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Ghostless Feynman rules in non-Abelian gauge theories

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We have found two sets of Feynman rules for non-Abelian gauge theories in which ghosts do not appear. These Feynman rules are derived from the canonical formalism which has the advantage (over the path-integral formalism) of the propagators having explicit boundary conditions. These boundary conditions are not necessarily those of Feynman. The new Feynman rules are, however, equivalent to the conventional ones in the Feynman gauge. This demonstrates that ghosts can be eliminated at the cost of Feynman's boundary conditions.

I. INTRODUCTION

It is usually argued,[1] incorrectly,[2] that in non-Abelian gauge field theories there is no ghost in the axial gauge ($A_\mu n^\mu=0$, where n^μ is an arbitrary vector). In this paper we shall give two sets of Feynman rules which are truly ghostless.[3] The derivation of these Feynman rules are on the basis of canonical quantization.

In the past, the main difficulty in canonically deriving Feynman rules for gauge theories is the non-normalizability of the wave function. (See, for example, Ref. 4 for a discussion of this point.) We shall give two tricks for handling this non-normalizability of the wave function, obtaining the two sets of ghostless Feynman rules aforementioned. One feature of the propagator is that it does not necessarily satisfy Feynman's boundary condition which determines the sign of $i\epsilon$ in the denominator.

II. THE WEYL GAUGE

In the Weyl gauge formulation for the pure Yang-Mills theory, the Hamiltonian is

$$H_W=\int \tfrac{1}{2}(\boldsymbol{\pi}^a\cdot\boldsymbol{\pi}^a+\mathbf{B}^a\cdot\mathbf{B}^a)d^3x \ , \tag{2.1}$$

with

$$B_i^a=\tfrac{1}{2}\epsilon_{ijk}F_{jk}^a \tag{2.2}$$

and

$$F_{ij}^a=\partial_i A_j^a-\partial_i A_j^a-gf^{abc}A_i^bA_j^c \ , \tag{2.3}$$

where $A^{i,a}$ is the gauge field operator with polarization i and color index a, $\boldsymbol{\pi}^a$ is the operator conjugate to $\mathbf{A}^a$, g is the coupling constant, and f^{abc} is the structure constant of the non-Abelian group. The Gauss law is imposed on the physical states:

$$G^a\,|\,\psi,t\,\rangle=0 \ , \tag{2.4}$$

where

$$G^a=\nabla\cdot\boldsymbol{\pi}^a+gf^{abc}\mathbf{A}^b\cdot\boldsymbol{\pi}^c \ . \tag{2.5}$$

Let us assume that we can adiabatically switch off the coupling constant at the distant past and at the distant future. This means, in particular, that the state satisfying (2.4) satisfies

$$\lim_{t\to\pm\infty}\nabla\cdot\boldsymbol{\pi}\,|\,\psi,t\,\rangle=0 \ ; \tag{2.6}$$

Eq. (2.6) says that $\langle A^a\,|\,\psi,\pm\infty\,\rangle$, the wave function at $|\,t\,|=\infty$ is independent of A_L^a, the longitudinal component of $\mathbf{A}^a$. This explicitly demonstrates that the wave function is not normalizable in the measure $\int\mathscr{D}\mathbf{A}^a$. Such non-normalizable wave functions pose difficulties for the canonical derivation of Feynman rules, as the Green's functions can no longer be expressed as vacuum expectation values of time-ordered products of field operators. We shall, therefore, perform a *similarity* transformation

$$|\,\psi\,\rangle\to|\,\overline{\psi}\,\rangle=S\,|\,\psi\,\rangle \tag{2.7}$$

and

$$H_W\to\overline{H}_W=SH_WS^{-1} \ , \tag{2.8}$$

where

$$S\equiv\exp\left[\tfrac{1}{2}\int\lambda(A_L^a)^2d^3x\right] \ , \tag{2.9}$$

with λ a positive c number and with

$$A_L^a\equiv-\frac{i}{(-\nabla^2)^{1/2}}\nabla\cdot\mathbf{A}^a \ .$$

Note that A_L^a is anti-Hermitian; thus, its eigenvalues are purely imaginary. By (2.7) and (2.9), $\langle\,\mathbf{A}\,|\,\overline{\psi},t\,\rangle$ is normalizable in the measure $\int\mathscr{D}\mathbf{A}^a$. Furthermore, since $\langle\,\mathbf{A}^a\,|\,\overline{\psi},\pm\infty\,\rangle$ *factorizes* into the product $\langle\,\mathbf{A}^a\,|\,\psi,\pm\infty\,\rangle$, a function of $\mathbf{A}_T^a$ only, and S, a function of A_L^a only, we have

$$\int\mathscr{D}\mathbf{A}_T^a\langle\,\psi_f\,|\,\mathbf{A}^a\,\rangle\langle\,\mathbf{A}^a\,|\,\Psi,\infty\,\rangle$$
$$=\frac{\int\mathscr{D}\mathbf{A}^a\langle\,\overline{\psi}_f\,|\,\mathbf{A}^a\,\rangle\langle\,\mathbf{A}^a\,|\,\overline{\psi},\infty\,\rangle}{\int\mathscr{D}A_L^aS^2} \ , \tag{2.10}$$

where $\langle\,\mathbf{A}^a\,|\,\psi_f\,\rangle$ depends on $\mathbf{A}_T^a$ only. Therefore, the S matrix, which is the left side of (2.10), can be expressed as the right side of (2.10). In other words, we may deduce the S matrix from the wave function in the representation

(2.7) together with the Hamiltonian in the representation (2.8) and the measure $\int \mathscr{D}\mathbf{A}^a$.

The Hamiltonian $\overline{H}_W$ given by (2.8) can be explicitly written as

$$\overline{H}_W=\int\left[\tfrac{1}{2}(\pi^a)^2+\tfrac{1}{2}\lambda^2(A_L^a)^2+\tfrac{1}{2}(\mathbf{B}^a)^2+i\lambda\frac{1}{\nabla^2}(\nabla\cdot\mathbf{A}^a)[\nabla\cdot\pi^a-i\lambda(\nabla\cdot\mathbf{A}^a)]\right]d^3x\ . \tag{2.11}$$

And the Gauss law (2.4) becomes

$$\overline{G}^a|\overline{\psi}\rangle=0\ , \tag{2.12}$$

where

$$\overline{G}^a=\nabla\cdot\pi^a-i\lambda(\nabla\cdot\mathbf{A}^a)+gf^{abc}\mathbf{A}^b\cdot\left[\pi^c-i\nabla\lambda\frac{1}{\nabla^2}(\nabla\cdot\mathbf{A}^c)\right]\ . \tag{2.13}$$

Equations (2.11) and (2.13) can be easily derived by utilizing

$$S\pi^aS^{-1}=\pi^a-i\nabla\lambda\frac{1}{\nabla^2}(\nabla\cdot\mathbf{A}^a)\ . \tag{2.14}$$

Applying the Gauss law (2.12), we may express $\overline{H}_W$ in (2.11) as

$$\overline{H}_W=\int\left[\tfrac{1}{2}(\pi^a)^2+\tfrac{1}{2}\lambda^2(A_L^a)^2+\tfrac{1}{2}(\mathbf{B}^a)^2-igf^{abc}\lambda\left[\frac{1}{\nabla^2}\nabla\cdot\mathbf{A}^a\right]\mathbf{A}^b\cdot\left[\pi^c-i\nabla\lambda\frac{1}{\nabla^2}(\nabla\cdot\mathbf{A}^c)\right]\right]d^3x\ . \tag{2.15}$$

The expression (2.15) has an advantage over (2.11): the unperturbed part of $\overline{H}_W$ (obtained by setting $g=0$) in (2.15) is simply that of a system of uncoupled harmonic oscillators, with the frequency of a transverse mode of momentum $\mathbf{k}$ equal to $|\mathbf{k}|$, and that of the longitudinal mode of momentum $\mathbf{k}$ equal to λ. Thus the unperturbed propagator can be easily calculated to be

$$\widetilde{D}^{ij}(k)=\frac{i(\delta_{ij}-k^ik^j/|\mathbf{k}|^2)}{k^2+i\epsilon}+\frac{k^ik^j}{|\mathbf{k}|^2}\frac{i}{k_0^2-\lambda^2+i\epsilon}\ . \tag{2.16}$$

(The group factors in the propagators are not and will not be exhibited whenever there is little chance for confusion.)

The interaction terms contained in $\frac{1}{2}(\mathbf{B}^a)^2$ give all three-point vertices and four-point vertices in the usual Feynman rules with all polarization indices spatial. Furthermore, although there is no A_0 in the Hamiltonian (2.15), we may easily identify an operator which plays the same role as A_0. Let us put

$$A_0^a\equiv-i\lambda\frac{1}{\nabla^2}\nabla\cdot\mathbf{A}^a\ , \tag{2.17}$$

then the last term in (2.15) is

$$gf^{abc}A_0^a\mathbf{A}^b\cdot(\pi^c+\nabla A_0^c) \tag{2.18}$$

which accounts for all other three-point vertices and four-point vertices. From the definition (2.17), the other propagators are easily deduced as

$$\widetilde{D}^{00}(k)=-\frac{\lambda^2}{\mathbf{k}^2}\frac{i}{k_0^2-\lambda^2+i\epsilon}\ , \tag{2.19}$$

$$\widetilde{D}^{0i}(k)=-\frac{\lambda k^i}{\mathbf{k}^2}\frac{i}{k_0^2-\lambda^2+i\epsilon}\ , \tag{2.20}$$

and

$$\widetilde{D}^{i0}(k)=\frac{\lambda k^i}{\mathbf{k}^2}\frac{i}{k_0^2-\lambda^2+i\epsilon}\ . \tag{2.21}$$

Equations (2.16) and (2.19)–(2.21) can be summarized as

$$\widetilde{D}_{\mu\nu}(k)=-\frac{i}{k^2+i\epsilon}\left[g_{\mu\nu}-d(k)n_\mu k_\nu-e(k)k_\mu n_\nu+\frac{k^2-\lambda^2}{k_0^2-\lambda^2+i\epsilon}\frac{k_\mu k_\nu}{\mathbf{k}^2}\right]\ , \tag{2.22a}$$

where

$$d(k)=-e(-k)=\frac{\mathbf{k}^2-\lambda k_0}{\mathbf{k}^2(k_0-\lambda+i\epsilon)}\ , \tag{2.22b}$$

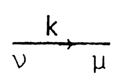
and where n^μ is the unit four-vector in the time direction. Note that

$$\widetilde{D}_{\mu\nu}(k)=\widetilde{D}_{\nu\mu}(-k)\ , \tag{2.23}$$

but

$$\widetilde{D}_{\mu\nu}(k)\neq\widetilde{D}_{\mu\nu}(-k)\ .$$

Thus the designation of the indices μ and ν are correlated with the direction of k. In our notation, $\widetilde{D}_{\mu\nu}(k)$ is diagrammatically represented by



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i.e., the direction of k is from index ν to index μ.

Next we shall prove that, for all choices of λ (λ can even depend on $\mathbf{k}$), the Feynman rules with the propagator (2.22) without ghosts are equivalent to the conventional Feynman rules.[1–3] This is done by invoking the theorem[4–6] that, if the gluon propagator is

$$\frac{i}{k^2+i\epsilon}[g_{\mu\nu}-a_\mu(k)k_\nu-b_\nu(k)k_\mu+c(k)k_\mu k_\nu]\ , \tag{2.24}$$

where

$$a_\mu(k)=-b_\mu(-k) \tag{2.25}$$

and

$$c(k)=c(-k)\ ,$$

the ghost propagator is

$$\frac{i}{k^2+i\epsilon}$$

and the ghost-ghost-gluon vertex is

$$[(a\cdot k)-1]k^{\mu}-k^2a^{\mu}\ , \tag{2.26}$$

then the physical amplitudes are independent of a_μ and c. In the present case, the ghost-ghost-gluon vertex in (2.26) is easily calculated to be

$$-\frac{k^2}{\mathbf{k}^2(k_0-\lambda+i\epsilon)}[\lambda k^{\mu}+n^{\mu}(\mathbf{k}^2-\lambda k_0)]\ . \tag{2.27}$$

For such a vertex, the integrand for a ghost loop vanishes as $|k_0|\to\infty$, and has singularities in the lower-half k_0 plane only. Therefore, we may choose to close the contour integration in the upper-half k_0 plane, obtaining zero. In other words, there is no contribution from the ghosts of (2.27) and the Feynman rules with (2.22) without ghosts are equivalent to the Feynman rules (2.24)—(2.26) with $a_\mu=b_\mu=c=0$, which are the conventional Feynman rules in the Feynman gauge.

Two special cases of (2.22) are worth mentioning.

(i) The case $\lambda=0$. Then (2.22) becomes

$$\tilde{D}_{\mu\nu}(k)=-\frac{i}{k^2+i\epsilon}\left[g_{\mu\nu}-\frac{n_\mu k_\nu}{k_0+i\epsilon}-\frac{k_\mu n_\nu}{k_0-i\epsilon}+\frac{k_\mu k_\nu}{k_0{}^2+i\epsilon}\right] \tag{2.28}$$

which can be regarded as the gluon propagator in the Weyl gauge. (Note, however, that $\tilde{D}_{0\mu}\neq 0$.)

(ii) The case $\lambda=|\mathbf{k}|$. Then (2.22) becomes

$$\tilde{D}_{\mu\nu}(k)=-\frac{i}{k^2+i\epsilon}\left[g_{\mu\nu}+\frac{n_\mu k_\nu-n_\nu k_\mu}{|\mathbf{k}|}\right]\ . \tag{2.29}$$

III. THE FEYNMAN GAUGE

In this formulation, the Hamiltonian is given by[4]

$$\mathscr{H}=\int H\,d^3x\ , \tag{3.1a}$$

where

$$H=-\tfrac{1}{2}\pi_0^a\pi_0^a+\tfrac{1}{2}\boldsymbol{\pi}^a\cdot\boldsymbol{\pi}^a-\tfrac{1}{2}(\nabla A_0^a)\cdot(\nabla A_0^a)+\tfrac{1}{2}(\nabla\cdot\mathbf{A}^a)(\nabla\cdot\mathbf{A}^a)+\tfrac{1}{2}\mathbf{B}^a\cdot\mathbf{B}^a+gf^{abc}A_0^a\mathbf{A}^b\cdot(\boldsymbol{\pi}^c+\nabla A_0^c)\ . \tag{3.1b}$$

The state is required to satisfy the Lorentz condition

$$l^a(\mathbf{x})\,|\,\psi,t\rangle=0\ , \tag{3.2}$$

where

$$l^a=-\pi_0^a+\nabla\cdot\mathbf{A}^a\ . \tag{3.3}$$

Equation (3.2), satisfied for all t, implies the Gauss law

$$G^a(\mathbf{x})\,|\,\psi,t\rangle=0\ , \tag{3.4}$$

where

$$G^a\equiv(\nabla\delta^{ab}-gf^{abc}\mathbf{A}^c)\cdot(\boldsymbol{\pi}^b+\nabla A_0^b)\ . \tag{3.5}$$

This is because we may easily calculate the commutator between $\mathscr{H}$ and l^a and get

$$[\mathscr{H},l^a]=-iG^a\ , \tag{3.6}$$

i.e., in the Heisenberg representation, G^a is equal to $\dot{l}^a$.

As a consequence of (3.2),

$$\langle A_\mu^a\,|\,\psi,t\rangle=\exp\left[i\int A_0^a\nabla\cdot\mathbf{A}^a d^3x\right]\psi_W(\mathbf{A}^a,t)\ . \tag{3.7}$$

Note that ψ_W is independent of A_0^a. Furthermore, we may prove from (3.4) and (3.7) that, if $g=0$, ψ_W is independent of A_L, thus $\psi_W(\mathbf{A}^a,t)$ is independent of A_L^a as $t\to\pm\infty$, when the coupling constant is adiabatically switched. Therefore, the dependence of $\langle A^a\,|\,\psi,\pm\infty\rangle$ on A_0^a and A_L^a is contained entirely in the exponential factor in (3.7). Consequently, in the measure $\int\mathscr{D}A_\mu^a$, $\langle A_\mu\,|\,\psi,\pm\infty\rangle$ is not normalizable, if we integrate over *real* values of A_0^a and $\nabla\cdot\mathbf{A}^a$.

Instead, let us take A_0^a and $\nabla\cdot\mathbf{A}^a$ to be complex. More precisely, let us put

$$A_0^a\equiv\frac{u^a+v^a}{\sqrt{2}} \tag{3.8}$$

and

$$A_L^a\equiv\frac{u^2-v^a}{\sqrt{2}}\ . \tag{3.9}$$

The unperturbed part of H in (3.1) for the A_L^a and A_0^a becomes

$$-\tfrac{1}{2}[\pi_u{}^2+\pi_v{}^2+(\nabla u)^2+(\nabla v)^2]\ , \tag{3.10}$$

and the exponential factor in (3.7) becomes

$$\exp\left[\int[\tfrac{1}{2}v^a(-\nabla^2)^{1/2}v^a-\tfrac{1}{2}u^a(-\nabla^2)^{1/2}u^a]d^3x\right]\ . \tag{3.11}$$

The factor in (3.11) is normalizable if we choose v to be purely imaginary and u to be real. The S matrix can be obtained with the normalization as $\langle A_\mu^a\,|\,\psi,\pm\infty\rangle$ has the factorized form aforementioned.

With this normalization, we easily obtain the free propagators for u and v as

$$\langle Tu(x)u(0)\rangle=-i\int\frac{d^4k}{(2\pi)^4}\frac{e^{-ikx}}{k^2-i\epsilon}\ , \tag{3.12}$$

$$\langle Tv(x)v(0)\rangle=-i\int\frac{d^4k}{(2\pi)^4}\frac{e^{-ikx}}{k^2+i\epsilon}\ , \tag{3.13}$$

and

$$\langle Tv(x)u(0)\rangle=0\ , \tag{3.14}$$

where T denotes time ordering and $\langle\ \rangle$ denotes the expectation value with respect to the unperturbed vacuum. Note that the denominator of the integrand in (3.12) is $(k^2-i\epsilon)$, while that in (3.13) is $(k^2+i\epsilon)$. This is because the energy in a u mode is negative, while that in a v mode, with v purely imaginary, is positive. It follows from (3.8), (3.9), (3.12), and (3.13) that the free propagators for A_0, A_L are

$$\langle TA_0(x)A_0(0)\rangle = -i\int\frac{d^4k}{(2\pi)^4}e^{-ikx}\times\frac{1}{2}\left[\frac{1}{k^2-i\epsilon}+\frac{1}{k^2+i\epsilon}\right], \tag{3.15}$$

$$\langle TA_L(x)A_L(0)\rangle = -i\int\frac{d^4k}{(2\pi)^4}e^{-ikx}\times\frac{1}{2}\left[\frac{1}{k^2-i\epsilon}+\frac{1}{k^2+i\epsilon}\right], \tag{3.16}$$

$$\langle TA_0(x)A_L(0)\rangle = -i\int\frac{d^4k}{(2\pi)^4}e^{-ikx}\times\frac{1}{2}\left[\frac{1}{k^2-i\epsilon}-\frac{1}{k^2+i\epsilon}\right], \tag{3.17}$$

and

$$\langle TA_L(x)A_0(0)\rangle = -i\int\frac{d^4k}{(2\pi)^4}e^{-ikx}\times\frac{1}{2}\left[\frac{1}{k^2-i\epsilon}-\frac{1}{k^2+i\epsilon}\right]. \tag{3.18}$$

One notices that, had we ignored the sign of $i\epsilon$ in (3.15)—(3.18), we would have arrived at the conclusion that the gluon propagator of momentum k is $-ig_{\mu\nu}/k^2$, and that there are no ghosts in the Feynman rules in this gauge. Such an error underlines the importance of the boundary conditions dictated by the initial and the final wave functions, which have been ignored in the path-integral formulation for gauge fields.

Equations (3.15)—(3.18), together with the usual propagators in the transverse modes, can be summarized as

$$\tilde{D}_{\mu\nu}(k) = -\frac{ig_{\mu\nu}}{k^2+i\epsilon}+\frac{i}{\mathbf{k}^2}(ak^\nu n^\mu + bk^\mu n^\nu + ck^\mu k^\nu), \tag{3.19}$$

where

$$a(k) = \frac{|\mathbf{k}|-k_0}{2}\left[\frac{1}{k^2-i\epsilon}-\frac{1}{k^2+i\epsilon}\right],$$

$$c(k) = \frac{1}{2}\left[\frac{1}{k^2-i\epsilon}-\frac{1}{k^2+i\epsilon}\right],$$

and

$$b(k) = -a(-k).$$

Note that we have used the identity

$$\frac{k_0{}^2}{|\mathbf{k}|^2}\left[\frac{1}{k^2-i\epsilon}-\frac{1}{k^2+i\epsilon}\right] = \left[\frac{1}{k^2-i\epsilon}-\frac{1}{k^2+i\epsilon}\right]$$

in deriving (3.19). It is easy to verify, by using the rules given in (2.24)—(2.26), that the ghost loops give zero amplitude after integrating over the time component of the loop momentum. Therefore, using the propagator (3.19) with no ghosts is equivalent to using the propagator $-ig_{\mu\nu}/(k^2+i\epsilon)$ with the ghost-ghost-gluon vertex $-k^\mu$, which are the conventional rules in the Feynman gauge.

One notes that the propagator given by (3.19) does not satisfy

$$k^\mu \tilde{D}_{\mu\nu} = 0 .$$

This does not contradict the Lorentz condition (3.2). The reason is that the Lorentz condition is imposed as a supplementary condition on the wave function, not as an operator equation.

IV. CONCLUSION

The quantum theory of non-Abelian gauge fields can be formulated either via canonical quantization or via path integration.[1-3] The path-integral formulation is merely heuristic. One of the reasons is that, in the path-integral approach, the form of the propagator is deduced from the form of the unperturbed Lagrangian. However, the Lagrangian only provides the differential equation satisfied by the propagator. To determine the propagator uniquely, one needs, in addition, boundary conditions for the propagators. In the case of scalar field theories, for example, such boundary conditions are provided by the initial and final wave functions (wave functions at the distant past and at the distant future), and turn out to be those of Feynman. In the case of gauge field theories, the conventional expression of the path integral does not specify the initial and the final wave functions. As a matter of fact, the wave functions one should use in such an expression are not normalizable, making the path integral infinite. Such an infinity cannot be handled by the Faddeev-Popov trick. Furthermore, this non-normalizability remains in the Euclidean space; hence, this difficulty cannot be resolved by the formulation in the Euclidean space. Thus the boundary conditions for the propagator in a gauge field theory formulated via path integrals are ambiguous. In contrast, we derive the Feynman rules on the basis of canonical quantization. The non-normalizability of the wave function is handled by either multiplying the wave function with a cutoff factor, or continuing into the complex space. No additional auxiliary variables are needed, and hence there are no ghosts.

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