Monopoles in topologically massive gauge theories

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In the presence of a Chern-Simons term, a single Dirac monopole induces a current that extends throughout space-time. The Dirac string acquires a constant charge per unit length, which should confine monopoles and antimonopoles by a linear potential. The monopole of 't Hooft and Polyakov is generalized to include a Chern-Simons term for the gauge field, and the string tension for these monopoles is calculated.

Topologically massive gauge theories—gauge theories in $2+1$ dimensions with a Chern-Simons[1] mass term—provide examples of some extraordinary phenomena.[2–4] In this paper I consider what happens to monopoles in these theories.

Dirac monopoles[5] in the Abelian theory are the subject of Sec. I. Henneaux and Teitelboim[6] have shown that for a Dirac monopole to be a solution of the field equations, a current induced by the monopole must be included. I extend their analysis to show that for a single Dirac monopole the induced current is of infinite range: for a spherical shell of radius R about the monopole, the current flowing in through this shell is a constant, balanced by the current flowing out through the Dirac string.

The Abelian example helps both to motivate and to explain the results of Sec. II. 't Hooft[7] and Polyakov[8,9] showed that magnetic monopoles are produced when an SO(3) gauge theory is spontaneously broken to U(1). I consider how the monopole changes when a Chern-Simons term for the SO(3) gauge field is added to the action. I argue that there is a monopole solution that is regular everywhere, but find that the action for a single monopole, S_M, is infinite: if R is the radius of space-time, the action diverges linearly in R, $S_M \sim \sigma R$ as $R \to \infty$.

This divergence can be understood by remembering that over large distances, by a gauge rotation a non-Abelian monopole can always be treated as if it were effectively Abelian. From Sec. I the total charge induced by a Dirac monopole grows linearly in R, so it is not surprising to find that the action of the underlying non-Abelian field behaves in the same way. Alternately, this linear divergence can be viewed as the self-energy of a charged Dirac string. A monopole-antimonopole pair does have finite action $S_{M\bar{M}}$, which is linear in their separation R for large R, $S_{M\bar{M}} \sim \sigma R$. Because the action for a single monopole is infinite, I term this the "confinement" of monopoles. From $S_{M\bar{M}}$ we see that monopoles and antimonopoles are confined by a linear potential, with a string tension that is just σ.

For the Abelian theory without monopoles there is no relation between the Chern-Simons mass m and the gauge coupling e. In the non-Abelian theory, topological gauge invariance requires $4\pi m/g^2(\equiv q)$ to be an integer; g is the non-Abelian coupling constant. For the Abelian theory with Dirac monopoles, following Alvarez[10] and Henneaux and Teitelboim[6] it can be shown that gauge invariance requires $\pi m/e^2$ to be an integer. In Sec. III I show how in the presence of a $Z(2)$ monopole in an (unbroken) SU(2) gauge theory, q must be not just an integer, but an even integer. While the usual quantization condition on q is related to the one cocycle of the gauge group, in the presence of a $Z(2)$ monopole the two cocycle of the group also enters.

Throughout this work I assume the conventions and notation of Refs. 3 and 4.

I. DIRAC MONOPOLES

I first review how to introduce Dirac monopoles[5] in $2+1$ dimensions when there is no Chern-Simons term. Given the field strength $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$, and its dual $\tilde{F}_\mu=\frac{1}{2}\epsilon_{\mu\nu\lambda}F^{\nu\lambda}$, the field equations in the absence of monopoles are

$$\partial_\mu F^{\mu\nu}=j^\nu,\quad \partial_\mu \tilde{F}^{\mu}=0\ . \tag{1.1}$$

j^ν is the usual current density and is conserved, $\partial_\mu j^\mu=0$.

In $2+1$ dimensions, monopoles are points not in space but in space-time, with a density k that is a space-time pseudoscalar. For the time being, I prefer to avoid Dirac strings by using a (pseudoscalar) monopole potential ϕ. I define a new field strength $G_{\mu\nu}$:

$$G_{\mu\nu}=F_{\mu\nu}+\epsilon_{\mu\nu\lambda}\partial_\lambda\phi\ . \tag{1.2}$$

For $k\neq 0$, the field equations are

$$\partial_\mu G^{\mu\nu}=j^\nu,\quad \partial_\mu \tilde{G}^{\mu}=k\ . \tag{1.3}$$

Of course ϕ is an unphysical degree of freedom and could be eliminated by introducing a Dirac string for each monopole. This is possible because the equations for A_μ and ϕ decouple $-\partial_\mu G^{\mu\nu}=\partial_\mu F^{\mu\nu}=j^\nu$ and $\partial_\mu\tilde{G}^{\mu}=\partial^2\phi=k$.

It is worth emphasizing that monopoles in $2+1$ dimensions are rather different from those in $3+1$ dimensions. Since monopoles in $2+1$ dimensions are points in space-time, they cannot be viewed as a type of particle, but rather as a form of "instanton." Further, monopoles in $2+1$ dimensions are characterized by nonzero $\widetilde{G}^{\mu}$, so they carry both electric and magnetic fields. I nevertheless refer to such objects as (Dirac) monopoles, for they do share an essential property with their cousins in $3+1$ dimensions: the divergence of the dual of the field strength tensor is only nonzero at monopole sources, Eq. (1.3).

With a Chern-Simons term, but without monopoles the field equations in Euclidean space-time are

$$\partial_{\mu}F^{\mu\nu}+im\widetilde{F}^{\nu}=j^{\nu},\quad \partial_{\mu}\widetilde{F}^{\mu}=0\ . \tag{1.4}$$

To include monopoles, the natural thing to do in Eq. (1.4) is to replace $F^{\mu\nu}$ by $G^{\mu\nu}$, and $\widetilde{F}^{\mu}$ by $\widetilde{G}^{\mu}$. This cannot be the whole story, however, for if one takes

$$\partial_{\mu}G^{\mu\nu}+im\widetilde{G}^{\nu}=j^{\nu}\ ,$$

then the divergence of this equation gives $im\,\partial_{\nu}\widetilde{G}^{\nu}=0$—so $k=0$. To avoid this, Henneaux and Teitelboim[6] add a new term to the current, j_M^{ν}. The complete field equations are then

$$\partial_{\mu}G^{\mu\nu}+im\widetilde{G}^{\nu}=j^{\nu}+j_M^{\nu}\ , \tag{1.5}$$

$$\partial_{\mu}\widetilde{G}^{\mu}=k\ . \tag{1.6}$$

As before, $\partial_{\mu}\widetilde{G}^{\mu}=\partial^2\phi=k$, but for Eq. (1.5) to consistent, j_M^{ν} must satisfy

$$\partial_{\mu}j_M^{\mu}=im\,\partial^2\phi=imk\ . \tag{1.7}$$

j_M^{ν} represents a current which is induced by the monopole. I take the obvious solution of Eq. (1.7):

$$j_M^{\mu}=im\,\partial^{\mu}\phi\ . \tag{1.8}$$

Then Eq. (1.5) reduces to $\partial_{\mu}F^{\mu\nu}+im\widetilde{F}^{\nu}=j^{\nu}$, and as before the equations for A_{μ} and ϕ decouple. Note that the induced current j_M^{μ} is imaginary in Euclidean space-time and real in Minkowski space-time.

The field equations of Eqs. (1.5) and (1.6) are generated by the Lagrangian

$$\mathscr{L}=\tfrac{1}{4}G_{\mu\nu}{}^2-\frac{im}{2}\epsilon^{\mu\nu\lambda}A_{\mu}\partial_{\nu}A_{\lambda}+j^{\mu}A_{\mu}+(j_M^{\mu}-im\,\partial^{\mu}\phi)A_{\mu}+\phi k\ . \tag{1.9}$$

The total monopole current that couples to the vector potential A_{μ} is $J_M^{\mu}=j_M^{\mu}-im\,\partial^{\mu}\phi$. While the induced current j_M^{μ} is not conserved, the total monopole current J_M^{μ} is; indeed, from Eq. (1.8) J_M^{μ} vanishes identically. Thus the action formed from Eq. (1.9) is gauge invariant— even at points where there are monopoles—up to the usual terms that depend upon the boundary of space-time. As will be seen in Sec. III, these surface terms can be significant if there are monopoles about.

It is instructive to solve for the example of a single Dirac monopole. Given a monopole of strength g_M at the origin of space-time,

$$k=g_M\delta^3(\mathbf{x})\ . \tag{1.10}$$

Then

$$\phi=\frac{g_M}{4\pi}\frac{1}{r}\ , \tag{1.11}$$

$r=\sqrt{\mathbf{x}^2}$, and

$$j_M^{\mu}=-\frac{im}{4\pi}g_M\frac{\hat{x}^{\mu}}{r^2}\ , \tag{1.12}$$

$\hat{x}^{\mu}=x^{\mu}/r$. If a surface encloses the monopole, the induced current flowing through the surface is $\int j_M^{\mu}d^2S^{\mu}=-img_M$.

ϕ can be eliminated at the expense of a Dirac string. (For a given time slice, the string is at best a "Dirac point.") Then $\widetilde{F}^{\mu}=(g_M/4\pi)\hat{x}^{\mu}/r^2$ and j_M^{μ} is as in Eq. (1.12), except on the string. As the string carries $\widetilde{F}^{\mu}$, it must also carry an induced current, $j_M^{\mu}\sim+img_Mt^{\mu}$, where t^{μ} is the tangent vector to the string. In the string description of the monopole, current conservation is preserved because the induced current flowing into a surface which encloses the monopole (but not the string) equals $-img_M$, which is canceled by an induced current equal to $+img_M$ flowing out through the string.

The necessity of introducing the induced current j_M^{μ} is best seen in the string description of the monopole. The crucial point is that with a Chern-Simons term, $\widetilde{F}^{\nu}$ appears directly in the field equation $\partial_{\mu}F^{\mu\nu}+im\widetilde{F}^{\nu}=j^{\nu}$. Because of this term, when $m\neq0$ currents j^{ν} produce a flux for $\widetilde{F}^{\nu}$.[3] What happens with a monopole is the converse of this: flux producing a current. To have $\partial_{\mu}\widetilde{F}^{\mu}\sim\delta^3(\mathbf{x})$, $\widetilde{F}^{\mu}\sim\hat{x}^{\mu}/r^2$, so away from the monopole and its string, if the monopole is to be a solution of the equations of motion, there must be an induced current. Current conservation then dictates the current carried by the Dirac string.

There is an analogy to the current j_M^{μ} in $3+1$ dimensions. Witten[11] has shown that when the Θ parameter is nonzero, monopoles acquire an electric charge $\sim\Theta$. While this is something like what happens here, there is an important and obvious distinction. In $3+1$ dimensions for $\Theta\neq0$, the charge is concentrated on the monopole. With a Chern-Simons term in $2+1$ dimensions, the current induced by the monopole runs throughout space-time, including along the string.

As the Dirac string carries a current when $m\neq0$, it becomes a physical entity. It is natural to guess that, since the string has a given charge per unit length, the electromagnetic self-energy of the string will give rise to some mass per unit length σ. σ is then the coefficient of a confining, linear potential for a straight string between a monopole antimonopole pair.

This confinement of monopoles by physical strings is similar to what happens to monopoles from the Higgs effect. If one has Dirac monopoles in an U(1) gauge group that is spontaneously broken, the monopoles are confined by a linear potential. However the photon gets a mass—from either the Chern-Simons term or the Higgs effect—naively the confinement of monopoles can be viewed as the result of the physical vacuum being unable to support the long-range U(1) fields of a monopole.

This explanation is a bit glib. When the U(1) symmetry is broken, the Dirac string represents a tube of unbroken U(1) flux. The degrees of freedom for this string are directly related to those fields which are responsible for the symmetry breaking; e.g., the monopoles' string tension is proportional to the difference in the action density between the broken and unbroken phases.

In contrast, with a Chern-Simons term the string is only visible because it carries a current. The Chern-Simons term does not break the gauge symmetry nor does it add new degrees of freedom—it is simply inconsistent to have an invisible Dirac string in its presence. Further, there is no string tension between the monopoles until electromagnetic self-energies are computed.

Dirac monopoles are point objects, so in the Abelian theory all self-energies diverge. In the next section I consider an SO(3) gauge theory which is spontaneously broken to U(1). For the monopoles in this theory, the vacuum expectation value of the Higgs field provides a natural length scale for their size. This cuts off the ultraviolet divergence in the monopole self-energy, and so allows for the direct computation of the string tension σ.

II. 't HOOFT-POLYAKOV MONOPOLES

For an SO(3) gauge theory which is spontaneously broken to U(1) by an isotriplet Higgs field, 't Hooft[7] and Polyakov[8] showed that certain static solutions in 3 + 1 dimensions represented magnetic monopoles in the unbroken U(1). As static solutions of the field equations in 3 + 1 dimensions, these monopoles carry over unaltered to 2 + 1 dimensions. Polyakov[9] has shown that in 2 + 1 dimensions, a dilute plasma of these monopoles produces the confinement of external U(1) charges. In this section I add a Chern-Simons term for the gauge field to the action and find the monopole analogous to that of 't Hooft and Polyakov. Contrary to the case without a Chern-Simons term, it is the monopoles and not external (electric) charges which are confined. I remark that the solution I find has no counterpart in 3 + 1 dimensions: while there are close similarities between the Chern-Simons term and Θ vacua in 3 + 1 dimensions, only the Chern-Simons term contributes to the local equations of motions—the Θ term does not.

To look for a monopole solution, I take as an ansatz

$$h^a(\mathbf{x})=\hat{x}^a h \,, \tag{2.1}$$

for the Higgs boson field, and

$$A^a_\mu(\mathbf{x})=\frac{1}{r}[\epsilon_{a\mu\nu}\hat{x}^\nu(1-\phi_1)+\delta^{a\mu}\phi_2+(rA-\phi_2)\hat{x}^a\hat{x}^\mu] \tag{2.2}$$

for the gauge field. I assume that the functions h, ϕ_1, ϕ_2, and A all depend only on the radius $r=\sqrt{\mathbf{x}^2}$.

Equations (2.1) and (2.2) are the most general solution which are invariant under combined rotations of isospin and space-time. The form of this ansatz is preserved under an Abelian subgroup of gauge rotations generated by Ω_{ab},

$$\Omega_{ab}=\exp(if\hat{x}\cdot\sigma/2) \,, \tag{2.3}$$

$f=f(r)$, σ denotes the Pauli matrices. The gauge-transformed $\tilde{A}_\mu=\Omega^{-1}{}_{ab}(\partial_\mu+A_\mu)\Omega_{ab}$, with

$$\begin{aligned}\tilde{\phi}_1&=\phi_1\cos f+\phi_2\sin f \,,\\ \tilde{\phi}_2&=-\phi_1\sin f+\phi_2\cos f \,,\\ \tilde{A}&=A-f' \,,\end{aligned} \tag{2.4}$$

$f'\equiv df/dr$.

For the monopole of 't Hooft[7] and Polyakov,[8] only h and ϕ_1 are nonzero, with $\phi_2=A=0$. It will be seen that when a Chern-Simons term for the gauge field is added to the action, it is crucial to consider the full ansatz of Eq. (2.2).

The total action S is a sum of three terms, $S=S_0+S_m+S_h$. S_0 is the usual action for the gauge field:[7,8]

$$\begin{aligned}S_0&=-\frac{1}{2g^2}\int d^3x\,\mathrm{tr}(F_{\mu\nu}F^{\mu\nu})\\ &=\frac{4\pi}{g^2}\int_0^\infty dr\left[(\phi_1'+A\phi_2)^2+(\phi_2'-A\phi_1)^2\right.\\ &\qquad\left.+\frac{1}{2r^2}(1-\phi_1{}^2-\phi_2{}^2)^2\right] .\end{aligned} \tag{2.5}$$

S_m is the contribution of the Chern-Simons term:[12]

$$\begin{aligned}S_m&=\frac{1}{g^2}\int d^3x(-im)\epsilon^{\mu\nu\lambda}\mathrm{tr}(A_\mu\partial_\nu A_\lambda+\tfrac{2}{3}A_\mu A_\nu A_\lambda)\\ &=\frac{4\pi}{g^2}\int_0^\infty dr[im(\phi_1'\phi_2-\phi_2'\phi_1+\phi_2')\\ &\qquad-A(1-\phi_1{}^2-\phi_2{}^2)] .\end{aligned} \tag{2.6}$$

Lastly, S_h involves the Higgs fields:

$$\begin{aligned}S_h&=\frac{1}{g^2}\int d^3x[\tfrac{1}{2}(D_\mu h^a)^2-\tfrac{1}{2}\mu^2h^ah^a+\tfrac{1}{4}\lambda(h^ah^a)^2]\\ &=\frac{4\pi}{g^2}\int_0^\infty dr\left[\frac{r^2}{2}(h')^2+h^2(\phi_1{}^2+\phi_2{}^2)-\frac{\mu^2}{2}r^2h^2\right.\\ &\qquad\left.+\frac{\lambda r^2}{4}h^4\right] .\end{aligned} \tag{2.7}$$

In Eqs. (2.5)–(2.7), A^a_μ and h^a are scaled by $1/g$ relative to Refs. 3 and 4. I work in the regime of weak coupling, where the dimensional couplings g^2 and λg^2 (λ is dimensionless) are much less than the masses m and μ, m and $\mu>0$.

Under the gauge transformation of Eq. (2.4), $S_m\to\tilde{S}_m$,

$$\tilde{S}_m=S_m+\frac{4\pi m}{g^2}i[\phi_2(\cos f-1)-\phi_1\sin f+f]\,|_{r=0}^{\infty} . \tag{2.8}$$

Consider the class of gauge transformations for which $f(0)$ and $f(\infty)$ are each equal to 2π (integer). From Eq. (2.4), such transformations do not alter ϕ_1 or ϕ_2 at either $r=0$ or ∞; from Eq. (2.8), they do change the action by $2\pi i(4\pi m/g^2)$ (integer). Thus within the ansatz of Eq. (2.2) these are the topological gauge transformations which are responsible for quantizing $q=4\pi m/g^2$ as an integer. This is the usual condition on q, and it is not altered by the presence of a monopole. This is unlike Dirac monopoles,[6,10] or Z_2 monopoles in SU(2) (Sec. III), where monopoles do change the quantization condition.

From Eq. (2.8) it is possible to choose a "unitary" gauge in which $\phi_2=0$ everywhere. Henceforth I set $\phi_2=0$, and redefine $\phi_1 \to \phi$. The Chern-Simons term becomes

$$S_m = \frac{4\pi}{g^2}\int_0^\infty dr[-imA(1-\phi^2)] \ . \tag{2.9}$$

Equation (2.9) shows why it is necessary to consider the general ansatz of Eq. (2.2). If the usual monopole ansatz was assumed, $\phi_2=A=0$, one would have concluded that the monopole was unaffected by the Chern-Simons term, $S_m=0$. Because S_m is linear in A, S_m will not be extremal with respect to small variations in A at $A=0$ unless $\phi=1-$ and $A_\mu^a=0$.

This just means that when $m\neq 0$, the stationary point of the action has $A\neq 0$. For $\phi_2=0$, S_0 has a term $\sim A^2\phi^2$. Since no derivatives of A appear we can solve for it:

$$A = \frac{im}{2}\frac{1-\phi^2}{\phi^2} \ . \tag{2.10}$$

Substituting this back gives the gauge field action found by d'Hoker and Vinet:[12]

$$S_0+S_m = \frac{4\pi}{g^2}\int_0^\infty dr\left[(\phi')^2+\frac{(1-\phi^2)^2}{4}\left[\frac{2}{r^2}+\frac{m^2}{\phi^2}\right]\right] . \tag{2.11}$$

Because A is imaginary for real ϕ, the vector potential A_μ^a is complex. While worrisome, since the action itself is complex in Euclidean space-time this seems inescapable. Surprisingly, the action we are left with in unitary gauge, Eqs. (2.7) and (2.11), is real, and has solutions with real ϕ and h.

The equations of motion for ϕ and h are

$$\phi'' = \frac{\phi(\phi^2-1)}{r^2}+\frac{m^2}{4}\frac{(\phi^4-1)}{\phi^3}+h^2\phi \ , \tag{2.12}$$

$$h''+\frac{2}{r}h' = \frac{2h}{r^2}\phi^2+(-\mu^2+\lambda h^2)h \ , \tag{2.13}$$

$\phi''=d^2\phi/dr^2$, etc.

I claim that there is a regular solution of Eqs. (2.12) and (2.13) which is a U(1) monopole. That a solution is a monopole is actually guaranteed by the form of the ansatz. Consider the Abelian field strength $F'_{\mu\nu}$ (Ref. 7):

$$F'_{\mu\nu} = \frac{h^a}{h}F^a_{\mu\nu}-\frac{1}{h^3}\epsilon_{abc}h^aD_\mu h^bD_\nu h^c = -\epsilon_{\mu\nu a}\frac{\hat{x}^a}{r^2} \ , \tag{2.14}$$

as long as $h\neq 0$; $\tilde{F}'_\mu = -\hat{x}^\mu/r^2$ is the dual field strength for a monopole at the origin.

About the origin, the m-dependent term in Eq. (2.12) is small relative to that $\sim 1/r^2$. Thus the limiting behavior as $r\to 0$ is the same as when $m=0$: $\phi\sim 1+O(r^2)$, and $h\sim r$.

At large r, the terms involving $1/r$ and derivatives d/dr can be neglected, so the Higgs field behaves as expected: $-h(r)\to h_\infty \equiv (\mu^2/\lambda)^{1/2}$ as $r\to\infty$. By the same reasoning, from Eq. (2.12) when $m\neq 0$, ϕ is also nonzero at infinity: $\phi(r)\to \pm\phi_\infty$ as $r\to\infty$,

$$\phi_\infty = \left[\frac{m^2}{m^2+h_\infty{}^2}\right]^{1/4} ; \tag{2.15}$$

by convention I choose $\phi(r)\to +\phi_\infty$. From Eq. (2.10), this means that $A(r)\to A_\infty$ at infinity:

$$A_\infty = \frac{im}{2}\frac{1-\phi_\infty{}^2}{\phi_\infty{}^2} \ . \tag{2.16}$$

As m is tuned to zero, $\phi_\infty \sim \sqrt{m}$ and vanishes smoothly, whereas $A_\infty \sim ih_\infty/2$ does *not*. This is not a contradiction: if g^2 is taken as given, by the topological quantization condition m must be an integer q times $g^2/4\pi$, and so m cannot vanish continuously.

For large r, the corrections to h_∞ and ϕ_∞ are well behaved:

$$h(r) \underset{r\to\infty}{\sim} h_\infty\left[1-\frac{\phi_\infty{}^2}{\mu^2}\frac{1}{r^2}+\cdots\right] ,$$
$$\phi(r) \underset{r\to\infty}{\sim} \phi_\infty\left\{1+\frac{\phi_\infty{}^4}{m^2}\left[1-\phi_\infty{}^2\left[1-\frac{2}{\lambda}\right]\right]\frac{1}{r^2}+\cdots\right\} . \tag{2.17}$$

As for the 't Hooft-Polyakov monopole, I cannot find an exact solution, but surely there is a regular solution that smoothly interpolates between the behavior found at large and small r.

While this solution is regular, it does not have finite action. To evaluate the action, the contribution of the trivial vacuum, $h(\mathbf{x})=h_\infty$, $A_\mu^a(\mathbf{x})=0$, must be subtracted. The remainder has a divergence that is linear in R, the radius of space-time:

$$S_0+S_m+S_h \sim \frac{4\pi}{g^2}\int_0^R\left[h^2\phi^2+\frac{m^2}{4}\frac{(1-\phi^2)^2}{\phi^2}+\cdots\right] \tag{2.18}$$

$$\sim \sigma R + O(1) \ . \tag{2.19}$$

The terms in the action which are not written in Eq. (2.18) do not contribute to the linear divergence, Eq. (2.19).

This linear divergence occurs because when $m\neq 0$, $A_\mu^a(\mathbf{x})$ is not a pure gauge rotation at space-time infinity: ϕ_∞ and $A_\infty \neq 0$. For the Dirac monopoles of Sec. I, there is a current induced by the monopole at infinity. In the present example, this induced current is carried by the Higgs field. For the ansatz of Eqs. (2.1) and (2.2), the current for the Higgs field is

$$j_\mu^a = \epsilon_{\mu ab}\hat{x}^b\frac{h^2\phi}{r} \ .$$

Hence, because $\phi_\infty \neq 0$, there is a Higgs current at infinity, $j_\mu^a \sim \epsilon_{\mu ab}\hat{x}^b h_\infty{}^2\phi_\infty/r$ as $r\to\infty$.

As discussed in my introductory remarks, σ is the string tension for a monopole antimonopole pair. From Eq. (2.18), it is easy to read off σ:

$$\sigma = \frac{4\pi}{g^2}\left[h_\infty{}^2\phi_\infty{}^2+\frac{m^2}{4}(1-\phi_\infty{}^2)^2\right] . \tag{2.20}$$

For small m,

$$\sigma \underset{m \ll h_\infty}{\sim} \frac{4\pi}{g^2} m h_\infty \; , \tag{2.21}$$

while for large m,

$$\sigma \underset{m \gg h_\infty}{\sim} \frac{4\pi}{g^2} h_\infty{}^2 \; . \tag{2.22}$$

Without a Chern-Simons term, in perturbation theory the static potential for the unbroken U(1) gauge field is governed by single photon exchange, and is logarithmic in space. Polyakov[9] has demonstrated how even a dilute plasma of monopoles changes the U(1) potential from logarithmic into linear. This occurs because the monopoles have Coulombic interactions over large distances, and respond to external currents as a charged plasma.

With a Chern-Simons term, monopoles and antimonopoles are bound to each other so tightly that they do not respond as a plasma, but as a gas of "molecules." Because the U(1) field has a Chern-Simons mass, the perturbative U(1) potential is Yukawa-like. Over large distances, this is not altered by the (small) monopole molecules.

This result is not as surprising as it might first appear. Without the Chern-Simons term, the photon is massless and there are long-range correlations between fields. These correlations build up to the produce the confining plasma of monopoles.

The situation is very different when there is a Chern-Simons term. All gauge fields, including the photon, are massive, so long-range correlations are not expected. As argued by d'Hoker and Vinet,[12] this lack of long-range correlations should mean that topologically massive theories do not confine external charges. My results here illustrate that, although it is amusing to see how in detail this comes about—by the confinement of monopoles.

There is one point of subterfuge to which I should confess. Equation (2.12) only determines the value of ϕ^4 at infinity, so the field equations allow not only $\phi \to \pm\phi_\infty$ as $r \to \infty$, but $\phi \to \pm i\phi_\infty$. The solutions I constructed have real ϕ and imaginary A; if $\phi \to \pm i\phi_\infty$, ϕ, A, and so A^a_μ will all be purely imaginary. Unlike the σ of Eq. (2.20), which is always positive, imaginary ϕ can give $\sigma < 0$; physically, this is not sensible.

I suggest the purely imaginary A^a_μ should not be included as saddle points in the functional integral. To calculate the quantum fluctuations about a complex background field, it is necessary to consider how to deform the contour of integration over $A^a_\mu(\mathbf{x})$. For some solutions, the required deformations of contour will be allowed, while for others, not. I believe that one could sensibly expand about the monopole solutions with real ϕ that I have constructed (or more precisely, a monopole antimonopole pair), but that an analysis of small fluctuations would exclude *any* purely imaginary gauge potentials—such as the monopole with imaginary ϕ.

I argue this by analogy. Consider a real scalar field ϕ with a potential $V(\phi) = \mu^2\phi^2/2 + \lambda\phi^4/4$, μ^2 and $\lambda > 0$. A solution to the field equations is $\phi_i = \pm i(\mu^2/\lambda)^{1/2}$, with an action density $V(\phi_i) = -\mu^4/4\lambda$. While $V(\phi_i)$ is less than that of the perturbative vacuum $\phi = 0$, surely ϕ_i has nothing to do with the physical vacuum for positive μ^2 and λ.

This reasoning also excludes some of the solutions found by d'Hoker and Vinet,[12] who found purely imaginary $A^a_\mu(\mathbf{x})$ that are constant in space-time. While these solution have an action density that is negative, because A^a_μ is purely imaginary I do not think they represent a physical vacuum state.

III. $Z(2)$ MONOPOLES

't Hooft[13] has argued that in (unbroken) SU(N) gauge theories, quark confinement in 2 + 1 dimensions occurs due to a condensate of $Z(N)$ monopoles. These $Z(N)$ monopoles are effectively Abelian configurations, so by Sec. I, adding a Chern-Simons term for the gauge field should result in the confinement of the $Z(N)$ monopoles. This strongly suggests[12] that quarks are *not* confined in topologically massive SU(N) gauge theories—certainly for weak coupling, $m \gg g^2$, and probably for any (renormalized) $m \neq 0$.

If $Z(N)$ monopoles are inserted by hand into an SU(N) gauge theory, they will alter the allowed values of $q = 4\pi m/g^2$. I specialize to an SU(2) gauge group without matter fields, and work at temperature $T \neq 0$ (Ref. 4). At finite temperature, $A_\mu(\mathbf{x},t)$ must be strictly periodic in time,

$$A_\mu(\mathbf{x}, t+\beta) = +A_\mu(\mathbf{x},t) \; ,$$

$\beta = T^{-1}$, but since there are no fields in the fundamental representation, the allowed gauge transformations Ω need only be periodic up to a constant element of $Z(2)$:

$$\Omega(\mathbf{x}, t+\beta) = \pm\Omega(\mathbf{x},t) \; .$$

Let $\Omega = \exp(\omega)$, and assume that at spatial infinity $(\mathbf{x} \to \infty)$, ω approaches a constant value.

I consider a $Z(2)$ monopole at the origin of space $(\mathbf{x} = 0)$ at a time t', and an antimonopole at $\mathbf{x} = 0$ and a time t'', $t'' > t'$. There are two different ways to run the Dirac string between these monopoles; for simplicity I assume that the string runs along the time axis. The first is just to run it from $t = t'$ up to $t = t''$. There is a second, however: run the string up the time axis from $t = t''$ to $t = \beta$. Then, using the periodicity in time, run the rest of the string from $t = 0$ up to $t = t'$. These two ways are topologically distinct, and I assume the second.

In a background field A_μ, under a gauge transformation the action for the Chern-Simons term transforms as[2,3]

$$S_m \to S_m + 2\pi i \left(\frac{4\pi m}{g^2} \right) (w + \Delta w) \; . \tag{3.1}$$

The winding number w depends only on Ω and is an integer for any compact manifold.[1,4] Δw is a surface term, and is a function of both Ω and the background field A_μ:

$$\Delta w = \frac{1}{8\pi^2} \int d^2S^\mu \epsilon^{\mu\nu\lambda} \mathrm{tr}(\partial_\nu \Omega \Omega^{-1} A_\lambda) \; , \tag{3.2}$$

d^2S^μ is the surface element for space-time.

I evaluate Δw for a special Ω and the A_μ of a monopole. I take the monopole and antimonopole to be very close to each other, $t'' \sim t'$. As only the boundary of space-time enters into Eq. (3.2), for $t'' \sim t'$ I can neglect the monopole fields, and consider only the effects of the Dirac string. Thus in Eq. (3.2) I can take A_μ to be pure gauge, $A_\mu = \Omega_M{}^{-1}\partial_\mu \Omega_M / g$;

$$\Omega_M = \exp(\omega_M) = \exp\left[-\frac{i}{2}\alpha\sigma^3\right] . \quad (3.3)$$

α is the angular direction in space, $\mathbf{x} = r_s(\cos\alpha, \sin\alpha)$. Because of the $Z(2)$ string, Ω_M is multivalued:[13]

$$\Omega_M(\alpha + 2\pi) = -\Omega_M(\alpha) . \quad (3.4)$$

Since Ω and Ω_M are periodic in time up to ± 1, in Eq. (3.2) the integrals over space at $t=0$ and β cancel, leaving only the integrals over the spatial boundaries. This includes not only spatial infinity, but possibly a small circle of infintesimal radius ϵ about the Dirac string at $r_s = 0$. Over these boundaries, ω and ω_M are constant elements of the Lie algebra, so

$$\Delta w = \frac{1}{8\pi^2} \int dS^i \int_0^\beta dt\, \epsilon^{ij} \mathrm{tr}(+\partial_0\omega\partial_j\omega_M - \partial_j\omega\partial_0\omega_M)\Big|_{r_s=\epsilon}^{\infty} . \quad (3.5)$$

dS^i is a surface element for space at $r_s = \epsilon$ or ∞. Integrating ∂_0 by parts, and dropping terms $\sim \partial_0\partial_j$ (this is allowed for the ω and ω_M I use),

$$\Delta w = \frac{1}{8\pi^2} \int dS^i \epsilon^{ij} \mathrm{tr}(\omega\partial_j\omega_M - \omega_M\partial_j\omega)\Big|_{r_s=\epsilon}^{\infty}\Big|_{t=0}^{\beta} . \quad (3.6)$$

Δw in Eq. (3.6) is manifestly an integral over the two-cocycle of the gauge group.

I choose an Ω similar to that used in Ref. 4:

$$\Omega = \exp\left[-i\pi\frac{t}{\beta}\hat{\Theta}\cdot\sigma\right] . \quad (3.7)$$

Unlike the Ω of Ref. 4, this Ω is antiperiodic in time. That is allowed here, because there are no fields in the fundamental representation. I take $\hat{\Theta} = \hat{\Theta}(\mathbf{x})$ to be a two-dimensional instanton[14] centered on the origin with instanton number$=1$. I require that as $\mathbf{x} \to \infty$, $\hat{\Theta}\cdot\sigma \to +\sigma^3$; then $\hat{\Theta}\cdot\sigma \to -\sigma^3$ as $\mathbf{x} \to 0$.

Evaluating Δw for this Ω gives $\Delta w = \frac{1}{2}$. From Eq. (3.1), for the partition function to be gauge invariant, $q = 4\pi m / g^2$ must be not just an integer, but an *even* integer.

The quantization condition on q differs in the presence of monopoles because the Dirac string carries flux, and when the string pierces the boundary of space-time, surface terms such as Eq. (3.2), which are usually negligible, become important.

For $Z(N)$ monopoles in SU(N), I suspect that the analogous condition is that q must be a multiple of N. My direct but inelegant approach is not the best way to show this.

Similar arguments can be used to derive the quantization condition on m/e^2 for Dirac monopoles in the Abelian theory.[6,10] Under a gauge transformation $A_\mu \to A_\mu + \partial_\mu\Lambda/e$, the Abelian Chern-Simons term transforms as

$$-\int d^3x \left[\frac{im}{2}\right] \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda \to \text{same} - \frac{im}{2e}\int d^2S^\mu \Lambda \tilde{F}_\mu ,$$

I consider a Λ which wraps around the time direction in a nontrivial way: $\Lambda(\mathbf{x},\beta) = 2\pi$, $\Lambda(\mathbf{x},0) = 0$. Taking a monopole-antimonopole pair as above, only the contribution of the Dirac string cutting through $t=\beta$ matters. For a Dirac monopole of unit charge, the flux through the string is $2\pi/e$; the action changes by $-2\pi i(\pi m/e^2)$, so $\pi m/e^2$ must be an integer.

I have been a bit sloppy in these arguments, since I have neglected the induced current which by Sec. I must run through the string. Including it, however, does not alter the conclusions: in the Abelian case, this current contributes $+2\pi i(2\pi m/e^2)$ to the action, while for the non-Abelian, it gives $-2\pi i(4\pi m/g^2)$.

In the Abelian theory, an alternate argument can be given following Henneaux and Teitelboim.[6] In Minkowski space-time, the Dirac string carries a real charge per unit length $e_M = m g_M$. For a monopole of unit strength, $g_M = 2\pi/e$; requiring $e_M g_M$ to be 2π (integer) implies that $2\pi m/e^2$ must be an integer. This is less restrictive than my condition that $\pi m/e^2$ be an integer.

In this section I have only considered topological gauge invariance at the classical level. For the sake of discussion, let me assume that the mass and charge are renormalized in the same way when there are monopoles around as not. Because monopoles are large gauge fields, this assumption could well be wrong.

In an SU(N) gauge theory with N_f flavors of adjoint fermions of mass m_f, about the trivial vacuum q is renormalized[3,4] as $q_{\mathrm{ren}} = q + N + N N_f \mathrm{sgn}(m_f)$. [The fermions must lie in a representation such as the adjoint so that they do not spoil the $Z(N)$ symmetry.] Consequently, if q is a multiple of N, so is q_{ren}: topological gauge invariance is still maintained in the quantum theory with $Z(N)$ monopoles.

In the Abelian theory without monopoles, Coleman and Hill[15] have shown that $p \equiv \pi m/e^2$ is only renormalized to one-loop order, $p_{\mathrm{ren}} = p + \frac{1}{4}N_f \mathrm{sgn}(m_f)$ for fermions of charge e. Apparently, for p_{ren} to remain an integer N_f must be a multiple of 4.

More generally, perhaps a consideration of the quantum theory with monopoles could yield a unified—and topological—understanding of all of these results: in the Abelian theory, a nonperturbative proof of the theorem of Coleman and Hill, and in the non-Abelian SU(N) theory, why $q_{\mathrm{ren}} - q$ is proportional to N.

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