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Classical dynamics of strings with rigidity

T. L. Curtright
Department of Physics, University of Florida, Gainesville, Florida 32611

G. I. Ghandour
Department of Physics, Kuwait University, Kuwait

C. K. Zachos
High Energy Physics Division, Argonne National Laboratory, Argonne, Illinois 60439
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We explore the classical dynamics of strings with rigidity, i.e., with terms added to the string action which depend on the extrinsic curvature of the world sheet. We study classical solutions of the string equations of motion using both analytical and numerical methods, and we determine the leading Regge trajectories $J(E^2)$ for a set of classical rotating string configurations. We observe that for open strings the dominant classical solutions are identical to those for the conventional Nambu string, and correspondingly give linear trajectories. However, for closed strings we find new solutions that include finite-energy, static configurations, and that give trajectories which are nonlinear as the lowest-energy solution is approached, but become linear asymptotically as $J(E^2)\rightarrow\infty$.

I. INTRODUCTION

Polyakov[1] has recently emphasized the interesting possibility of adding a rigidity term to the action of the relativistic string. Such a term represents the extrinsic curvature of the string's world sheet as embedded in higher-dimensional spacetime, and produces interactions containing quartic derivatives. Indeed, such terms may appear in effective actions of strings as a result of integrating out fermions in functional integrals.[2] Polyakov also noted that the coupling corresponding to the inverse rigidity is asymptotically free, although these renormalization effects had previously been investigated in other contexts.[3] For example, it was known that lipid membranes with small surface tension have their effective rigidity reduced at long distances (long-wavelength ripples) due to thermal fluctuations in the fluid in which the membranes are immersed.

As discussed by us in an earlier Letter,[4] interactions which are sensitive to the extrinsic properties of the world-sheet embedding in spacetime dramatically alter the spin content of the string, especially for closed-string configurations. In addition, such terms should have a significant bearing on the compactification of extra spatial dimensions, if indeed this is induced uniquely by string dynamics. Also, besides controlling a "crumpled phase" of strings,[1] these interactions are expected to suppress longitudinal kink/fold modes.[5] Finally, it is not clear what happens to general covariance in the embedding spacetime in models containing such extrinsic-curvature terms, since it seems possible to remove the graviton from the zero-mass excitation spectrum. ($m_{\text{graviton}}{}^2$ might be used as the independent parameter, instead of the rigidity.) Of course, from the point of view of hadronic physics, this may be a desirable feature for Pomeron phenomenology.

In this paper we present the details of our nonperturbative, classical study of the dynamical effects of extrinsic curvature terms by exploring the solutions of the string equations of motion when nontrivial rigidity is present. (A reader might like to contemplate a rubber band as an intuitive, albeit limited, mechanical analog.) As will become clear, our study is in several ways analogous to the construction of classical monopole solutions in non-Abelian gauge theories.

If we define classical Regge trajectories, as is customary, to be the angular momentum (J) versus energy squared (E^2) relationship for solutions of the classical equations of motion, then there are linear leading trajectories for the open rigid string. The corresponding lowest-energy solution is the usual straight-line, pinwheel motion[6] familiar from the conventional, pliable, Nambu string. Extrinsic curvature vanishes for this motion. However, for the closed string, there is another motion which supplants, for low angular momentum, the standard folded-over pinwheel rotation of the Nambu string. This new motion is the rotation of an oblate closed loop, which reduces to a circle in the static limit, with finite energy at zero angular momentum. For nonzero angular momentum, the rotation rate for this string configuration first increases, reaches a maximum, and then decreases again as the energy and angular momentum increase monotonically. The result is a $J(E^2)$ trajectory which is approximately a hyperbola for $J>0$. For very high angular momentum, the string configuration elongates considerably as the rotation rate decreases. As the rotation rate now goes to zero, there is an approach to a second limiting configuration of two infinite, parallel straight lines, with infinite energy.

The outline of this paper is as follows. In Sec. II we review the area and rigidity terms in the action for the string and discuss some of their symmetries and geometrical properties. In Sec. III we go over the variational analysis of higher-derivative theories, specifically to obtain the equations of motion and the energy-momentum density for the rigid string. We discuss how the rigid-string model shares with the Nambu theory the property of general coordinate invariance on the world sheet, and we show explicitly that the model has a vanishing energy-momentum density before gauge fixing is used to break both world-sheet reparametrization and spacetime Lorentz invariance.

In Sec. IV we begin our study of the solutions of the classical equations of motion for the rigid string. We establish a lemma to the effect that all *nonsingular* classical solutions for the Nambu string (which has vanishing rigidity) are solutions for the rigid-string equations as well, and that these solutions also carry the same energy and angular momentum as in the Nambu case. We then review the conventional pinwheel motion of the open string which is indeed a configuration for both the Nambu and rigid strings, as suggested by our lemma. We discuss how closed string configurations for the Nambu model are simply folded-over elaborations of the basic pinwheel motion. These configurations are sufficiently singular at the folds, however, that the rigid string does *not* admit the same closed-string motions as the Nambu model. We then describe the basic closed rigid string configurations. These are rotating planar hoops, with nonsingular planar curvature. We construct the static, nonrotating solution exactly (a circular hoop), and perturbatively analyze the effects of small rotation rates (small J).

In Sec. V we carry our investigation of the closed-string solutions a little farther analytically. We change variables to the point that we obtain a simple action, and a numerically tractable, nonlinear, second-order differential equation. The equations are shown to be of the Painlevé type. However, we can construct exact analytic solutions only for special limiting cases. In Sec. VI we then present a numerical analysis of the closed-string equations, which allows us to extend the previous small-J perturbative results to arbitrary angular momentum and energy. We find numerical solutions by using boundary-value methods, and also by extremizing an effective action considered as a function of initial data. Both methods agree within numerical uncertainties. We define numerically the angular momentum and energy, and hence compute classical Regge trajectories for the closed rigid strings. These are approximated by a hyperbolic relation between E^2 and J.

Finally, in Sec. VII we examine the consequences of the resulting nonlinear trajectory structure and speculate on the physical significance of our results. However, we leave open the fundamental question of quantization. A thoroughgoing incorporation of quantum mechanics is the major outstanding problem for this class of string theories.

In an appendix we discuss in passing the (global) spacetime supersymmetric extensions of the model investigated in the paper, and we comment on a first-order formulation. The problem of finding a κ (local) world-sheet supersymmetry is not solved, however.

II. EXTRINSIC CURVATURE AND THE RIGID-STRING ACTION

Recall the conventional Nambu-Goto "area law" action in second-order formalism:

$$I_1=-T_0\int d^2\xi\sqrt{-g}\,,\quad g_{ab}\equiv\partial_aX^\mu\partial_bX_\mu\,, \tag{2.1}$$

where T_0 is the tension, ξ^a $(a,b=0,1)$ are the world-sheet parameters, and X^μ $(\mu=0,\ldots,D-1)$ are the spacetime string coordinates. The corresponding equation of motion is the covariant wave equation

$$\partial_a\frac{\partial\mathscr{L}}{\delta\partial_aX_\mu}=-T_0\sqrt{-g}\;\Box X^\mu=0\,, \tag{2.2}$$

where

$$\Box X^\mu=\frac{1}{\sqrt{-g}}\partial_a(\sqrt{-g}\;g^{ab}\partial_bX^\mu)=g^{ab}D_aD_bX^\mu\,, \tag{2.3}$$

the last step following from $D_aD_bX^\mu=(\partial_a\partial_b-\Gamma^c_{ab}\partial_c)X^\mu$ and the well-known identity $\partial_a(\sqrt{-g}\;g^{ab})=-\sqrt{-g}\;g^{ac}\Gamma^b_{ac}$. Since the action I_1 depends only on the metric g_{ab} of the world sheet, it is sensitive only to the intrinsic geometry of the sheet, and is impervious to the extrinsic curvature which characterizes the embedding of the sheet in spacetime. Thus, I_1 does not distinguish between flat and corrugated sheets, for example.

However, one may also contemplate[1-3] strings with interaction terms which depend on the gradients of the tangents to the world sheet (i.e., the derivatives of the "vielbein" ∂_aX^μ) through the second fundamental form:[7]

$$K^i_{ab}=n^i_\mu\partial_a\partial_bX^\mu\,. \tag{2.4}$$

Here n^i_μ are the $D-2$ unit normals to the sheet:

$$n^i_\mu n^{j\mu}=\delta^{ij},\quad n^i_\mu\partial_aX^\mu=0,\quad i=1,\ldots,D-2\,. \tag{2.5}$$

In general, the Gauss-Weingarten formulas give the complete gradients of the vielbein,

$$\partial_a\partial_bX^\mu=\Gamma^c_{ab}\partial_cX^\mu+K^i_{ab}n^{i\mu}\,, \tag{2.6}$$

including the components tangent to the sheet. These components follow from using the form of the induced metric, as given in (2.1), in the definition of the Christoffel symbol to obtain

$$\Gamma_{abc}\equiv(\partial_ag_{bc}+\partial_bg_{ac}-\partial_cg_{ab})/2=\Gamma^d_{ab}g_{dc}$$
$$=\partial_a\partial_bX^\mu\partial_cX_\mu\,. \tag{2.7}$$

The gradients of the normals also split into two types of components,

$$\partial_an^i_\mu=-(n^i_\nu\partial_an^{j\nu})n^j_\mu-K^i_{ab}g^{bc}\partial_cX_\mu\,, \tag{2.8}$$

as may be easily checked by differentiating (2.5). Finally, rewriting the Gauss-Weingarten formulas as

$$D_aD_bX^\mu\equiv\partial_a\partial_bX^\mu-\Gamma^c_{ab}\partial_cX^\mu=K^i_{ab}n^{i\mu} \tag{2.9}$$

immediately allows us to reexpress the covariant wave equation (2.3) as

$$\Box X^\mu=g^{ab}K^i_{ab}n^{i\mu}\,. \tag{2.10}$$

We now use this last relation to give two alternate forms for the new extrinsic curvature-dependent term to be appended to the string action:

$$I_2 = S_0 \int d^2\xi \sqrt{-g}\,(\Box X^\mu)^2 = S_0 \int d^2\xi \sqrt{-g}\,(g^{ab}K^i_{ab})^2 . \quad (2.11)$$

The coupling S_0 is called the "rigidity," or "stiffness," since it tends to prevent curving of the world sheet in the enveloping spacetime.

Note that the other bilinear which could be formed from K^i_{ab} is not independent when integrated over the world sheet, since by Gauss's equation[7] it is related to the above bilinear in (2.11) and the intrinsic sheet curvature:

$$(K^{ia}_a)^2 - K^{ia}_b K^{ib}_a = 2\sum_i \det K^i = R . \quad (2.12)$$

By the Gauss-Bonnet theorem, the integral of $\sqrt{-g}\,R$ is 4π times the Euler characteristic of the sheet, i.e., the topological invariant [2−No. of boundaries−2×(No. of handles)].

A reader may wish to visualize these concepts in ordinary Euclidean space by a sheet of paper, whose intrinsic curvature vanishes. The paper's extrinsic curvature also vanishes when it lies flat, but not when it is rolled up into a right cylinder of radius r (Fig. 1). Essentially, the extrinsic curvature represents the sum of the inverses of the two principal radii of curvature, while the intrinsic curvature represents the product of these inverses. Hence, the latter vanishes when *either* of the radii becomes infinite.

The action for the model of interest is the sum of I_1 and I_2:

$$I_R = -S_0 \int d^2\xi \sqrt{-g}\left[\frac{1}{R_0{}^2} - (\Box X^\mu)^2\right], \quad R_0{}^2 \equiv S_0/T_0 , \quad (2.13)$$

where we have factored S_0 from the expression and defined a radius R_0. This is correct dimensionally (e.g., S_0 is dimensionless, T_0 is not) as follows from considering the scaling properties of I_1 and I_2. By construction, of course, I_R is world-sheet reparametrization invariant. However, more to the point, I_1 and I_2 are each separately invariant under scaling of the sheet parameters ξ, but only I_2 is invariant under $X^\mu \to \lambda X^\mu$.

After substituting $\partial_a X^\mu \partial_b X_\mu$ for g_{ab}, we may regard the theory defined by (2.13) as a two-dimensional field theory in flat space with higher derivative couplings and with the D-dimensional spacetime serving as the internal-symmetry space.[1] Given the obvious $SO(D)/SO(2) \times SO(D-2)$ symmetry possessed by the model, it is natural to identify it with a Grassmannian σ model. However, the latter would require $2(D-2)$ degrees of freedom, while (2.13) exhibits D, so $D-4$ fields would need to be constrained away in order to validate this identification.

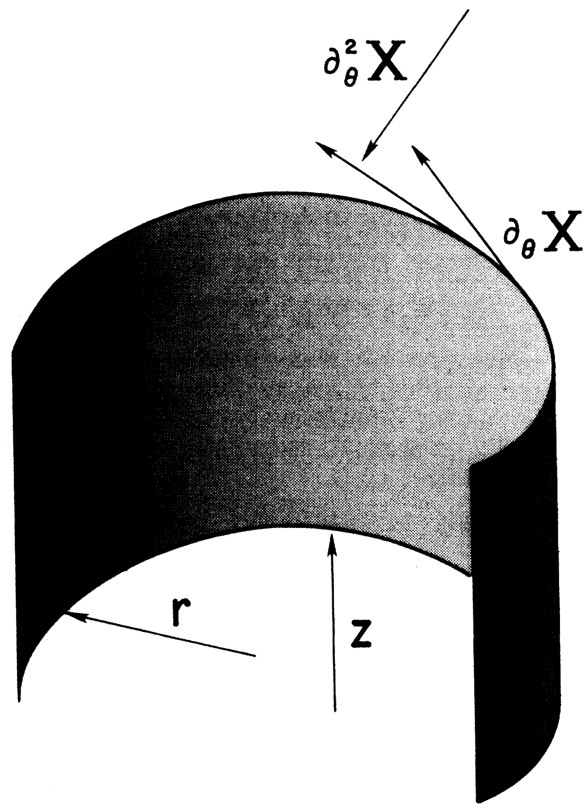

FIG. 1. Extrinsic curvature of a rolled sheet of paper: z points along the axis of symmetry, and $\hat{n} = \hat{r}$. In cylindrical coordinates $g_{zz} = 1$, $g_{rr} = 1$, $g_{\theta\theta} = r^2$, and all components of K_{ab} vanish except for $K_{\theta\theta} = -r$, whence $K^a_a = -1/r$. This is the extrinsic curvature. The Euclidean version of (2.11) thus reduces to $I_2 = -S_0 \int dz\, d\theta\, r\, 1/r^2$.

III. VARIATIONS OF $\mathscr{L}$, CONSERVATION LAWS, AND THE ENERGY-MOMENTUM DENSITY

To establish the classical equations of motion and conservation laws, we briefly review general variations of multiderivative actions such as I_R. As usual, $\delta I_R = \int d^2\xi\, \delta\mathscr{L}$, where the variation of the Lagrangian density is

$$\begin{aligned}\delta\mathscr{L} &= (\delta X^\mu)\delta\mathscr{L}/\delta X^\mu + (\partial_a \delta X^\mu)\delta\mathscr{L}/\delta(\partial_a X^\mu) + (\partial_a\partial_b \delta X^\mu)\delta\mathscr{L}/\delta(\partial_a\partial_b X^\mu) \\ &= (\delta X^\mu)[\delta\mathscr{L}/\delta X^\mu - \partial_a \delta\mathscr{L}/\delta(\partial_a X^\mu) + \partial_a\partial_b \delta\mathscr{L}/\delta(\partial_a\partial_b X^\mu)] \\ &\quad + \partial_a[(\delta X^\mu)\delta\mathscr{L}/\delta(\partial_a X^\mu) - (\delta X^\mu)\partial_b \delta\mathscr{L}/\delta(\partial_a\partial_b X^\mu) + \partial_b(\delta X^\mu)\delta\mathscr{L}/\delta(\partial_a\partial_b X^\mu)] . \quad (3.1)\end{aligned}$$

In this expression, off-diagonal pairs of indices are understood to occur only once in the sum over a and b. Demanding that $\delta\mathscr{L}=0$ gives the equations of motion [first brackets in (3.1)] and the proper boundary conditions [second brackets in (3.1)]. Also from (3.1), by positing specific variations δX^μ which leave the Lagrangian density invariant, we obtain the higher derivative form of Noether's on-shell conservation theorem: $\partial_a J^a=0$, where

$$J^a=(\delta X^\mu)\delta\mathscr{L}/\delta(\partial_a X^\mu)-(\delta X^\mu)\partial_b\delta\mathscr{L}/\delta(\partial_a\partial_b X^\mu)+\partial_b(\delta X^\mu)\delta\mathscr{L}/\delta(\partial_a\partial_b X^\mu)\ . \tag{3.2}$$

For example, when δX^μ is an infinitesimal spacetime rotation, the corresponding conserved charge, $J\equiv\int d\xi^1 J^0$, is the spacetime angular momentum of the string.

Similarly, demanding translational invariance on the world sheet, i.e.,

$$\partial_c\mathscr{L}=(\partial_c X^\mu)\delta\mathscr{L}/\delta X^\mu+(\partial_c\partial_a X^\mu)\delta\mathscr{L}/\delta(\partial_a X^\mu)+(\partial_c\partial_a\partial_b X^\mu)\delta\mathscr{L}/\delta(\partial_a\partial_b X^\mu)\ , \tag{3.3}$$

leads to the on-shell energy-momentum conservation law: $\partial_a\theta^a{}_b=0$, where the canonical energy-momentum tensor is

$$\theta^a{}_b=-\delta^a{}_b\mathscr{L}+(\partial_b X^\mu)\delta\mathscr{L}/\delta(\partial_a X^\mu)-(\partial_b X^\mu)\partial_c\delta\mathscr{L}/\delta(\partial_c\partial_a X^\mu)+\partial_c\partial_b X^\mu\delta\mathscr{L}/\delta(\partial_c\partial_a X^\mu)\ . \tag{3.4}$$

Thus, $E=\int d\xi^1\theta^0{}_0$ is the conserved energy. (This particular result will be used in determining the classical Regge trajectories for the completely gauge-fixed action in the later sections.)

Actually, for the string theory defined by (2.13), the energy-momentum tensor $\theta^a{}_b$ is degenerate, as in the case of the Nambu string. If we apply (3.4) to the Lagrangian in the form (2.13) which is completely covariant (i.e., before we fix a gauge which breaks world-sheet reparametrization invariance and spacetime Lorentz invariance) we find

$$\theta^a{}_b=0\ . \tag{3.5}$$

This may be seen by explicit calculation as follows.

First, without violating the conclusions, we may choose a class of world-sheet coordinates which satisfy the condition

$$\dot X\cdot X'=0\ , \tag{3.6}$$

where

$$\dot X_\mu\equiv\partial X_\mu/\partial\xi^0,\quad X'_\mu\equiv\partial X_\mu/\partial\xi^1\ . \tag{3.7}$$

Such a condition is Lorentz invariant in the embedding spacetime. Now, to compute the energy-momentum density, we must expand the Lagrangian density in (2.13) keeping terms to first order in $\dot X\cdot X'$, since (3.4) involves first variations. For this, we require $\sqrt{-g}$ and $\sqrt{-g}\,g^{ab}$ to order $\dot X\cdot X'$:

$$g_{ab}=\begin{bmatrix}\dot X^2 & \dot X\cdot X'\\ \dot X\cdot X' & X'^2\end{bmatrix}\ , \tag{3.8}$$

$$\sqrt{-g}\,g^{ab}=\frac{1}{\sqrt{-g}}\begin{bmatrix}-X'^2 & \dot X\cdot X'\\ \dot X\cdot X' & -\dot X^2\end{bmatrix}\ , \tag{3.9}$$

and

$$\sqrt{-g}=[(\dot X\cdot X')^2-\dot X^2X'^2]^{1/2}=(-\dot X^2X'^2)^{1/2}+O((\dot X\cdot X')^2)\ . \tag{3.10}$$

It is now straightforward to use these and expand the Lagrangian density for the rigid string to first order in $\dot X\cdot X'$. Multiplying by $\sqrt{-g}$ for later convenience, we obtain

$$\sqrt{-g}\ \mathscr{L}_R=T_0\dot X^2X'^2+S_0[\mathscr{L}_0+\mathscr{L}_1+O((\dot X\cdot X')^2)]\ , \tag{3.11}$$

where

$$\mathscr{L}_0=-\frac{X'^2}{\dot X^2}\ddot X^2-\frac{\dot X^2}{X'^2}X''^2-2\ddot X\cdot X''+\frac{1}{\dot X^2}\left[X'\cdot\ddot X+\frac{\dot X^2}{X'^2}X'\cdot X''\right]^2+\frac{1}{X'^2}\left[\dot X\cdot X''+\frac{X'^2}{\dot X^2}\dot X\cdot\ddot X\right]^2\ , \tag{3.12}$$

$$\mathscr{L}_1=(\dot X\cdot X')\left[\frac{6\dot X\cdot X''X'\cdot\ddot X}{X'^2\dot X^2}+\frac{2\dot X\cdot\ddot X X'\cdot\ddot X}{\dot X^4}+\frac{4\dot X'\cdot\ddot X}{\dot X^2}-\frac{2\dot X\cdot\ddot X X'\cdot X''}{\dot X^2X'^2}+\frac{2\dot X\cdot X''X'\cdot X''}{X'^4}+\frac{4\dot X'\cdot X''}{X'^2}\right]\ . \tag{3.13}$$

The Nambu term in (3.10) gives no contribution to the canonical energy-momentum tensor, since

$$\dot X_\mu\delta\sqrt{-g}\,/\delta\dot X_\mu=X'_\mu\delta\sqrt{-g}\,/\delta X'_\mu=\sqrt{-g}$$

and (3.14)

$$\dot X_\mu\delta\sqrt{-g}\,/\delta X'_\mu=X'_\mu\delta\sqrt{-g}\,/\delta\dot X_\mu=0\ .$$

These are true whether we use (3.6) or not. We therefore need to consider only the contributions from $\mathscr{L}_0$ and $\mathscr{L}_1$, which simplify if we make use of (3.6). Note that

$$\delta(\sqrt{-g}\ \mathscr{L}_{0,1})/\delta\dot X'_\mu=0\ , \tag{3.15}$$

and that

$$\dot X_\mu\delta(\sqrt{-g}\ \mathscr{L}_{0,1})/\delta\ddot X_\mu=0=X'_\mu\delta(\sqrt{-g}\ \mathscr{L}_{0,1})/\delta X''_\mu\ , \tag{3.16}$$

$$X'_\mu\delta(\sqrt{-g}\ \mathscr{L}_{0,1})/\delta\ddot X_\mu=0=\dot X_\mu\delta(\sqrt{-g}\ \mathscr{L}_{0,1})/\delta X''_\mu\ ,$$

where the condition (3.6) is imposed after evaluating variations. We incorporate these results into (3.4) to obtain

$$\sqrt{-g}\,\theta^0{}_0 = S_0[-2\sqrt{-g}\,\mathscr{L}_0 + 2\ddot{X}_\mu \delta(\sqrt{-g}\,\mathscr{L}_0)/\delta\ddot{X}_\mu + \dot{X}_\mu \delta(\sqrt{-g}\,\mathscr{L}_0)/\delta\dot{X}_\mu]\,, \tag{3.17}$$

$$\sqrt{-g}\,\theta^1{}_1 = S_0[-2\sqrt{-g}\,\mathscr{L}_0 + 2X''_\mu \delta(\sqrt{-g}\,\mathscr{L}_0)/\delta X''_\mu + X'_\mu \delta(\sqrt{-g}\,\mathscr{L}_0)/\delta X'_\mu]\,.$$

Because of the choice in (3.6), $\mathscr{L}_1$ does not contribute to the diagonal components of $\theta^a{}_b$. Evaluating (3.17) now reveals complete cancellation of terms:

$$\theta^0{}_0 = \theta^1{}_1 = 0\,. \tag{3.18}$$

We stress that equations of motion were *not* used to obtain these results.

Next, consider the off-diagonal components of $\theta^a{}_b$. Again incorporating (3.14)–(3.16) into (3.4), we have

$$\sqrt{-g}\,\theta^1{}_0 = S_0[2\dot{X}'_\mu \delta(\sqrt{-g}\,\mathscr{L}_0)/\delta X''_\mu + \dot{X}_\mu \delta(\sqrt{-g}\,(\mathscr{L}_0+\mathscr{L}_1))/\delta X'_\mu]\,, \tag{3.19}$$

$$\sqrt{-g}\,\theta^0{}_1 = S_0[2\dot{X}'_\mu \delta(\sqrt{-g}\,\mathscr{L}_0)/\delta\ddot{X}_\mu + X'_\mu \delta(\sqrt{-g}\,(\mathscr{L}_0+\mathscr{L}_1))/\delta\dot{X}_\mu]\,.$$

For the off-diagonal elements, the $\mathscr{L}_1$ term does contribute, but only when the $\dot{X}\cdot X'$ condition is varied in $\delta(\sqrt{-g}\,\mathscr{L}_1)/\delta\dot{X}_\mu$ and $\delta(\sqrt{-g}\,\mathscr{L}_1)/\delta X'_\mu$. Again, a straightforward evaluation of (3.19) reveals complete cancellation of terms. Hence

$$\theta^0{}_1 = \theta^1{}_0 = 0\,. \tag{3.20}$$

To summarize, all the components of the canonical energy-momentum tensor vanish before we completely break reparametrization invariance and spacetime Lorentz invariance through more specific choices of the world-sheet coordinates. (The canonical form of $\theta^a{}_b$ does *not* vanish after more specific coordinate choices are made: cf. Sec. IV, where the temporal gauge $\xi^0 = X^0$ is used.) Although we used the condition in (3.6) to expedite the calculation of $\theta^a{}_b$, this was not necessary. The reader may verify that $\theta^a{}_b = 0$ when no conditions are imposed on $\dot{X}_\mu$ and X'_μ.

IV. CLASSICAL STRING CONFIGURATIONS

We now consider classical motions of the rigid string. First, we establish a useful lemma. Since I_2 is quadratic in $\Box X^\mu$, solutions of the covariant wave equations for the Nambu action, I_1, receive no modifications from I_2, including boundary conditions. To be more precise, we should include the caveat that singular points in the metric [for example, points where $\det(g_{ab})=0$] should be examined with due care, since additional derivatives of g_{ab} and $\Box X^\mu$ are required in the wave equation resulting from I_R

Lemma: All nonsingular classical solutions for the Nambu action are also solutions for I_R, with the same values for conserved quantities.

This lemma will serve, at the very least, as a guide in searching for solutions of the rigid-string equations using known results from the case of the Nambu string.

Following the treatment of the Nambu string,[6] we shall, for convenience, work in the timelike orthogonal gauge, exploiting world-sheet reparametrization invariance to choose coordinates not only satisfying (3.6) but also

$$X^0 = \xi^0\,. \tag{4.1}$$

Then (3.6) becomes

$$\dot{\mathbf{X}}\cdot\mathbf{X}' = 0\,, \tag{4.2}$$

where

$$X^\mu \equiv (X^0, \mathbf{X}),\quad \dot{\mathbf{X}} \equiv \partial\mathbf{X}/\partial\xi^0,\quad \mathbf{X}' \equiv \partial\mathbf{X}/\partial\xi^1\,. \tag{4.3}$$

In this gauge, the sheet metric in (3.8) reduces to

$$g_{ab} = \begin{bmatrix} 1-\dot{\mathbf{X}}\cdot\dot{\mathbf{X}} & 0 \\ 0 & -\mathbf{X}'\cdot\mathbf{X}' \end{bmatrix} \tag{4.4}$$

while the action becomes

$$I_R = -S_0 \int d^2\xi\, |\mathbf{X}'|(1-\dot{\mathbf{X}}^2)^{1/2} \left[\frac{1}{R_0{}^2} + \left[\frac{\ddot{\mathbf{X}}}{(1-\dot{\mathbf{X}}^2)} - \frac{\mathbf{X}''}{\mathbf{X}'^2} \right]^2 - \frac{1}{\mathbf{X}'^2(1-\dot{\mathbf{X}}^2)^2}[-\mathbf{X}'\cdot\ddot{\mathbf{X}} + (1-\dot{\mathbf{X}}^2)\mathbf{X}'\cdot\mathbf{X}''/\mathbf{X}'^2]^2 + \frac{1}{\mathbf{X}'^4(1-\dot{\mathbf{X}}^2)}[-\dot{\mathbf{X}}\cdot\mathbf{X}'' + \mathbf{X}'^2\dot{\mathbf{X}}\cdot\ddot{\mathbf{X}}/(1-\dot{\mathbf{X}}^2)]^2 \right]. \tag{4.5}$$

We now consider only those simple string motions which are uniform "rigid body" rotations about a fixed axis, $\hat{\boldsymbol{\omega}}$, with angular frequency ω, since in general these minimize E for a given J. Henceforth we make this ansatz for our solutions.

For such an ansatz, we have

$$\boldsymbol{\omega} = \omega\hat{\boldsymbol{\omega}},\quad \dot{\mathbf{X}} = \boldsymbol{\omega}\times\mathbf{X},\quad \ddot{\mathbf{X}} = \boldsymbol{\omega}(\boldsymbol{\omega}\cdot\mathbf{X}) - \omega^2\mathbf{X}\,, \tag{4.6}$$

$$\dot{\mathbf{X}}^2 = \omega^2\mathbf{X}^2 - (\boldsymbol{\omega}\cdot\mathbf{X})^2,\quad \mathbf{X}\cdot\dot{\mathbf{X}} = \dot{\mathbf{X}}\cdot\ddot{\mathbf{X}} = \mathbf{X}'\cdot\dot{\mathbf{X}}' = 0\,, \tag{4.7}$$

etc. In fact, any inner product involving derivatives of the spatial components $\mathbf{X}$ will vanish if an odd number of time derivatives appears, just as in the specific cases appearing in (4.7). Thus we achieve consistency with our gauge-fixing conditions, since the terms in parentheses vanish in $\mathscr{L}_1$, (3.13). We therefore only need to keep the Nambu and the $\mathscr{L}_0$ terms in determining the equations of motion for the ansatz. [Note that the last terms in parentheses in $\mathscr{L}_0$, (3.12)—or equivalently, the last bracketed terms in (4.5)—also vanish.]

If we use (4.6) to simplify the surviving terms in (4.5), we find that the ansatz action reduces to an expression linear in the time, $t=\int d\xi^0$,

$$I_R = tL \ , \tag{4.8}$$

where L is the Lagrangian

$$L=-S_0\int d\xi^1\,|\mathbf{X}'|\,[1-\omega^2\mathbf{X}^2+(\boldsymbol{\omega}\cdot\mathbf{X})^2]^{1/2} \times\left[\frac{1}{R_0{}^2}+\left[\frac{-\omega^2\mathbf{X}+\boldsymbol{\omega}\cdot\mathbf{X}\boldsymbol{\omega}}{[1-\omega^2\mathbf{X}^2+(\boldsymbol{\omega}\cdot\mathbf{X})^2]}-\frac{\mathbf{X}''}{\mathbf{X}'^2}\right]^2 -\frac{1}{\mathbf{X}'^2}\left[\frac{\omega^2\mathbf{X}\cdot\mathbf{X}'-\boldsymbol{\omega}\cdot\mathbf{X}\boldsymbol{\omega}\cdot\mathbf{X}'}{[1-\omega^2\mathbf{X}^2+(\boldsymbol{\omega}\cdot\mathbf{X})^2]}+\frac{\mathbf{X}'\cdot\mathbf{X}''}{\mathbf{X}'^2}\right]^2\right] . \tag{4.9}$$

The angular momentum also simplifies considerably for such uniform motions since $\delta\mathbf{X}=\dot{\mathbf{X}}/\omega$. For such variations, the Noether current in (3.2) gives the angular momentum as

$$\omega J=\int d\xi^1[\dot{\mathbf{X}}\cdot\delta\mathcal{L}/\delta\dot{\mathbf{X}}-\dot{\mathbf{X}}\cdot(\delta\mathcal{L}/\delta\ddot{\mathbf{X}})^{\cdot} -\tfrac{1}{2}\dot{\mathbf{X}}\cdot(\delta\mathcal{L}/\delta\dot{\mathbf{X}}')'+\ddot{\mathbf{X}}\cdot\delta\mathcal{L}/\delta\ddot{\mathbf{X}} +\tfrac{1}{2}\dot{\mathbf{X}}'\cdot\delta\mathcal{L}/\delta\dot{\mathbf{X}}'] . \tag{4.10}$$

Now, when the shape of the string lies in a plane, and when the axis of rotation is also in this same plane, the above expression for the magnitude of the angular momentum reduces to

$$J=dL/d\omega \ . \tag{4.11}$$

In addition, for such a rotating configuration, the energy reduces to the Legendre transform of the Lagrangian:

$$E=\omega J-L \ . \tag{4.12}$$

This follows from inspection of (3.4), now applied to the gauge-fixed action (4.5), which represents a manifestly time-translationally invariant system. The reader should contrast this nontrivial energy to the vanishing energy-momentum tensor (3.18) of the nonfixed action, as discussed in Sec. III. It is easy to see, e.g., by merely examining the Nambu action, that the formal $\theta^a{}_b=0$ result of Sec. III is a consequence of making variations with respect to the unphysical parameter X^0.

It is now straightforward to verify that the classical open-string solution of the Nambu action, the rotating straight-line "pinwheel" motion,[6] is also a solution of the equations of motion resulting from (4.5), and also satisfies the same boundary conditions ($\delta\mathcal{L}/\delta X'=0$) at the ends of the string. Our lemma holds in this case, without qualifications.

For the pinwheel motion, the Lagrangian, angular momentum, and energy have the same integral expressions as for the Nambu string. Recall that those expressions are[6]

$$L=-T_0\int d\xi^1\,|\mathbf{X}'|\,[1-\omega^2\mathbf{X}^2+(\boldsymbol{\omega}\cdot\mathbf{X})^2]^{+1/2} , \tag{4.13}$$

$$\omega J=+T_0\int d\xi^1\,|\mathbf{X}'|\,[\omega^2\mathbf{X}^2-(\boldsymbol{\omega}\cdot\mathbf{X})^2] \times[1-\omega^2\mathbf{X}^2+(\boldsymbol{\omega}\cdot\mathbf{X})^2]^{-1/2} , \tag{4.14}$$

$$E=+T_0\int d\xi^1\,|\mathbf{X}'|\,[1-\omega^2\mathbf{X}^2+(\boldsymbol{\omega}\cdot\mathbf{X})^2]^{-1/2} . \tag{4.15}$$

From these integral expressions, the Legendre transform in (4.12) is easily checked.

The actual values for the physical quantities in (4.13)–(4.15) may be obtained using an explicit, simple form for the pinwheel solution:

$$\mathbf{X}=\xi^1\hat{\mathbf{e}},\quad -1/\omega\le\xi^1\le 1/\omega \ , \tag{4.16}$$

with $\hat{\mathbf{e}}\perp\hat{\boldsymbol{\omega}}$ and $\dot{\hat{\mathbf{e}}}=\boldsymbol{\omega}\times\hat{\mathbf{e}}$. The ends of the string therefore move at the speed of light, and both E and J decrease monotonically as ω is increased. The I_2 term in I_R does not modify anything in this case, even at the end points of the string. Upon evaluation of the above integrals using the pinwheel solution, the Lagrangian, energy, and angular momentum are given as usual by

$$L=-\pi T_0/2\omega,\quad E=\pi T_0/\omega \ , \qquad J=\pi T_0/2\omega^2=E^2/2\pi T_0 \ . \tag{4.17}$$

Thus, classical Regge trajectories for these open string configurations are linear with slope $(2\pi T_0)^{-1}$.

For the closed string case, however, the folded-over straight-line configuration (which extremizes the Nambu action and gives a Regge trajectory with half the slope of the open-string case) is *not* a solution. The above lemma is obviated in this case by the creased ends of the straight-line segments where the metric is singular. If the ends are rounded out slightly (we thank Eric Braaten for his help in establishing the following argument), with a radius ϵ, the rigidity term I_2 contributes $O(1/\epsilon)$ to the total action. To see this, write the infinitesimal rounding in the form

$$\mathbf{X}=\frac{1}{\omega}[\mathbf{e}(1-\epsilon^2)^{1/2}\cos(\xi^1)+\hat{\boldsymbol{\omega}}\epsilon\sin(\xi^1)] \ . \tag{4.18}$$

Then it can be shown that near the sharply folded ends the contribution to the Lagrangian goes like $\epsilon^2\int d\sigma\,\sigma^4/(\sigma^2+\epsilon^2)^4$. Consequently, ϵ is not a suppressed perturbation, and this configuration for the closed string with rigidity is not a classical solution as $\epsilon\to 0$.

The actual, planar, classical solution for the rotating closed string with rigidity is easy to visualize since it has a nonrelativistic limit, unlike the Nambu string. It *resembles* an ellipse, as in Fig. 2, with the axis of rotation along the "semiminor axis." In this oblate hooplike configuration, the rigidity and centripetal acceleration balance the string tension.

We may reparametrize the solution by working in the plane of the hoop, defining r to be the distance from the center of the hoop, and θ to be the angle from the "semimajor axis" (cf. Fig. 2). That is, we write

$$\mathbf{X}=(r(\theta)\cos\theta, r(\theta)\sin\theta) \ . \tag{4.19}$$

Using this parametrization, the Lagrangian for the hoop ansatz reduces to

$$L=-S_0\int d\theta[r^2+(dr/d\theta)^2]^{1/2}(1-\omega^2r^2\cos^2\theta)^{1/2}\left[\frac{1}{R_0{}^2}+\left[\kappa-\frac{\omega^2 r\cos\theta(r\cos\theta+dr/d\theta\sin\theta)}{[r^2+(dr/d\theta)^2]^{1/2}(1-\omega^2r^2\cos^2\theta)}\right]^2\right], \quad (4.20)$$

where κ is the conventionally defined curvature for a planar curve:

$$\kappa=\frac{1-\dfrac{d}{d\theta}\arctan(r^{-1}dr/d\theta)}{[r^2+(dr/d\theta)^2]^{1/2}} . \quad (4.21)$$

In this form it is easy to determine the static $\omega=0$ solution for the hoop using simple topological/geometrical arguments. In the static limit we write

$$L(\omega=0)=-S_0\int d\theta[r^2+(dr/d\theta)^2]^{1/2}\times\left[\frac{2\kappa}{R_0}+\left[\kappa-\frac{1}{R_0}\right]^2\right] . \quad (4.22)$$

The first term on the right-hand side (RHS), linear in κ, gives a topological invariant (Hopf's circulation theorem for closed planar curves[7]), which is clear from the explicit form for κ in (4.21). This term counts the number of times, N, the tangent to the curve rotates through 2π as the closed string is circumambulated.

The second term on the RHS of (4.22) is obviously extremized ($=0$) when $\kappa=1/R_0$, a constant. Geometrically, of course, this means the static hoop forms a circle of radius $r=R_0$. For such a circle, wound N times with string, (4.22) immediately gives the action and energy since $E=-L(\omega=0)$ in the static case:

$$E_N=4\pi NS_0/R_0=4\pi N\sqrt{S_0T_0} . \quad (4.23)$$

These results come as no surprise to anyone familiar with a rubber band, except perhaps for the feature that the length of this string is labile.

For the nonstatic $\omega\neq 0$ case, we do not have similar topological/geometrical arguments to obtain the classical solution. As a preliminary step, we may perform a small ω perturbative analysis by first substituting an expansion for $r(\theta)$ in terms of even harmonics, and then adjusting the coefficients to extremize L. The result is

$$r(\theta)=R_0\{1+\omega^2R_0{}^2[1+\tfrac{1}{3}\cos(2\theta)]+O(\omega^4)\} . \quad (4.24)$$

Thus, a slow rotation stretches both the semimajor and semiminor axes. Substituting this result into L, and evaluating the angular momentum and energy using (4.11) and (4.12), we obtain

$$L=-4\pi NS_0/R_0[1-\tfrac{3}{4}\omega^2R_0{}^2+O(\omega^4)] ,$$
$$\omega J=6\pi NS_0R_0\omega^2+O(\omega^4) , \quad (4.25)$$
$$E=4\pi NS_0/R_0[1+\tfrac{3}{4}\omega^2R_0{}^2+O(\omega^4)] .$$

This of course is what one would expect for a nonrelativistic rigid body rotation. The resulting Regge trajectories are nonlinear for small ω (Fig. 3):

$$J=[\tfrac{3}{2}R_0{}^2(E^2-16\pi^2N^2S_0T_0)]^{1/2}+O(\omega^4) . \quad (4.26)$$

The infinite slope, dJ/dE^2 at $J=0$, for this trajectory should be contrasted with the vanishing slope obtained for the "dumbbell" configuration of the massive string.[8]

V. CLASSIFICATION OF THE EQUATIONS OF MOTION

We next study the rotating simple hoop configuration ($N=1$) for larger rotation rates. Eventually, we will resort to numerical methods to complete such a treatment (cf. Sec. VI). Before numerical analysis, however, we will perform a few analytical transformations of the equations which will not only aid the later numerical work, but will also serve to classify our second-order nonlinear differential equations. (We thank Tom McCarty for discussions and calculations related to the material of this section.)

First, it is convenient to change variables to a modified angle variable which is just the curvature integrated over the arc length of the curve.

$$\varphi\equiv\theta-\arctan(r^{-1}dr/d\theta)=\int_0^{s(\theta)}ds\,\kappa(s) ,$$
$$ds\equiv d\theta[r^2+(dr/d\theta)^2]^{1/2} . \quad (5.1)$$

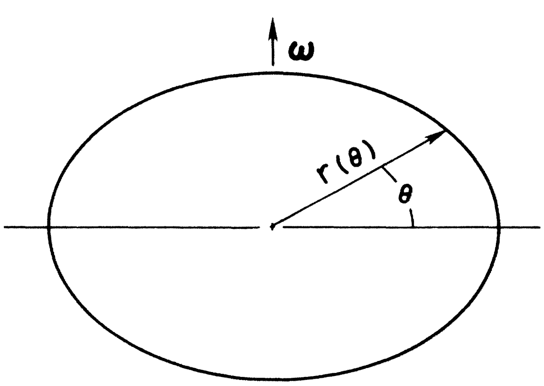

FIG. 2. Parametrization of the axially symmetric hoop configurations.

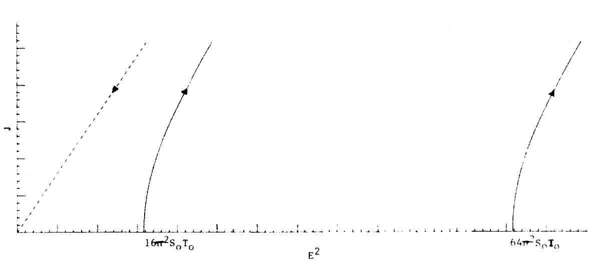

FIG. 3. The leading and sister Regge trajectories of the rotating hoop configuration for small J. Arrows indicate increasing ω. The dashed diagonal trajectory is that of the conventional Nambu closed string.

This angle is a natural independent variable to describe the planar string configuration. One way to see this is to reconstruct the positions of points on the planar string curve in terms of initial data and integrals of the planar curvature. Let us first switch to dependent variables which are rectangular coordinates giving the position of a point on the string:

$$x(\theta)\equiv r(\theta)\sin\theta,\ \ y(\theta)\equiv r(\theta)\cos\theta\ . \tag{5.2}$$

For these variables, the above-mentioned reconstruction of the curve leads to

$$\begin{aligned} x(\theta)=x(0)&+\frac{dx(0)/d\theta}{\{[dx(0)/d\theta]^2+[dy(0)/d\theta]^2\}^{1/2}}\int_0^{s(\theta)}ds_1\cos\left[\int_0^{s_1}ds_2\kappa(s_2)\right]\\ &-\frac{dy(0)/d\theta}{\{[dx(0)/d\theta]^2+[dy(0)/d\theta]^2\}^{1/2}}\int_0^{s(\theta)}ds_1\sin\left[\int_0^{s_1}ds_2\kappa(s_2)\right] \end{aligned} \tag{5.3}$$

and

$$\begin{aligned} y(\theta)=y(0)&+\frac{dy(0)/d\theta}{\{[dx(0)/d\theta]^2+[dy(0)/d\theta]^2\}^{1/2}}\int_0^{s(\theta)}ds_1\cos\left[\int_0^{s_1}ds_2\kappa(s_2)\right]\\ &+\frac{dx(0)/d\theta}{\{[dx(0)/d\theta]^2+[dy(0)/d\theta]^2\}^{1/2}}\int_0^{s(\theta)}ds_1\sin\left[\int_0^{s_1}ds_2\kappa(s_2)\right]\ . \end{aligned} \tag{5.4}$$

The naturalness of the angle φ is apparent. Note from the definition in (5.1) that when the planar curvature is constant, as for the static circular hoop of the last section, the angle φ is equal to θ.

For the oblate hoop configuration pictured in Fig. 2, with the string making a single winding of the curve, the points $\theta=0$, $\pi/2$, π, $3\pi/2$, and 2π coincide with $\varphi=0$, $\pi/2$, π, $3\pi/2$, and 2π, respectively. This follows from the assumed symmetry of the curve and the Hopf circulation theorem. Of course, when the hoop is not a circle (i.e., $\kappa\neq$const), points in between these special values for θ and φ have $\theta\neq\varphi$.

Also, for the configuration of Fig. 2, we have $dx(0)/d\theta=0$, $dy(0)/d\theta>0$, $x(0)>0$, $x(\pi/2)=0$, $y(0)=0$, and $y(\pi/2)>0$. Inserting these values in (5.3) and (5.4) leads to some simplifications:

$$\begin{aligned} x(\theta)&=x(0)-\int_0^{s(\theta)}ds_1\sin\left[\int_0^{s_1}ds_2\kappa(s_2)\right]\ ,\\ y(\theta)&=\int_0^{s(\theta)}ds_1\cos\left[\int_0^{s_1}ds_2\kappa(s_2)\right]\ . \end{aligned} \tag{5.5}$$

That is to say, for the configuration in Fig. 2, $dx/ds=-\sin\varphi$, $dy/ds=\cos\varphi$. Since (5.1) implies $d\varphi/ds=\kappa$, as well, we have

$$\frac{-1}{\sin\varphi}\frac{dx}{d\varphi}=\kappa^{-1},\ \ \frac{1}{\cos\varphi}\frac{dy}{d\varphi}=\kappa^{-1}\ . \tag{5.6}$$

This suggests a further variable change. Let

$$u\equiv\cos\varphi\ , \tag{5.7}$$

so that

$$\begin{aligned} \frac{du}{ds}&=-\kappa(1-u^2)^{1/2}\ ,\\ \frac{dx}{du}&=\kappa^{-1}\ ,\\ \frac{dy}{du}&=-\kappa^{-1}u(1-u^2)^{-1/2}\ . \end{aligned} \tag{5.8}$$

As we shall see below, this last change of variables leads to equations of motion with explicit algebraic singularities. Further, note that $u=1$ ($\varphi=0$) corresponds to a point on the hoop which lies on the semimajor axis, as in Fig. 2, while $u=0$ ($\varphi=\pi/2$) corresponds to a point on the semiminor ($\hat{\omega}$) axis.

It is important to realize that (5.3) and (5.4) [or (5.5) and (5.6)] have no dynamical content. They are only geometrical descriptions of planar curves in terms of the arc length and local curvature. To impose the dynamics of the rigid-string system, we need an additional equation, e.g., one which independently determines the local curvature, κ.

Such an additional equation comes, of course, from the Euler-Lagrange equations of motion. If we rewrite L, in (4.9), in terms of x and y, and use

$$ds=d\xi^1\,|\mathbf{X}'|\ , \tag{5.9}$$

as well as (5.8) to eliminate all derivatives of x and y, we obtain a simple form for the Lagrangian which has explicit dependence on κ, u, and x, but no explicit dependence on y. The absence of explicit y dependence is a consequence of translational invariance along the $\hat{\omega}$ axis of Fig. 2. The result for L is

$$L=-4S_0\int_0^1 du\left[\frac{1-\omega^2x^2}{1-u^2}\right]^{1/2}\times\frac{1}{\kappa}\left[\frac{1}{R_0{}^2}+\left[\kappa-\frac{\omega^2ux}{1-\omega^2x^2}\right]^2\right]\ , \tag{5.10}$$

where we have explicitly assumed a fourfold symmetry for the solution, i.e., invariance under $u\to-u$, $x\to-x$, as shown in Fig. 2.

The classical dynamics of the rigid string for the special case of rotating planar hoops is now obtained by varying x in L, recognizing that $\kappa=(dx/du)^{-1}$, as in (5.8). This gives a second-order, nonlinear equation for $x(u)$:

$$\frac{d}{du}\left[\kappa-\frac{\omega^2ux}{1-\omega^2x^2}\right]$$
$$=\frac{u}{1-u^2}\frac{1}{\kappa}\left[\frac{1}{R_0{}^2}-\kappa^2+\left[\frac{\omega^2ux}{1-\omega^2x^2}\right]^2\right]. \quad (5.11)$$

Alternatively, we may view $u(x)$ as the dependent variable, and x as the independent variable in (5.11). Upon doing so, and multiplying both sides of (5.11) by $\kappa=du/dx$, the differential equation may be identified as being of Painlevé type.[9] To see this, we absorb ω into the independent variable (assuming $\omega\neq0$), and rewrite the equation as

$$\frac{d}{dz}\left[\frac{du}{dz}-\frac{uz}{1-z^2}\right]=\frac{u}{1-u^2}\left[\frac{1}{\omega^2R_0{}^2}-\left[\frac{du}{dz}\right]^2+\left[\frac{uz}{1-z^2}\right]^2\right], \quad (5.12)$$

where $z=\omega x$.

Equations of the form

$$u''=A(z,u)u'^2+B(z,u)u'+C(z,u), \quad (5.13)$$

were investigated early in this century by Picard, Painlevé, Gambier, and others, in a search for nonlinear equations whose branch points and essential singularities were fixed, i.e., did not move as the initial data were changed. There are 50 canonical forms for equations with this property.[9] To put (5.12) in a form close to one in this canonical list, let

$$W=\frac{u-1}{u+1}. \quad (5.14)$$

Then (5.12) becomes

$$W''=\left[\frac{1}{2W}+\frac{1}{W-1}\right]W'^2+f(z)W'$$
$$+a(z)(1-W)^2(W^2-1)/W+b(z)(1-W^2), \quad (5.15)$$

where

$$f(z)=z/(1-z^2),$$
$$a(z)=\frac{1}{8}\left[f(z)^2+\frac{1}{\omega^2R_0{}^2}\right], \quad (5.16)$$
$$b(z)=\frac{f(z)^2}{2z^2}.$$

The coefficient of W'^2 in (5.15) is now in one of the standard canonical forms. Unfortunately, the $a(z)$ and $b(z)$ terms are not, and what is more, they cannot be brought to canonical form by changing either dependent or independent variables. The $a(z)$ and $b(z)$ terms are close to a canonical form (i.e., case XL, p. 341, Ref. 9) which admits an analytic solution through a reduction to a pair of Ricatti equations. So far, however, we have not found this proximity to a known equation to be useful. Perhaps it would be a useful starting point for an approximation scheme where certain terms in the equation are neglected. For example, in the limit $(\omega R_0)^{-1}\to0$, $a(z)$ is of a form so that (5.15) can be solved. $W(z)^{\pm1}=\tanh[K+\ln(1-z^2)/8]$ are two particular solutions in the limit. These particular solutions nicely illustrate a general feature of (5.15) before neglecting any terms. The equation is mapped into itself under $W\to W^{-1}$. Physically, this corresponds to a translation along the $\hat{\omega}$ axis of the solution pictured in Fig. 2, by an amount such that the lowest point of the figure is moved up to the highest point.

We close this section by noting that the angular momentum for the rotating hoop ansatz is given by (4.11), or explicitly in terms of κ, x, and u by

$$J=4\omega S_0\int_0^1 du\left[\frac{1}{(1-u^2)(1-\omega^2x^2)}\right]^{1/2}\frac{1}{\kappa}\left\{x^2\left[\frac{1}{R_0{}^2}+\left[\kappa-\frac{\omega^2ux}{1-\omega^2x^2}\right]^2\right]+\frac{4ux}{1-\omega^2x^2}\left[\kappa-\frac{\omega^2ux}{1-\omega^2x^2}\right]\right\}. \quad (5.17)$$

Similarly, the energy for the ansatz is simply related to L and J as in (4.12).

VI. NUMERICAL RESULTS AND METHODS

We have numerically solved the differential equation for $x(u)$, (5.11), treating ω as a parameter, and have thereby obtained the properties of the rotating planar hoops for the rigid string as functions of ω. From our data for $x(u)$, we numerically integrated (5.10) to obtain the action $L(\omega)$. We also integrated (5.17) to obtain the angular momentum $J(\omega)$ and then combined the results for L and J to find the energy of the solution $E(\omega)$ using the Legendre transform, (4.12). This allowed the numerical determination of the classical Regge trajectory for the rotating hoop ansatz.

The results for the trajectory are shown in Fig. 4. For a given ω, there are two classical solutions. We illustrate this in Fig. 5 by graphing the two planar configurations for $\omega=0.25$. One branch of solutions (that which contains the inner hoop in Fig. 5) of (5.11) originates with the static circular hoop, and develops for small ω as described above in (4.24) and (4.25). Both $E(\omega)$ and $J(\omega)$ increase monotonically for this branch up to a critical value ω_c ($\simeq3.5\sqrt{T_0}$, for $S_0=1$). At this critical value ω_c the energy, considered as a function of J, has an inflection point (since $\omega=dE/dJ$ and $d^2E/dJ^2=0$). This is shown in Fig. 6, again for $S_0=1$. The trajectory $E(J)$ continues to

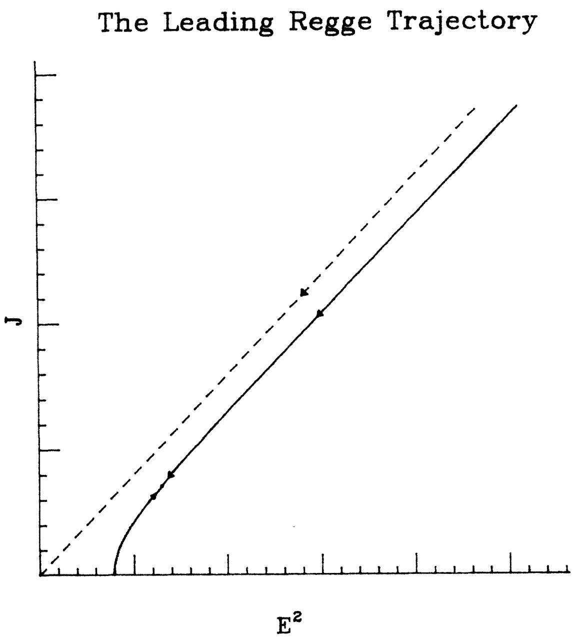

FIG. 4. The leading ($N=1$) Regge trajectory for $S_0=1$ (solid line) and for $S_0=0$ (dashed line). Arrows indicate increasing ω.

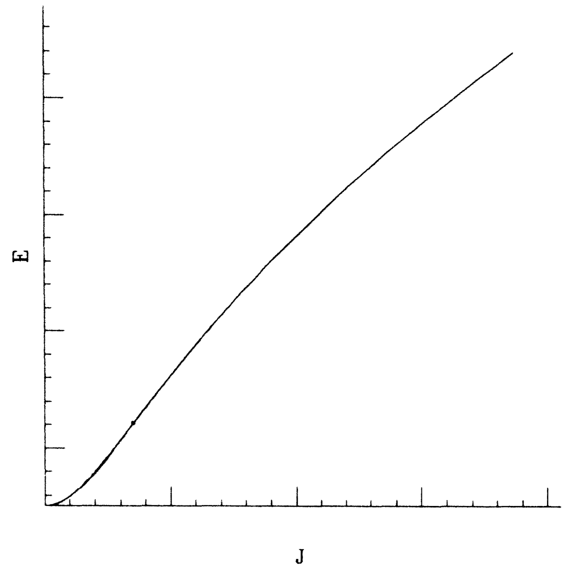

FIG. 6. Energy versus angular momentum for the rotating hoop configuration ($S_0=T_0=1$). The dot marks the inflection point $d^2E/dJ^2=0$, corresponding to ω_c.

higher E and J values through the second branch of solutions to (5.11), allowing ω to decrease back to zero.

The branch of solutions giving the upper segment of the trajectory (that which contains the outer hoop in Fig. 5) behaves in many ways similar to the motions of the pliable, Nambu string. As $\omega \to 0$, the length of the hoop along the semimajor axis, $x(u=1)$, grows as $1/\omega$. The trajectory seems to become linear in the same limit. Unlike the Nambu closed string, however, the ends of this configuration are not sharp folds. The elastic energy stored in these ends displaces the trajectory to the right of the Nambu straight-line trajectory, as evident in Fig. 4.

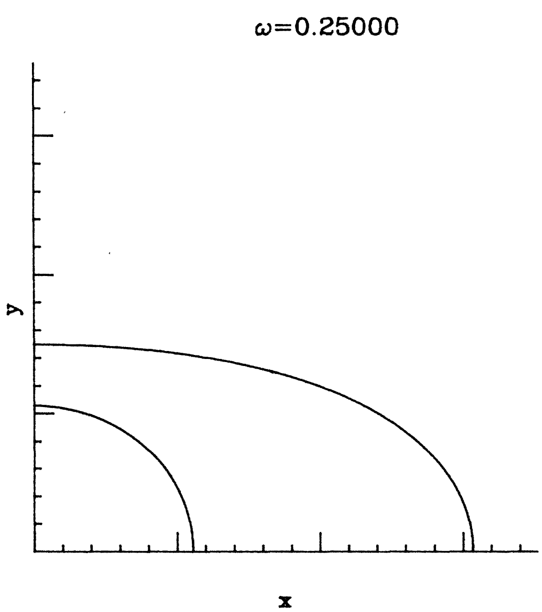

FIG. 5. Hoop profiles for the two classical solutions of the equations of motion, for $\omega=0.250$, $S_0=T_0=1$. Reflection symmetry about the x and y axes is understood.

As energy and angular momentum are pumped into the system, both minor axis and major axis of the configuration grow, although their ratio goes to zero. Along with this growth in size, the edges of the hoop farthest from the axis of rotation move with growing speed. This growth continues smoothly from the lower branch part of the trajectory into the upper branch, except that on the upper branch the growth of the edge speed lags the growth of the major-axis length. The critical frequency ω_c represents the maximum frequency attained by the ansatz solution. (For higher frequencies, there may in principle exist another type of solution, but we have not found it.) We suspect that the lack of a solution for $\omega>\omega_c$ represents an instability of the planar solution against warping of the plane, but we have not confirmed this. (Similar instabilities, associated with a loss of symmetry for the actual rotating solution, are well known in rotating, self-gravitating fluids.)

An analytic expression which approximates the trajectory fairly well is given by a hyperbola:

$$E^2(J)_{\text{approx}} = T_0[10\pi^2 S_0 + \pi J + (9\pi^2 J^2 - 12\pi^3 S_0 J + 36\pi^4 S_0{}^2)^{1/2}] \ . \tag{6.1}$$

The greatest disagreement between this expression, and the actual (numerical) values occurs near the critical value ω_c. For example, when $S_0=1=T_0$, and $\omega=0.35$ (which is very slightly below ω_c), the two solution branches (denoted $\pm$) compare with the above expression as follows:

$$E_+{}^2=262,\quad J_+=14.5,\quad E_{\rm approx}{}^2=274\ ,$$
$$E_-{}^2=254,\quad J_-=13.8,\quad E_{\rm approx}{}^2=266\ . \tag{6.2}$$

This gives a relative error $(E^2-E_{\rm approx}{}^2)/E^2=4\%$, in both cases.

We now discuss the methods by which the above numerical properties of the rotating hoop configuration were obtained. Two techniques were used to search for solutions of the equation (5.11).

In the first of these we treated the equation as a boundary value problem. Starting with $x(u=0)=0$ we used a "garden hose" method. We chose an initial value for $dx(0)/du=\kappa(0)^{-1}$ [recall (5.8)]. We then used the Runge-Kutta method to numerically integrate to the most rapidly moving point on the hoop, at $u=1$. At that point we required the boundary condition that $d\kappa/du$ remain finite at $u=1$. This required that the numerator on the RHS of (5.11) vanish, giving a relation between $\kappa(u=1)$ and $x(u=1)$:

$$\kappa^2(u=1)=\frac{1}{R_0{}^2}+\left[\frac{\omega^2 x(u=1)}{1-\omega^2x^2(u=1)}\right]^2 . \tag{6.3}$$

If the initial curvature was such that this condition was satisfied, we accepted the numerical data as a solution, and hence we had determined $x(u)$ and $\kappa(u)$ on the interval $u\in[0,1]$. Given $\kappa(u)$, we then determined $y(u)$, up to an irrelevant translation along the $\hat\omega$ axis. The rest of the configuration was given in terms of the data on this interval using the assumed fourfold symmetry evident in Fig. 2.

The practical implementation of this boundary value method is illustrated in Fig. 7. We have plotted

$$V\equiv\frac{1}{R_0{}^2}-\kappa^2(1)+\left[\frac{\omega^2 x(1)}{1-\omega^2x^2(1)}\right]^2 \tag{6.4}$$

as a function of the initial curvature $\kappa(0)$. Only when $V=0$ do we have an acceptable numerical solution. The existence of two solutions, i.e., two allowed initial values for κ, is revealed by the parabolic shape of V, which has two zeros for sufficiently small ω. As ω increases, however, V lies entirely above the axis, and there are no configurations which satisfy the boundary condition of finite $d\kappa/du$.

Our second numerical technique for finding classical solutions of (5.11) consisted again of taking $x(0)=0$, choosing a starting curvature, and integrating the second-order differential equation from $u=0$ to $u=1$ using Runge-Kutta methods. Rather than directly checking a boundary condition at $u=1$, however, this time we simply calculated L in (5.10), using Simpson's rule for numerical integration. This gave us an effective Lagrangian as a function of the starting curvature $\kappa(u=0)$, with $x(u=0)=0$. We then found the extrema of this effective Lagrangian, again numerically. The situation is illustrated in Fig. 8, where we have plotted the effective Lagrangian as a function of the initial curvature, for the case $S_0=1$. There are two extrema, for any $\omega<\omega_c$, for which value the extrema coalesce at an inflection point. For higher values of ω, the slope of the effective Lagrangian never vanishes, indicating again that there are no solutions.

Insofar as we have checked, the two techniques which we have used agree within numerical uncertainties. However, the latter variational method appears to be better.

We finally note that as the rigidity S_0 is decreased, tending to the limit of the pliable string ($S_0=0$), the E^2 intercept of the trajectory moves toward the origin. The critical point on the trajectory also moves toward the ori-

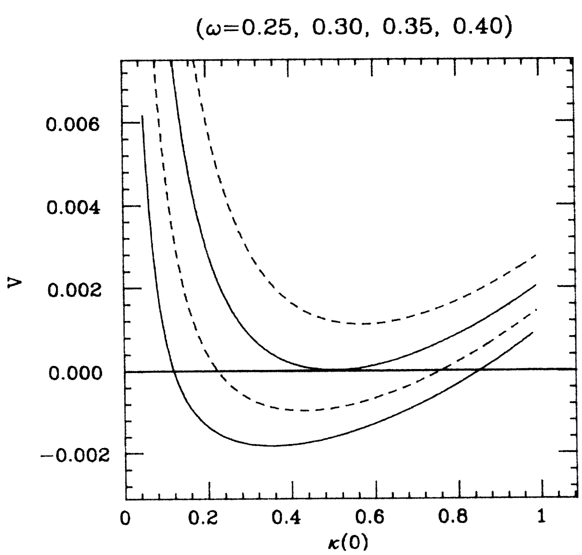

FIG. 7. The boundary condition function

$$V\equiv\frac{1}{R_0{}^2}-\kappa^2(1)+\left[\frac{\omega^2 x(1)}{1-\omega^2x^2(1)}\right]^2 ,$$

versus the initial curvature $\kappa(0)$, for four values of ω. (Lower ——, $\omega=0.25$; lower – – – – –, $\omega=0.30$; upper ——, $\omega=0.35$; upper – – – – –, $\omega=0.40$.) Classical solutions correspond to zeros of V.

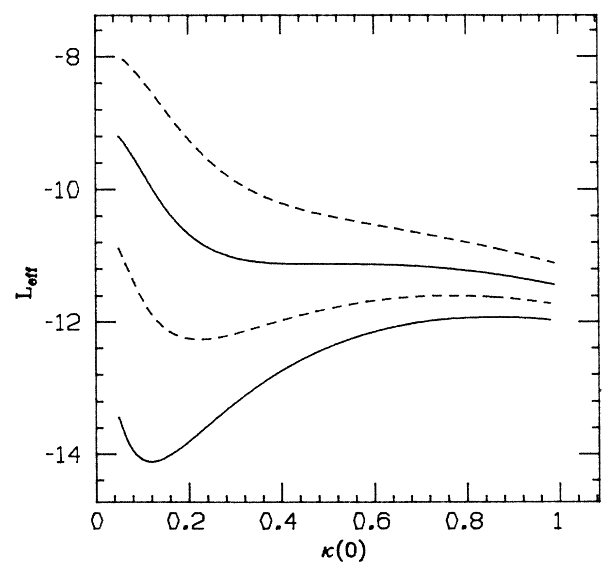

FIG. 8. The effective Lagrangian $L_{\rm eff}$ as a function of the initial curvature $\kappa(0)$ for four values of ω. (Lower ——, $\omega=0.25$; lower – – – – –, $\omega=0.30$; upper ——, $\omega=0.35$; upper – – – – –, $\omega=0.40$.) Classical solutions correspond to extrema of $L_{\rm eff}$.

gin, so that only the upper solution branch survives. This yields the diagonal, straight-line trajectory which characterizes the Nambu string. In fact, for fixed string tension T_0, the critical rotation rate is inversely proportional to S_0, while the angular momentum and energy squared for the critically rotating hoop are proportional to S_0:

$$\omega_c{}^2 S_0 = \text{const}_1 \simeq 0.350 \ ,$$
$$J_c / S_0 = \text{const}_2 \simeq 14.1 \ , \qquad (6.5)$$
$$E_c{}^2 / S_0 = \text{const}_3 \simeq 258 \ .$$

These properties follow from a simple rescaling of the variables in the equations of motion, as in (5.12), and in the expressions for E and J. This behavior is consistent with the approach to the Nambu string in the limit $S_0 \to 0$, since $E^2 = 0 = J$ corresponds to $\omega = \infty$ [cf. (4.17)].

The approximate numerical values given in (6.5) are extracted from our data. To obtain these numbers, we approximated E and J for the upper and lower solution branches near the critical rotation rate using the empirically successful formulas

$$E_\pm = E_c \pm \text{const}_4 \times [1 - (\omega/\omega_c)^2]^{1/2} \ ,$$
$$J_\pm = J_c \pm \text{const}_5 \times [1 - (\omega/\omega_c)^2]^{1/2} \ . \qquad (6.6)$$

The best fit to our data, for $T_0 = 1 = S_0$, gave $\omega_c \simeq 0.350$, $J_c \simeq 14.1$, $E_c \simeq 16.1$, $\text{const}_4 \simeq 5.75$, and $\text{const}_5 \simeq 16.4$.

VII. DISCUSSION AND CONCLUSIONS

For the closed sector, the classical trajectories of the rigid string are nonlinear and are dominated at low energies (short strings) by the novel hoop configurations discussed above. At higher energies (long strings), the relevant configurations exhibit more conventional Regge behavior, although classically they lack the sharply folded ends of the Nambu string due to the rigidity which discourages bending of the string. Therefore, the rigidity interactions are expected to suppress the longitudinal kink/fold modes[5] of the conventional Nambu string.

Ultimately, this system must be quantized. At present we do not see how to carry this out beyond the usual methods of fluctuations about classical solutions, as developed in the context of soliton physics. (Related issues involve the classical stability of $N > 1$ static hoops.) Upon quantization, we expect a shift in the vacuum energy and some modifications of the low-energy spectrum. However, since the lowest energy state of the classical closed string is an adjustable parameter [as in (4.23)], we expect at least for some range of rigidity that there will be no massless states for the quantized rigid string. This raises several questions, which we cannot answer, concerning general covariance and gravitation in the embedding spacetime. Related issues concerning the compactification dynamics of such strings are easy to speculate upon.

Finally, the analytic continuation of the nonlinear trajectories to negative J and E^2, together with the resolutions of the intercepts of the sister trajectories [described classically in (4.23)], may have interesting implications in hadronic applications of string theory. It would also be interesting to see what relationship, if any, the nonlinear trajectories presented here have with those described in the earlier literature (e.g., see Ref. 10). Work on these issues is in progress.

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APPENDIX: GLOBALLY SUPERSYMMETRIC GENERALIZATIONS

It is not difficult to extend the action (2.13) to one with manifest, global, D-dimensional spacetime supersymmetry by using superspace techniques. Such a supersymmetrization of the Nambu-Goto term I_1 was achieved by Green and Schwarz.[11] However, their extension had the additional important property of invariance under a local κ world-sheet supersymmetry, which was achieved through the inclusion of specific relative amounts of superspace torsion terms in the action.[12] Here we indicate how to extend the rigidity term I_2 to its globally supersymmetric generalization, but we do not find the requisite mixture of terms necessary for local world-sheet supersymmetry. The reader is thus advised that the model in its present form will contain more physical fermionic degrees of freedom than the Green-Schwarz superstring.

The $N=1$ supervielbein in D-dimensional spacetime is

$$V_a^M \equiv (\partial_a X^\mu - i\bar\theta\Gamma^\mu\partial_a\theta, \partial_a\theta^\alpha) \equiv (V_a^\mu, V_a^\alpha) \ . \qquad (\text{A1})$$

In terms of the first component, the world-sheet metric is

$$g_{ab} \equiv V_a^\mu V_{b\mu} \ . \qquad (\text{A2})$$

The bosonic equation of motion of the $N=1$ covariant superstring generalization of the Nambu term (conventions are those of Curtright, Mezincescu, and Zachos in Ref. 12) is

$$B^\mu \equiv \partial_a(\sqrt{-g}\, g^{ab} V_b^\mu) + i\epsilon^{ab} V_a \Gamma^\mu V_b = 0 \ , \qquad (\text{A3})$$

where the second term, $i\epsilon^{ab}\partial_a\bar\theta\Gamma^\mu\partial_b\theta$, represents the superspace torsion. Given the usual projector, $P_-^{ab} \equiv (g^{ab} - \epsilon^{ab}/\sqrt{-g})/2$, the equation of motion may be rewritten as

$$B^\mu \equiv 2\partial_a(\sqrt{-g}\, P_-^{ab} V_b^\mu) = 0 \ . \qquad (\text{A4})$$

Consequently, the manifestly globally supersymmetric extension containing (2.11) in its bosonic reduction is

$$I_2^{GS} = B^\mu B_\mu / \sqrt{-g} \ . \qquad (A5)$$

In addition to I_2^{GS}, a number of further terms may be constructed, whose bosonic limit vanishes, similar to the case of the Green-Schwarz superspace torsion term. Perhaps a combination of such terms may restore the local world-sheet supersymmetry of the Green-Schwarz superstring, which is broken by I_2^{GS}. So far, however, such a combination has not been found. [What about a first-order formulation? One may, of course, contemplate a first-order formulation which incorporates an independent metric g_{ab} in the Nambu term, as usual, whereas the rigidity term is left as above, oblivious to the independent metric. In such a formulation, the equations of variation of g_{ab} yield the conventional expression determining the metric in terms of the X's and lead to the above second-order formalism. Furthermore, the *full* action trivially possesses the standard Weyl invariance (scaling g_{ab} while the X's are left unchanged). Note further that g_{ab} variations in such a formulation would isolate the vanishing energy-momentum tensor of the Nambu action, but not the full action including rigidity.]

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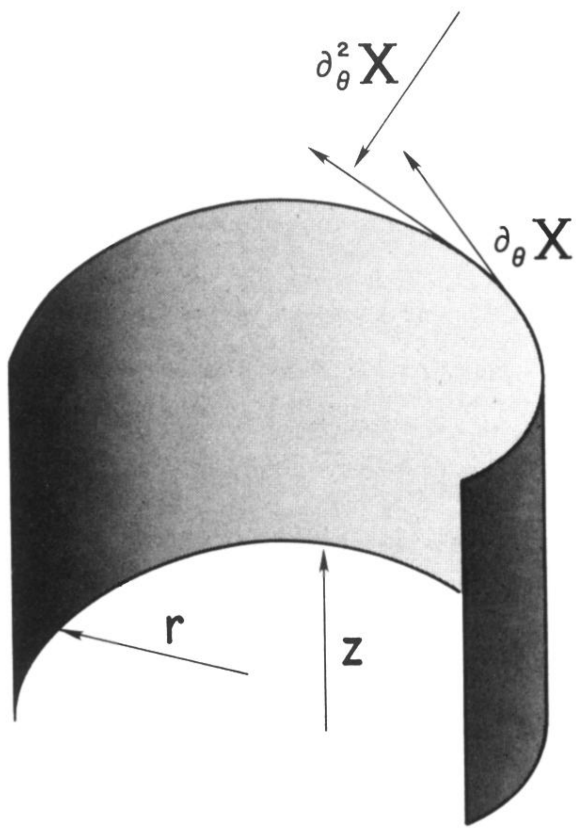

FIG. 1. Extrinsic curvature of a rolled sheet of paper: z points along the axis of symmetry, and $\hat{\mathbf{n}}=\hat{\mathbf{r}}$. In cylindrical coordinates $g_{zz}=1$, $g_{rr}=1$, $g_{\theta\theta}=r^2$, and all components of K_{ab} vanish except for $K_{\theta\theta}=-r$, whence $K_a^a=-1/r$. This is the extrinsic curvature. The Euclidean version of (2.11) thus reduces to $I_2=-S_0\int dz\,d\theta\,r\,1/r^2$.