\boldsymbol{\pi})$ is just $\frac{1}{2}\partial_a\mathrm{Tr}\boldsymbol{\pi}^2$ which vanishes due to the Kasner constraints, leaving the result

$$\mathrm{diag}(\partial_1 p_1,\ldots,\partial_N p_N)=-\mathrm{Tr}(\boldsymbol{\delta}_a\boldsymbol{\pi}) \qquad (63)$$

which are just N constraints on the choice of spatial frame.

One must then examine the effect of the spatial curvature on the situation. This is essentially the same analysis as in the spatially homogeneous case. The additional terms contributed by the gradients of the Kasner exponents are not "dangerous" in the same sense as the terms associated with the trace $C^b{}_{ab}$, and hence the interaction with the spatial curvature is the same as in the spatially homogeneous case. Either the approximate free state remains in an asymptotically free state or rebounds off a dangerous curvature term potential, after which it finds itself in a new Kasner state with a new spatial frame. Beyond this point the discussion itself becomes dangerous, in the sense that certain assumptions are made which are no longer trivial.[19]

Notationally, this discussion in the literature has been a bit cumbersome simply because of reluctance to use the machinery of noncoordinate frames. A so-called "Kasner frame" is just an orthogonal spatial frame of eigenvectors of the extrinsic curvature tensor. For the inhomogeneous Kasner metric above, the extrinsic curvature components are

$$(K^a{}_b)=-\tau^{-1}\mathrm{diag}(p_1,\ldots,p_N)\ , \qquad (64)$$

showing that $\{e_a\}$ are eigenvectors with eigenvalues $\{-\tau^{-1}p_a\}$. The dynamical effect of spatial curvature through the Einstein equations is to make the extrinsic curvature eigenvectors change with time. In the approximation of a scattering off a single dangerous term in the spatial curvature, one then has a linear transformation relating the initial and final extrinsic curvature eigenvectors which has been referred to as a "rotation of the Kasner axes." Of course it is the normalized eigenvectors which undergo a rotation.

V. CONCLUSIONS

An extensive literature exists treating not only the dynamics of highly symmetric cosmological models but also dealing with the problems of commuting symmetry imposition with derivation of field equations in variational approaches to the dynamics. All of these ideas extend naturally to higher dimensions without the necessity of repeating all of the trials and mistakes of the case $N=3$. This article has tried to point out how one can take advantage of already known results and ideas without going into too many unnecessary details, simply by understanding "the big picture."

APPENDIX

In analogy with the case $N=3$, one may introduce the natural dual of the antisymmetric pair of structure constant indices[27,29]

$$\begin{aligned} C^{aa_3\cdots a_N}&=\tfrac{1}{2}C^a{}_{bc}\epsilon^{bca_3\cdots a_N}\ ,\\ C^a{}_{bc}&=[(N-2)!]^{-1}C^{aa_3\cdots a_N}\epsilon_{bca_3\cdots a_N}\ . \end{aligned} \qquad (A1)$$

At first this might seem like an unprofitable move for $N>4$ due to the increase in the number of indices, but the totally contravariant density $C^{a_1a_3\cdots a_N}$ over $\mathcal{g}$ enables one to decompose the structure constant tensor into its irreducible parts under the action of the general linear group. The totally antisymmetric part is equivalent to the trace covector C_a

$$\begin{aligned} C_a&=C^b{}_{ab}=[(N-2)!]^{-1}C^{a_2\cdots a_N}\epsilon_{aa_2\cdots a_N}\ ,\\ (N-1)^{-1}C_f\delta^{fa}_{bc}&=[(N-2)!]^{-1}C^{[aa_3\cdots a_N]}\epsilon_{bca_3\cdots a_N}\ ,\\ C^{[a_2\cdots a_N]}&=(N-1)^{-1}C_b\epsilon^{ba_2\cdots a_N}\ , \end{aligned} \qquad (A2)$$

which suggests splitting off the trace part

$$\begin{aligned} C^a{}_{bc}&=C^{(0)a}{}_{bc}+(N-1)^{-1}C_f\delta^{fa}_{bc}\ ,\\ C^{(0)}_a&=C^{(0)b}{}_{ab}=0=C^{(0)[a_2\cdots a_N]}\ . \end{aligned} \qquad (A3)$$

For $N=3$ the trace-free part $C^{(0)ab}=C^{(ab)}\equiv n^{ab}$ is automatically symmetric and the decomposition is complete, but for $N>3$ one must decompose the tensor product representation of GL(N,R) on the space $R^N\otimes\Lambda^{N-2}(R^N)$, where the second factor in the tensor product is the $(N-2)$-fold wedge product of R^N with itself.

The Jacobi identity and its trace

$$C^d{}_{ab}C^e{}_{cd}+C^d{}_{bc}C^e{}_{ad}+C^d{}_{ca}C^e{}_{bd}=0\ ,\quad C_fC^f{}_{bc}=0 \tag{A4}$$

are equivalent to

$$C^a{}_{bc}C^{bca_4\cdots a_N}=0\ ,\quad C_fC^{fa_3\cdots a_N}=0\ , \tag{A5}$$

and the last equality is equivalent to either of the following:

$$C_fC^{(0)fa_3\cdots a_N}=0=C_fC^{(0)f}{}_{bc}\ . \tag{A6}$$

The pure trace part of the structure constant tensor trivially satisfies the Jacobi identity by itself and therefore corresponds to a Lie-algebra structure of its own; this Lie algebra generalizes the $N=3$ Bianchi type V Lie algebra. A natural question to ask is whether or not the trace-free part $C^{(0)a}{}_{bc}$ satisfies the Jacobi identity by itself and hence also corresponds to a Lie-algebra structure of its own, which will therefore be unimodular. If so, one can then arbitrarily scale the trace and trace-free parts independently and obtain a family of Lie-algebra deformations of the original Lie algebra. A calculation shows that

$$C^{(0)a}{}_{bc}C^{(0)bca_4\cdots a_N}=-(N-1)^{-1}C_fC^{afa_4\cdots a_N} \tag{A7}$$

and hence a further condition must be met

$$C_fC^{(0)afa_4\cdots a_N}=0 \tag{A8}$$

for the trace-free part alone to satisfy the Jacobi identity. For $N=3$, $C^{(0)ab}$ is symmetric and this is a consequence of the trace of the Jacobi identity.

This is not just a curious fact. One can ask whether or not the divergence matrices $\{\boldsymbol{\delta}_a\}$ generate a Lie algebra, in which case the supermomentum constraint functions are the moment functions for a canonical group action on the momentum phase space. A calculation gives the result

$$\begin{aligned}[\boldsymbol{\delta}_a,\boldsymbol{\delta}_b]&=(C^c{}_{ab}+C_f\delta^{fc}_{ab})\boldsymbol{\delta}_c-\mathbf{D}_{ab}\ ,\\ D_{ab}{}^f{}_g&=C^f{}_{ab}C_g+C^f{}_{bg}C_a+C^f{}_{ga}C_b\\ &=[(N-3)!]^{-1}C_eC^{fea_4\cdots a_N}\epsilon_{abga_4\cdots a_N}\ .\end{aligned} \tag{A9}$$

The extra term $\mathbf{D}_{ab}$ vanishes only if the same condition (A8) is satisfied. When this is so, the new structure constant tensor differs from $C^a{}_{bc}$ by the transformation $(C^{(0)a}{}_{bc},C_a)\to(C^{(0)a}{}_{bc},NC_a)$. When this condition is not satisfied it appears that the Poisson brackets of the supermomentum constraints do not close.

One may also decompose the Killing inner product into parts associated with the trace and trace-free parts of the structure constant tensor

$$\begin{aligned}\gamma_{ab}&=\mathrm{Tr}(\mathbf{k}_a\mathbf{k}_b)=C^b{}_{ac}C^c{}_{ab}\\ &=\gamma^{(0)}_{ab}+(N-1)^{-1}C_aC_b\ ,\\ \gamma^{(0)}_{ab}&=C^{(0)d}{}_{ac}C^{(0)c}{}_{bd}\ .\end{aligned} \tag{A10}$$

One may do the same thing for the divergence matrices

$$\boldsymbol{\delta}_a=\mathbf{k}^{(0)}_a+[(N-1)^{-1}(C_a\delta^b{}_c-N\delta^b{}_aC_c)] \tag{A11}$$

which enter into the nonpotential force

$$\begin{aligned}Q&=g^{1/2}C^a\mathrm{Tr}(\boldsymbol{\delta}_a\mathbf{g}^{-1}d\mathbf{g})\\ &=g^{1/2}C^a\mathrm{Tr}(\boldsymbol{\delta}_a\tilde{\mathbf{g}}^{-1}d\tilde{\mathbf{g}})\ .\end{aligned} \tag{A12}$$

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[1]A. Chodos and S. Detweiler, Phys. Rev. D **21**, 2167 (1980).
[2]J. Demaret and J.-L. Hanquin, Phys. Rev. D **31**, 258 (1985); J. Demaret, J.-L. Hanquin, M. Henneaux, and P. Spindel, Nucl. Phys. **B252**, 538 (1985).
[3]J. D. Barrow and J. Stein-Schabes, Phys. Rev. D **32**, 1595 (1985).
[4]Y. Hashimoto, Prog. Theor. Phys. **73**, 456 (1985).
[5]T. Furusawa and A. Hoyosa, Prog. Theor. Phys. **73**, 467 (1985).
[6]D. Lorenz-Petzold, Astrophys. Space Sci. **113**, 419 (1985); Class. Quantum Gravit. **2**, 829 (1985).
[7]P. Halpern, Phys. Rev. D **33**, 354 (1986).
[8]A. H. Taub and M. A. H. MacCallum, Commun. Math. Phys. **25**, 173 (1972).
[9]G. E. Sneddon, J. Phys. A **9**, 229 (1976).
[10]D. Eardley, Commun. Math. Phys. **37**, 287 (1974).
[11]R. Kantowski and R. K. Sachs, J. Math. Phys. **7**, 443 (1966).
[12]I. Fushiki, Gen. Relativ. Gravit. **17**, 1017 (1985).
[13]V. A. Belinsky and I. M. Khalatnikov, Zh. Eksp. Teor. Fiz. **63**, 1121 (1972) [Sov. Phys. JETP **36**, 591 (1973)].
[14]R. Arnowitt, S. Deser, and C. W. Misner, in *Gravitation: An Introduction to Current Research*, edited by L. Witten (Wiley, New York, 1962).
[15]C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973).
[16]B. S. DeWitt, Phys. Rev. **160**, 1113 (1967); see also W. R. Stoeger, A. P. Whitman, and R. Knill, J. Math. Phys. **26**, 2297 (1985).
[17]C. W. Misner, in *Magic Without Magic*, edited by J. R. Klauder (Freeman, San Francisco, 1970); in *Relativity*, edited by M. Carmeli, S. I. Fickler, and L. Witten (Plenum, New York, 1970); Phys. Rev. **186**, 1319 (1969).
[18]A. H. Taub, Ann. Math. **53**, 472 (1951).
[19]V. A. Belinsky, I. M. Khalatnikov, and E. M. Lifshitz, Ann. Phys. (N.Y.) **19**, 225 (1970); **31**, 639 (1983).
[20]R. T. Jantzen, Ann. Phys. (N.Y.) **145**, 378 (1983).
[21]B. G. Schmidt, Commun. Math. Phys. **15**, 329 (1969).
[22]R. Abraham and J. E. Marsden, *Foundations of Mechanics* (Benjamin/Cummings, Reading, Mass., 1980).
[23]R. T. Jantzen, Phys. Rev. D **33**, 2121 (1986).

[24]J. Demaret, M. Henneaux, and P. Spindel, Phys. Lett. **164B**, 27 (1985).
[25]R. Gilmore, *Lie Groups, Lie Algebras, and Some of Their Applications* (Wiley, New York, 1974).
[26]R. T. Jantzen, Ann. Phys. (N.Y.) **127**, 302 (1980).
[27]G. F. R. Ellis and M. A. H. MacCallum, Commun. Math. Phys. **12**, 108 (1969).
[28]V. Joseph, Proc. Cambridge Philos. Soc. **62**, 87 (1966).
[29]F. B. Estabrook, H. D. Wahlquist, and C. G. Behr, J. Math. Phys. **9**, 497 (1968).