Equivalence between the Thirring model and a derivative-coupling model

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We analyze the equivalence between the Thirring model and the fermionic sector of the theory of a Dirac field interacting via derivative coupling with two boson fields. For a certain choice of the parameters the two models have the same fermionic Green's functions.

In a recent work[1] we analyzed some properties of mass perturbation in the Thirring model as an example of a perturbative scheme in which the unperturbed system is not a free-field model but already incorporates some interaction. For practical reasons, instead of working directly with the Thirring model, it was convenient to use an equivalent theory, the derivative-coupling (DC) model. This theory describes an interaction of a Dirac field ψ with two fields, one scalar, η, and the other pseudoscalar, ϕ.

For specific values of the couplings, the fermionic Green's functions of the DC model turn out to be equal to those of the Thirring model as given, for example, by Klaiber.[2] This equivalence saved us a lot of technical complications making it possible, with relative ease to derive our results. However, in spite of this success, a basic question concerning the aforementioned equivalence still persists. Essentially, the problem is the following. The Thirring model has one degree of freedom, in the sense that the basic field can be written explicitly in terms of just one free scalar field. The DC model, on the other hand, has, in principle, three degrees of freedom, which can be taken to be η, ϕ, and the potential, c, of the free vector current. So, the numbers of degrees of freedom of the two models do not match. It is our purpose to clarify this situation and establish the precise way in which the equivalence of the two models should be understood. Anticipating our results, we are going to prove that in the fermionic sector a certain combination of the fields η, ϕ, and c is a spurion, or better it commutes with all the elements of the algebra generated by the fermionic components of the DC model. Besides that, to produce the same Green's functions as in the Thirring model we are forced to use another special combination of these fields. These two constraints effectively reduce the number of degrees of freedom from three to one.

The massless Thirring model is defined by the equations

$$i\partial\!\!\!/\psi(x)=-k\gamma^{\mu}N(j_{\mu}\psi)(x)\ , \tag{1}$$

$$j_{\mu}(x)=N(\bar{\psi}\gamma_{\mu}\psi)(x)\ , \tag{2}$$

$$\{\psi_{\alpha}(x),\psi^{\dagger}_{\beta}(0)\}_{\rm ET}=iZ\delta_{\alpha\beta}\delta(x^{1})\ , \tag{3}$$

where Z is a wave-function renormalization constant and the symbol N indicates a normal product prescription to be defined shortly. Both Klaiber's and Johnson's solutions can be written as

$$\psi(x)={:}\exp[i\alpha j(x)+i\tilde{\alpha}\gamma^{5}\tilde{j}(x)]\psi_{0}(x){:}\ , \tag{4a}$$

$$\begin{aligned} j_{\mu}(x)&=\tfrac{1}{4}\sum_{\epsilon,\tilde{\epsilon}}e^{-(a+\tilde{a})D^{-}(\epsilon)}\left[1+\frac{a+\tilde{a}}{4\pi}\right]^{-1/2}[\bar{\psi}(x+\epsilon)\gamma_{\mu}\psi(x)-\gamma_{\mu}\psi(x)\bar{\psi}(x-\epsilon)]\\ &=\left[1+\frac{a+\tilde{a}}{4\pi}\right]^{-1/2}\left[{:}\bar{\psi}_{0}\gamma_{\mu}\psi_{0}{:}(x)-\frac{\alpha}{2\pi}\partial_{\mu}j(x)-\frac{\tilde{\alpha}}{2\pi}\partial_{\mu}\tilde{j}(x)\right]\ , \end{aligned} \tag{4b}$$

$$N(j_{\mu}\psi)(x)=\tfrac{1}{2}[j_{\mu}(x+\epsilon)\psi(x)+\psi(x)j_{\mu}(x-\epsilon)]={:}j_{\mu}\psi{:}(x) \tag{4c}$$

with the constants a, $\tilde{a}$, α, and $\tilde{\alpha}$ given by

$$\begin{aligned} &a=\pi\left[\frac{4\pi}{\beta^{2}}-1\right],\quad \tilde{a}=\pi\left[\frac{\beta^{2}}{4\pi}-1\right]\ ,\\ &\alpha=\sqrt{\pi}\left[1-\frac{\sqrt{4\pi}}{\beta}\right],\quad \tilde{\alpha}=\sqrt{\pi}\left[1-\frac{\beta}{\sqrt{4\pi}}\right]\ , \end{aligned} \tag{5}$$

where β is the usual parameter of the sine-Gordon[3] model that is related to k by

$$k=\pi\left[\frac{\beta}{\sqrt{4\pi}}-\frac{\sqrt{4\pi}}{\beta}\right]\ . \tag{6}$$

The fields ψ_0, j, and $\tilde{j}$ are massless free fields. They are not independent fields. Indeed, as can be verified, they satisfy the commutation relations

$$[j^-(x),\tilde{j}^+(0)]=\tilde{D}^-(x)=-\frac{1}{4\pi}\ln\frac{x^0-x^1-i\epsilon}{x^0+x^1-i\epsilon}\ ,$$
$$[j^-(x),\psi_0(0)]=-i\sqrt{\pi}[\tilde{D}^-(x)+\gamma^5\tilde{D}^-(x)]\psi_0(0)\ , \tag{7}$$
$$[\tilde{j}^-(x),\psi_0(0)]=-i\sqrt{\pi}[\tilde{D}^-(x)+\gamma^5 D^-(x)]\psi_0(0)\ ,$$

where $D^-(x)=-(1/4\pi)\ln\mu^2[x^2-(x^0-i\epsilon)^2]$ is the two-point function of the scalar fields j and $\tilde{j}$. The $2n$-point Green's function

$$\begin{aligned}\langle T\psi(x_1)\cdots\psi(x_n)\bar{\psi}(y_1)\cdots\bar{\psi}(y_n)\rangle=&\exp\left[\sum_{j<k}[-(a+\tilde{a}\gamma_{xj}{}^5\gamma_{xk}{}^5)D_F(x_j-x_k)]\right]\\&\times\exp\left[\sum_{j<k}[-(a+\tilde{a}\gamma_{yj}{}^5\gamma_{yk}{}^5)D_F(y_j-y_k)]\right]\\&\times\exp\left[\sum_{j,k}[(a-\tilde{a}\gamma_{xj}{}^5\gamma_{yk}{}^5)D_F(x_j-y_k)]\right]\\&\times\langle T\psi_0(x_1)\cdots\psi_0(x_n)\bar{\psi}_0(y_1)\cdots\bar{\psi}_0(y_n)\rangle\end{aligned} \tag{8}$$

can then be computed, making intensive use of the above commutations relations and of the identities

$$\exp(A)\exp(B)=\exp(B)\exp(A)\exp([A,B])$$

and

$$C\exp(D)=\exp(d)\exp(D)C$$

which hold if $[A,B]=c$ number and $[C,D]=dD$ with $d=c$ number, respectively.

For future reference we also observe that the mass operator can be defined by

$$\begin{aligned}N(\bar{\psi}\psi)(x)&=\exp[-(a-\tilde{a})D^-(\epsilon)]\bar{\psi}(x+\epsilon)\psi(x)\\&=:\bar{\psi}_0\exp[i2\tilde{a}\gamma^5\tilde{j}]\psi_0:(x)\ .\end{aligned} \tag{9}$$

Let us now focus our attention on the DC model. Classically, the model is described by the Lagrangian[5]

$$\mathscr{L}=\bar{\psi}i\partial\!\!\!/\psi+\tfrac{1}{2}\partial_\mu\phi\partial_\mu\phi+\tfrac{1}{2}\partial_\mu\eta\partial_\mu\eta+(g\partial_\mu\eta-\tilde{g}\tilde{\partial}_\mu\phi)(\bar{\psi}\gamma_\mu\psi)\ . \tag{10}$$

and its quantum version corresponds to the equations

$$i\partial\!\!\!/\psi=-k\gamma^\mu N(g_\mu\psi)\ , \tag{11a}$$
$$\Box\eta=g\partial^\mu j_u\ , \tag{11b}$$
$$\Box\phi=-\tilde{g}\tilde{\partial}^\mu ju\ , \tag{11c}$$
$$g_\mu=-\frac{\tilde{g}}{k}\tilde{\partial}_\mu\phi+\frac{g}{k}\partial_\mu\eta\ , \tag{11d}$$
$$j_\mu=N(\bar{\psi}\gamma_\mu\psi)\ , \tag{11e}$$
$$\{\psi_\alpha(x),\psi_\beta^\dagger(0)\}_{\rm ET}=iZ_1\delta_{\alpha\beta}\delta(x^1)\ , \tag{11f}$$
$$[\phi(x),\dot{\phi}(0)]_{\rm ET}=iZ_2\delta(x^1)\ , \tag{11g}$$
$$[\eta(x),\dot{\eta}(0)]_{\rm ET}=iZ_3\delta(x^1)\ , \tag{11h}$$

where, as in the Thirring model, the Z's renormalization constants as well as the normal-product prescription, indicated by the symbol N, are specified together in the process of solving the model. Just for convenience, the coupling constant k was factorized on the right-hand side of (11a). Actually, the model is solved by the ansatz

$$\psi(x)=:\exp[ig\eta(x)+i\tilde{g}\gamma^5\phi(x)]:\psi_0(x)\ , \tag{12}$$

$$\begin{aligned}j_u(x)&=\tfrac{1}{4}\sum_{\epsilon,\tilde{\epsilon}}e^{-(g^2-\tilde{g}^2)D^-(\epsilon)}\left[1+\left|\frac{g^2-\tilde{g}^2}{4\pi}\right|\right]^{-1/2}[\bar{\psi}(x+\epsilon)\gamma_\mu\psi(x)-\gamma_\mu\psi(x)\bar{\psi}(x-\epsilon)]\\&=\left[1+\left|\frac{g^2+\tilde{g}^2}{4\pi}\right|\right]^{-1/2}\left[:\bar{\psi}_0\gamma_\mu\psi_0:(x)-\frac{g}{2\pi}\partial_\mu\eta(x)-\frac{\tilde{g}}{2\pi}\tilde{\partial}_\mu\phi(x)\right]\ ,\end{aligned} \tag{13a}$$

$$N(g_\mu\psi)(x)=\tfrac{1}{2}[g_\mu(x+\epsilon)\psi(x)+\psi(x)g_\mu(x-\epsilon]=:g_\mu\psi:(x)\ . \tag{13b}$$

The normalization factor for the current in (13a) was chosen to simplify the equations relating the DC and Thirring models. Moreover, it follows also from (13a), (11b), and (11c) that both η and ϕ are free fields. We can always redefine them, changing at the same time the couplings g and $\tilde{g}$ so that $Z_2{}^2=Z_3{}^2=1$. The case in which one of the Z's is negative, corresponding to a negative metric field, will be useful in our forthcoming discussion of the equivalence to the Thirring model.

Using the above results, one may derive the following expression for the $2n$-point function of the fermion field:

$$\begin{aligned}\langle T\psi(x_1)\cdots\psi(x_n)\bar{\psi}(y_1)\cdots\bar{\psi}(y_n)\rangle &= \exp\left[\sum_{j<k}[-(g^2+\tilde{g}^2\gamma_{xj}{}^5\gamma_{xk}{}^5)D_F(x_j-x_k)]\right]\\ &\times\exp\left[\sum_{j<k}[-(g^2+\tilde{g}^2\gamma_{yj}{}^5\gamma_{yk}{}^5)D_F(y_j-y_k)]\right]\\ &\times\exp\left[\sum_{j,k}[(g^2-\tilde{g}^2\gamma_{xj}{}^5\gamma_{yk}{}^5)D_F(x_j-y_k)]\right]\\ &\times\langle T\psi_0(x_1)\cdots\psi_0(x_n)\bar{\psi}_0(y_1)\cdots\bar{\psi}_0(y_n)\rangle\ . \end{aligned} \tag{14}$$

We may also define a mass operator

$$\begin{aligned}N(\bar{\psi}\psi)(x)&=\exp[-(g^2-\tilde{g}^2)D^-(\epsilon)]\bar{\psi}(x+\epsilon)\psi(x)\\ &=:\bar{\psi}_0\exp[i2\tilde{g}\gamma^5\phi]\psi_0:(x)\ ,\end{aligned}$$

which should be compared to the analogous expression for the Thirring model, Eq. (9).

It is now time to dissect the equivalence between the two models. Their fermionic Green's functions turn out to be equal after the following identification.

(1) For $k>0$ ($\beta^2>4\pi$),

$$\begin{aligned}\tilde{a}&=\tilde{g}^2,\quad [\phi(x),\phi(0)]=D(x)\ ,\\ a&=-g^2,\quad [\eta(x),\eta(0)]=-D(x)\ .\end{aligned} \tag{15}$$

(2) For $k<0$ ($\beta^2<4\pi$),

$$\begin{aligned}\tilde{a}&=-\tilde{g}^2,\quad [\phi(x),\phi(0)]=-D(x)\ ,\\ a&=g^2,\quad [\eta(x),\eta(0)]=D(x)\ .\end{aligned} \tag{16}$$

We have kept g and $\tilde{g}$ real at the expense of introducing an additional source of indefinite metric for the scalar fields.

The above Green's-functions identification does not hold at the operator level. To understand why this is so, it is convenient to employ a Mandelstam-type representation for the free Dirac field.[5] As mentioned earlier, in the case of the Thirring model the fields ψ_0, j, and $\tilde{j}$ are not independent. So, in order to be compatible with (6), we shall use the following boson representation:

$$j(x)=-\int_{-\infty}^{x^1}dx'^1\dot{\tilde{j}}(x'^1,x^0),\quad \partial_\mu j=\tilde{\partial}_\mu\tilde{j}\ , \tag{17}$$

$$\psi_0(x)=\mathcal{N}_0{:}\exp\left[-i\sqrt{\pi}\gamma^5\tilde{j}(x)+i\sqrt{\pi}\int_{-\infty}^{x^1}dx'^1\dot{\tilde{j}}(x')\right]{:}\chi\ , \tag{18}$$

where $\mathcal{N}_0$ is a normalization constant and χ a column matrix satisfying $\bar{\chi}\chi=1$.

Using the above expressions, the fields $\psi_{\rm Th}$, $j^\mu_{\rm Th}$, and $N(\bar{\psi}\psi)_{\rm Th}$ can be written entirely in terms of the potential $\tilde{j}$,

$$\psi_{\rm Th}(x)=\mathcal{N}{:}\exp\left[-i\frac{\beta}{2}\gamma^5\tilde{j}(x)+i\frac{2\pi}{\beta}\int_{-\infty}^{x^1}dx'^1\dot{\tilde{j}}(x')\right]{:}\chi\ , \tag{19}$$

$$J^\mu_{\rm Th}(x)=\frac{1}{\sqrt{\pi}}\tilde{\partial}_\mu\tilde{j}\ , \tag{20}$$

$$N(\bar{\psi}\psi)_{\rm Th}(x)=\mathcal{N}'{:}\cos[\beta\tilde{j}(x)]{:}\ . \tag{21}$$

For the DC model we use a boson representation similar to (18) employing a new independent field c (remember that in the DC model ψ_0, η, and ϕ are independent). Introducing a field $\tilde{\eta}$ related to η in the same way as $\tilde{j}$ is related to j in (17), we get

$$\begin{aligned}\psi_{\rm DC}=\mathcal{N}{:}\exp\Big[&-i\gamma^5(\sqrt{\pi}c-\tilde{g}\phi)\\ &+i\int_{-\infty}^{x^1}dx'^1(\sqrt{\pi}\dot{c}-g\dot{\tilde{\eta}})\Big]{:}\chi\ ,\end{aligned} \tag{22}$$

$$j^\mu_{\rm DC}=\left[1+\left|\frac{g^2-\tilde{g}^2}{4\pi}\right|\right]^{-1/2}\left[\frac{1}{\sqrt{\pi}}\tilde{\partial}_\mu C-\frac{g}{2\pi}\tilde{\partial}_\mu\tilde{\eta}-\frac{\tilde{g}}{2\pi}\tilde{\partial}_\mu\tilde{\phi}\right]\ , \tag{23}$$

$$g^\mu_{\rm DC}=-\frac{\tilde{g}}{k}\tilde{\partial}_\mu\phi+\frac{g}{k}\tilde{\partial}_\mu\tilde{\eta}\ , \tag{24}$$

$$N(\bar{\psi}\psi)_{\rm DC}=\mathcal{N}'{:}\cos(\sqrt{4\pi}c-2\tilde{g}\phi){:}\ . \tag{25}$$

Defining the fields

$$J=\left[1+\left|\frac{g^2-\tilde{g}^2}{4\pi}\right|\right]^{-1/2}\left[c-\frac{g}{\sqrt{4\pi}}\tilde{\eta}-\frac{\tilde{g}}{\sqrt{4\pi}}\phi\right]\ , \tag{26}$$

$$\begin{aligned}\sigma=&\left[1+\left|\frac{g^2-\tilde{g}^2}{4\pi}\right|\right]^{-1/2}\\ &\times\left\{-c+\frac{g}{\sqrt{4\pi}}\left[1+\frac{2\pi}{k}\left[1+\left|\frac{g^2-\tilde{g}^2}{4\pi}\right|\right]^{+1/2}\right]\eta\right.\\ &\left.+\frac{\tilde{g}}{\sqrt{4\pi}}\left[1-\frac{\tilde{g}}{\sqrt{4\pi}}\left[1+\left|\frac{g^2-\tilde{g}^2}{4\pi}\right|\right]^{+1/2}\right]\phi\right\}\ ,\end{aligned} \tag{27}$$

we can rewrite (22)—(25) as

$$\psi_{\rm DC}=\mathcal{N}\!:\exp\left[-i\frac{\beta}{2}\gamma^5 J+i\frac{2\pi}{\beta}\int_{-\infty}^{x^1}dx'^1\dot{J}(x')\right]\times\exp\left[-i\frac{k}{\sqrt{4\pi}}\left[\gamma^5\sigma+\int_{-\infty}^{x^1}dx'^1\dot{\sigma}\right]\right]:\chi\ , \tag{28}$$

$$j^\mu_{\rm DC}=\frac{1}{\sqrt{\pi}}\tilde{\partial}_\mu J\ , \tag{29}$$

$$g^\mu_{\rm DC}=\frac{1}{\sqrt{\pi}}(\tilde{\partial}_\mu J+\tilde{\partial}_\mu\sigma)\ , \tag{30}$$

$$N(\bar{\psi}\psi)_{\rm DC}=\mathcal{N}\!:\cos\left[\beta J+\frac{k}{\sqrt{\pi}}\sigma\right]:\ . \tag{31}$$

The model's equivalence in the situation specified by (15) and (16) follows from the similar roles played by J and $\tilde{j}$ in (19)—(27) and in (28)—(31). The extra field σ, relevant outside the fermionic sector of the DC model, is a spurion in the fermionic sector having no role there. Really, it is easily verified that

$$[J(x),J(0)]=D(x)\ ,\quad [J(x),\sigma(0)]=0\ ,\quad [\sigma(x),\sigma(0)]=0\ . \tag{32}$$

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