# Classification of closed-fermionic-string models

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A classification of ten-dimensional closed-fermionic-string models with world-sheet superconformal invariance, modular invariance, and physical states with proper space-time statistics is presented. This classification is based on an analysis of the spin structures of the Neveu-Schwarz-Ramond fermions and the fermionic variables used in representing the internal-symmetry space.

## I. INTRODUCTION

One of the most remarkable features of superstring theories is their near uniqueness; only a very few models appear to be mathematically consistent. Indeed at one time superstrings garnered little interest because it was believed that any such theories with pretensions of being physically realistic unified models must contain gauge and gravitational anomalies.[1] The discovery by Green and Schwarz[2] of at least one exception to this rule, the SO(32) type-I model, opened the floodgates of string research—in particular, the search for other consistent theories. This search met with one notable success. A clever hybrid construction of closed-string theories by Gross, Harvey, Martinec, and Rohm[3] led to two heterotic string models, with the gauge groups $\mathrm{Spin}(32)/Z_2$ or $E_8 \times E_8$ apparently uniquely singled out by the requirement of modular invariance of one-loop amplitudes.

A consistent fermionic string theory should be invariant under reparametrizations, superconformal transformations, and two-dimensional (world-sheet) Lorentz and supersymmetry transformations. In general these symmetries are spoiled by anomalies at the quantum level. The absence of such anomalies impose stringent constraints on string-model building; for example, the vanishing of conformal and superconformal anomalies requires the space-time dimension to be 10 and, if the string possesses an internal gauge symmetry described by fermionic fields, the number of such internal fermion fields must be 32. Modular transformations are discrete reparametrizations (or equivalently global Lorentz transformations) and so modular invariance is included in the requirements listed above. In calculating modular invariant loop amplitudes, the integration over moduli (which describe inequivalent surfaces) should be restricted to the fundamental domain of the modular group. This restriction is the key to the ultraviolet finiteness of string theory so that modular invariance is indeed a necessary requirement of any sensible closed-string theory.

Recently several groups[4–9] have realized that using a fermionic representation for the internal-symmetry group of a closed string and considering sometimes intricate correlations of the spin structures appearing in one-loop amplitudes, new models may be constructed which are also one-loop modular invariant. Some of these, in particular, a tachyon-free model with gauge group $SO(16)\times SO(16)$,[5,6] are possible unified theories; others, however, must be discarded because they have statistics inappropriate for a physical theory; i.e., either the space-time fermions and bosons have the same statistics, or different sectors of the excitations in the internal space appear with opposite statistics and so cannot be sensibly interpreted as giving rise to an internal-symmetry group. In the present work we systematically consider the possibilities for such constructions with periodic or antiperiodic boundary conditions for the fermion fields, finding closed-fermionic-string spin structures which give modular invariant one-loop amplitudes and a projection onto the subspace of physical states which have the appropriate spacetime statistics for a physically sensible theory; i.e., Fermi-Dirac statistics for space-time fermions and Bose-Einstein statistics for space-time bosons and internal-symmetry degrees of freedom. The work of Seiberg and Witten[4] clearly suggests that these requirements, one-loop modular invariance, and proper space-time statistics, are enough to ensure modular invariance of multiloop amplitudes. Our results are summarized in Table I, which we believe probably represents a complete list of possible ten-dimensional closed-fermionic-string models for which the fermionic fields representing the internal-symmetry space are noninteracting and satisfy either periodic or antiperiodic boundary conditions. The gauge group identification for the tachyonic models is tentative, since we have not checked the complete spectra and/or constructed the corresponding vertex operators. Each of the models also has a graviton multiplet (or supermultiplet if space-time supersymmetry is present) which is not included in Table I. The first 12 models are known (see, in particular, the elegant work of Dixon and Harvey[5] in which some of these models were discovered by modifying the bosonic Frenkel-Kac construction of the internal-symmetry group), while the final model is new. Unlike the known models with internal-symmetry groups, this new model has a gauge group of rank 8 rather than 16. A more de-

 

TABLE I. Ten-dimensional modular-invariant closed-fermionic-string theories with proper space-time statistics. The references are GS, Green and Schwarz (Ref. 12); SW, Seiberg and Witten (Ref. 4); GHMR, Gross, Harvey, Martinec, and Rohm (Ref. 3); DH, Dixon and Harvey (Ref. 5); AGMV, Alvarez-Gaume, Ginsparg, Moore, and Vafa (Ref. 6).

| Identification | Number of gauge bosons | Number of massless fermions[a] | Number of tachyons | Space-time SUSY | Chiral | Reference |
|---|---|---|---|---|---|---|
| Type IIA | | 0 | 0 | Yes | No | GS |
| Type IIB | | 0 | 0 | Yes | Yes | GS |
| Type A | | 0 | 1 | No | No | SW |
| Type B | | 0 | 1 | No | No | SW |
| SO(32) heterotic | 496 | 496 | 0 | Yes | Yes | GHMR |
| $E_8 \times E_8$ heterotic | 496 | 496 | 0 | Yes | Yes | GHMR |
| $O(16)\times O(16)$ | 240 | 512 | 0 | No | Yes | DH, AGMV |
| SO(32) | 496 | 0 | 32 | No | No | DH, SW |
| $O(16)\times E_8$ | 368 | 256 | 16 | No | Yes | DH, SW |
| $O(8)\times O(24)$ | 304 | 384 | 8 | No | Yes | DH |
| $(E_7\times SU_2)^2$ | 272 | 448 | 4 | No | Yes | DH |
| U(16) | 256 | 480 | 2 | No | Yes | DH |
| $E_8$ | 248 | 496 | 1 | No | No | |

[a]Not counting those in the graviton (super)multiplet.

tailed examination of the spectra and properties of these models will be considered elsewhere.

This paper is organized as follows. In Sec. II we collect some general results concerning modular invariance and spin structure which provide the starting point for the discussion following. In Sec. III we find modular invariant, physically sensible spin structures and consider the spectra of these allowed models in Sec. IV. Section V concludes with some overall discussions of our work. Appendix A contains a theorem key to the argument in Sec. III, and Appendix B includes some details extending and clarifying the main discussion.

## II. MODULAR INVARIANCE AND SPIN STRUCTURE

The possibilities for closed-fermionic-string theories are severely limited by the symmetries required for their consistency. This is perhaps best exemplified by the heterotic string action which, while it can be rewritten in many forms, is essentially unique and in some sense the most general such action that one can consider. Employing the Neveu-Schwarz-Ramond (NSR) formalism[10] for the space-time fermions, as we will throughout the present work, the covariant first quantized action is[3,11]

$$S(e,\psi,\lambda,X,\chi)=\int d^2\xi e\left[\tfrac{1}{2}g^{mn}\partial_m X^\mu \partial_n X^\mu+\frac{i}{2}\lambda_R^\mu \gamma^m \partial_m \lambda_R^\mu+\frac{i}{2}\psi_{nR}\gamma^m\gamma^n\lambda_R^\mu\left[\partial_m X^\mu-\frac{i}{4}\psi_{mR}\lambda_R^\mu\right]+\frac{i}{2}\chi_L^J\gamma^m\partial_m\chi_L^J\right]. \tag{2.1}$$

In this covariant formalism the zweibein $e_m^a$ and the metric tensor $g_{mn}=e_m^a e_{an}$ appear in the above form to guarantee reparametrization invariance, which is required to eliminate the timelike modes of the string coordinate $X^\mu$. In similar fashion the elimination of the timelike fermionic modes necessitates two-dimensional local supersymmetry which in turn requires the gravitino $\psi$. In the sector of right moving fields $(X_R^\mu,\lambda_R^\mu,\psi_{nR})$ we must have $\mu=1,2,\ldots,D=10$ in order for the superconformal anomaly to vanish. In the bosonic left-moving sector, the vanishing of the conformal anomaly requires 26 dimensions; here 16 of these dimensions are compactified to produce an internal-symmetry group given in a fermionic representation by the $\chi_L^J$, $J=1,\ldots,32$. If both right- and left-moving sectors are NSR strings the resulting theory is a type-II superstring; if both right- and left-moving sectors are bosonic, one has a pure bosonic theory with an internal-symmetry group. Apart from these modifications (and the possibilities for world-sheet fermion interactions or nontrivial background fields), (2.1) appears to uniquely specify possible closed-string theories. This is not the case, however, because we have yet to specify the boundary conditions on the fermions, i.e., the spin structures. At this level the only constraint that (2.1) places on these is that, due to the form of the interactions [specifically the third term in (2.1)] the right-moving gravitino and the ten right-moving fermions, $\lambda_R^\mu$, must have the same spin structure. *A priori* the 32 $\chi_L^J$ are not so limited, but by examining yet another symmetry, modular invariance of one-loop amplitudes, we will find that all of the spin structures must be intricately related.

To calculate a string one-loop amplitude we must sum over world sheets with the topology of a torus. In the conformal gauge all dependence on the world-sheet metric of the torus is contained in the complex modular parameter $\tau$ which parametrizes inequivalent tori. A consistent string theory should not depend on how we choose to parametrize its world sheet. At the one-loop level this requires that amplitudes be invariant under the modular group which consists of discrete reparametrizations (or equivalently global Lorentz transformations) and is generated by the transformations

$$\tau \to \frac{-1}{\tau}\,, \tag{2.2a}$$

$$\tau \to \tau + 1\,. \tag{2.2b}$$

Essentially (2.2a) interchanges the $\sigma$ and $t$ directions on the torus, while (2.2b) corresponds to cutting the torus at some constant $t$ slice, twisting one end through $2\pi$, and reconnecting. The path integrals of the bosonic and fermionic degrees of freedom on the torus must give functions of $\tau$ which are invariant under (2.2). The bosonic integration is already modular invariant if the Regge intercept is the usual one chosen to ensure unitarity; our concern here is with the fermionic sector. The fermion path integral depends on what spin structures we choose for the two noncontractable loops (with winding number one) on the torus, i.e., whether the fermions are chosen to have periodic or antiperiodic boundary conditions when traversing the $\sigma$ or $t$ directions on the torus. More complicated boundary conditions, e.g., some phase other than $\pm 1$, will be considered in a future paper.[13] Under modular transformations, functions of different spin structures may be mapped into each other;[4–9] thus a modular-invariant path integral must, in general, include a sum over spin structures.

To be explicit, let us introduce a notation which will prove useful later on. $Z_b^a$ is the contribution to the vacuum loop path integral from integrating out a single left-moving fermionic degree of freedom. The superscript indicates the spin structure in the $\sigma$ direction on the torus, $a=0$ (1) for antiperiodic (periodic) boundary conditions corresponding to Neveu-Schwarz (Ramond) sectors. Similarly the subscript $b$ indicates the spin structure in the $t$ direction. The phases of these partition functions may be chosen such that under modular transformations they become,[9] for $\tau \to -1/\tau$,

$$\begin{aligned} &Z_1^1 \to Z_1^1\,,\\ &Z_0^0 \to Z_0^0\,,\\ &Z_0^1 \to Z_1^0\,,\\ &Z_1^0 \to Z_0^1\,; \end{aligned} \tag{2.3a}$$

for $\tau \to \tau+1$,

$$\begin{aligned} &Z_1^1 \to e^{i\pi/12} Z_1^1\,,\\ &Z_0^0 \to e^{i\pi/12} Z_1^0\,,\\ &Z_0^1 \to e^{i\pi/12} Z_0^1\,,\\ &Z_1^0 \to e^{i\pi/12 - i\pi/4} Z_0^0\,. \end{aligned} \tag{2.3b}$$

Right-moving fermionic degrees of freedom give results which are the complex conjugates of those in (2.3); notice that as a necessary prerequisite for modular invariance the overall factor of $e^{i\pi/12}$ in (2.3b) must cancel between the left- and right-moving degrees of freedom, giving a constraint on the total number of fermionic modes:

$$N_R - N_L = 0 \quad (\mathrm{mod}\,24)\,. \tag{2.4}$$

Actually (2.4) follows rigorously from (2.3) only if none of the fermions have spin structure $\binom{1}{1}$. This is because $Z_1^1$ in fact vanishes for a vacuum loop due to the presence of a fermion zero mode; thus if one or more fermions have spin structure $\binom{1}{1}$ then the product of fermion determinants will vanish and modular invariance at this level is trivially satisfied [the transformations for $Z_1^1$ in (2.3) are hence merely symbolic]. Even so we can make some demands on the allowed spin structure in these sectors by considering the factorization of multiple-loop scattering amplitudes where $Z_1^1$ need not vanish yet the entire amplitude must still be modular invariant. In addition, we will see in Sec. III and Appendix A that proper space-time statistics requires the presence of sectors of the theory where no fermions have spin structure $\binom{1}{1}$; thus (2.4) will hold. Since the number of fermions must be the same for all spin structures (2.4) is true in all sectors; it follows that, for a consistent theory, the transformation for $Z_1^1$ given in (2.3b) should hold even for the appropriate scattering amplitudes. The known consistency of the NSR model[10,12] shows that this is in fact the case at least for fermions in groups of eight with the same spin structure.

Earlier we alluded to three general possibilities for closed-string theories; we can check that all of these satisfy (2.4). For type-II and purely bosonic models,[12] where the number of left- and right-moving fermionic modes is the same, this is trivially satisfied. For heterotic-type theories in the light-cone gauge, the gravitino and two fermionic modes drop out leaving 8 right-moving fermions which, with the 32 left-moving fermions, also satisfies (2.4). The methods we will present in Sec. III work equally well for all three types of models; however, we will restrict ourselves to the last case with 8 right-moving and 32 left-moving fermionic modes. The possibilities for type-II theories have been exhausted elsewhere[4] and we merely include the results in Table I. As for the purely bosonic models with internal-symmetry groups we simply mention in passing that from our analysis it is clear that all of these models are tachyonic. In general (leaving aside the possibilities for nontrivial background fields, etc.) the string in $D=4n+2$ space-time dimensions will have $8(6-n)$ left-moving fermions and an equal number of right-moving fermions in the internal-symmetry space.

In Sec. III we will determine the spin structures consistent with (2.3); for now we limit ourselves to deriving one general constraint which follows from (2.3) and (2.4). Applying (2.3b) twice gives (dropping the overall factor which cancels between left- and right-moving sectors)

$$\begin{aligned} &Z_1^1 \to Z_1^1\,,\quad Z_0^0 \to e^{-i\pi/4} Z_0^0\,,\\ &Z_0^1 \to Z_0^1,\quad Z_1^0 \to e^{-i\pi/4} Z_1^0\,. \end{aligned} \tag{2.5}$$

Thus modular invariance requires that in any given contribution to the path integral the number of fermions with spin structure $\binom{0}{0}$ and $\binom{0}{1}$ [denote these by $\Gamma\binom{0}{0}$ and $\Gamma\binom{0}{1}$, respectively] must satisfy,

$$\Gamma\begin{bmatrix}0\\0\end{bmatrix}+\Gamma\begin{bmatrix}0\\1\end{bmatrix}=0 \pmod 8 . \tag{2.6a}$$

Using (2.3) this can be extended to

$$\Gamma\begin{bmatrix}0\\0\end{bmatrix}+\Gamma\begin{bmatrix}1\\0\end{bmatrix}=0 \pmod 8 , \tag{2.6b}$$

$$\Gamma\begin{bmatrix}1\\0\end{bmatrix}+\Gamma\begin{bmatrix}0\\1\end{bmatrix}=0 \pmod 8 . \tag{2.6c}$$

If in some sector $\Gamma\binom{1}{1}=0$ (mod8) then (2.6) taken with (2.4) implies that (recall that the 8 right-handed fermions must already have the same spin structure)

$$\Gamma\begin{bmatrix}0\\0\end{bmatrix},\ \Gamma\begin{bmatrix}1\\0\end{bmatrix},\ \Gamma\begin{bmatrix}0\\1\end{bmatrix}=0 \pmod 8$$

$$\text{if } \Gamma\begin{bmatrix}1\\1\end{bmatrix}=0 \pmod 8 . \tag{2.7}$$

In Sec. III we will in fact restrict ourselves to this case:

$$\Gamma\begin{bmatrix}0\\0\end{bmatrix},\ \Gamma\begin{bmatrix}0\\1\end{bmatrix},\ \Gamma\begin{bmatrix}1\\0\end{bmatrix},\ \Gamma\begin{bmatrix}1\\1\end{bmatrix}=0 \pmod 8 . \tag{2.8}$$

The remaining possibility permitted by (2.6), that of spin structures identified in groups of four, will be considered elsewhere.[13]

One final piece of formalism will prove necessary for the discussion of Sec. III. We represent the results of fermion path integrals in Hamiltonian form as a sum over states,[4,12]

$$Z_b^a=\phi_b^a\,\mathrm{Tr}[e^{i\tau\hat{H}_a}(-1)^{b\hat{N}_a}] , \tag{2.9}$$

where

$$\hat{H}_0=\hat{H}_{\rm NS}=2\pi\left[\sum_{r=1/2}^{\infty} r b_r^\dagger b_r-\frac{1}{48}\right] , \tag{2.10a}$$

$$\hat{H}_1=\hat{H}_R=2\pi\left[\sum_{n=1}^{\infty} n d_n^\dagger d_n+\frac{1}{24}\right] , \tag{2.10b}$$

$$\hat{N}_0=\hat{N}_{\rm NS}=\sum_{r=1/2}^{\infty} b_r^\dagger b_r , \tag{2.11a}$$

$$(-1)^{\hat{N}_1}=\pm d_0(-1)^{\sum_{n=1}^{\infty} d_n^\dagger d_n} , \tag{2.11b}$$

and $\phi_b^a$ is chosen so that (2.3) holds. For our purposes we need only that $(\phi_b^a)^8=(-1)^{(a+b)}$. The Neveu-Schwarz and Ramond Hamiltonians in (2.10) are for single fermionic degrees of freedom. All we need to know about the bosonic contributions for the present work is that for the combined eight bosonic degrees of freedom the contribution to the normal-ordering constant is $-\frac{1}{3}$. Strictly speaking, Eq. (2.11b) is correct only when we consider products of an even number of fermions, i.e., $\prod_{i=1}^{2n}(-1)^{\hat{N}_1}$; for a single fermion it is simply a formal expression whose precise meaning is explained in Appendix B. The $\pm$ in (2.11b) corresponds to left or right chirality in the Ramond sector and may be absorbed into the representation chosen for $d_0$.

## III. CLASSIFICATION OF SPIN STRUCTURES

The most general possible contribution to the one-loop string path integral from the fermionic degrees of freedom is a sum of products of fermion determinants with arbitrary coefficients over all possible combinations of spin structures:

$$Z=\sum_{\{\mathbf{a},\mathbf{b}\}} C_{b_0b_1\cdots b_{32}}^{a_0a_1\cdots a_{32}} Z_{b_0b_1\cdots b_{32}}^{a_0a_1\cdots a_{32}} . \tag{3.1}$$

Here we use a natural generalization of the notation introduced in the preceding section; $Z_{b_0b_1\cdots b_{32}}^{a_0a_1\cdots a_{32}}$ is a product of the fermion determinant from the right-moving modes with spin structure $\binom{a_0}{b_0}$ and the 32 fermion determinants from the left-moving modes with spin structures $\binom{a_i}{b_i}$. Often we will employ an obvious vector notation, e.g., $Z_{\mathbf{b}}^{\mathbf{a}}$. The sum in (3.1) runs over all combinations of $a_i$ and $b_j$ equal to 0 or 1. Explicitly we have

$$Z_{\mathbf{b}}^{\mathbf{a}}=(-1)^{a_0+b_0+(1/8)\sum_{i=1}^{32}(a_i+b_i)}\,\mathrm{Tr}\left[\exp\left[i\tau\sum_{j=1}^{32}\hat{H}_{a_j}-i\bar{\tau}\hat{H}_{a_0}\right](-1)^{\mathbf{b}\cdot\hat{\mathbf{N}}_{\mathbf{a}}}\right] . \tag{3.2}$$

The components of $\hat{\mathbf{N}}_{\mathbf{a}}$ are the appropriate fermion number operators; for the right-moving sector $\hat{H}_{a_0}$ and $\hat{N}_{a_0}$ are the eight fermion analogs of the single fermion Hamiltonian and number operators of (2.10) and (2.11):

$$\hat{H}_{a_0=0}=2\pi\left[\sum_{r=1/2}^{\infty}\sum_{i=1}^{8} r b_r^{\dagger i} b_r^i-\frac{1}{6}\right] , \tag{3.3a}$$

$$\hat{H}_{a_0=1}=2\pi\left[\sum_{n=1}^{\infty}\sum_{i=1}^{8} n d_n^{\dagger i} d_n^i+\frac{1}{3}\right] , \tag{3.3b}$$

$$\hat{N}_{a_0=0}=\sum_{r,i} b_r^{\dagger i} b_r^i , \tag{3.4a}$$

$$(-1)^{\hat{N}_{a_0=1}}=\pm d_0^1 d_0^2\cdots d_0^8(-1)^{\sum_{n,i} d_n^{\dagger i} d_n^i} . \tag{3.4b}$$

The factor of $\frac{1}{8}$ in (3.2) causes no problems since from (2.8)

$$C_{\mathbf{b}}^{\mathbf{a}}=0 \text{ unless } \Gamma\begin{bmatrix}a\\b\end{bmatrix}=0 \pmod 8 \tag{3.5}$$

for any particular choice of $a,b\in Z_2$.

The phase in (3.2) is chosen so that under modular transformations the different $Z_{\mathbf{b}}^{\mathbf{a}}$ transform into each other without picking up any additional phases. For $Z$ to be modular invariant it is necessary and sufficient that the coefficients of terms in (3.1) related by modular transformations be identified. In our notation this condition is [cf. (2.3)]

$$C_{\mathbf{b}}^{\mathbf{a}}=C_{\mathbf{a}}^{\mathbf{b}} \tag{3.6a}$$

for invariance under (2.3a)

$$C_{\mathbf{b}}^{\mathbf{a}}=C_{\mathbf{1}+\mathbf{a}+\mathbf{b}}^{\mathbf{a}} \tag{3.6b}$$

for invariance under (2.3b), where $\mathbf{1}$ is the 33-dimensional vector with all components equal to one, and in such manipulations the subscripts and superscripts are elements of $Z_2$.

For (3.1) to be interpreted as a sum of $e^{i\tau H}$ over string states the $C_{\mathbf{b}}^{\mathbf{a}}$ must be chosen so that in each physical sector we obtain a sensible projection operator, i.e.,

$$\sum_{\{\mathbf{b}\}}(-1)^{(1/8)\sum_{i=1}^{32}(a_i+b_i)+\mathbf{b}\cdot\mathbf{n}}C_{\mathbf{b}}^{\mathbf{a}}=\eta_{\mathbf{n}}^{\mathbf{a}}=0 \text{ or } 1 . \tag{3.7}$$

For a given collection of fermion numbers $(\mathbf{n})$ a string state either appears in the sum of $e^{i\tau H_{\mathbf{a}}}$ contributing to $Z$ $(\eta_{\mathbf{n}}^{\mathbf{a}}=1)$ or it does not $(\eta_{\mathbf{n}}^{\mathbf{a}}=0)$. The sign accompanying the term in the sum is determined by requiring the states to have physically sensible statistics. In particular, in writing (3.7) we must define $\mathbf{n}$ to be the eigenvalue of $\hat{\mathbf{N}}$ (mod 2) except for the zeroth component $n_0$, which differs from the eigenvalue of $\hat{N}_0$ by 1; thus, the phase factor in (3.7) differs from that in (3.2) by a factor of $(-1)^{a_0}$ which ensures the desired statistics. Space-time bosons (Neveu-Schwarz sector, $a_0=0$) contribute to loop amplitudes with a positive sign while space-time fermions (Ramond sector, $a_0=1$) have the opposite statistics, contributing with a negative sign. On the other hand, for the left-moving degrees of freedom the states in the two sectors $a_i=0$ or 1 contribute to the loop amplitude with the same sign, i.e., have the same statistics, as is appropriate for an internal-symmetry group.

Our present goal is to solve (3.7) subject to the condition (3.6). A general solution to (3.7) is easy enough to obtain; multiplying through by $(-1)^{\mathbf{b}'\cdot\mathbf{n}}$ and summing over all $n_i=0,1$ one finds

$$(-1)^{(1/8)\sum_1^{32}(a_i+b_i')}C_{\mathbf{b}'}^{\mathbf{a}}=2^{-33}\sum_{\{\mathbf{n}\}}(-1)^{\mathbf{b}'\cdot\mathbf{n}}\eta_{\mathbf{n}}^{\mathbf{a}} . \tag{3.8}$$

It remains to determine for which choices of $\eta_{\mathbf{n}}^{\mathbf{a}}$ (3.6) holds. Consider first the pure Neveu-Schwarz sector, $\mathbf{a}=\mathbf{0}$. Equation (3.8) becomes (dropping the primes on $\mathbf{b}$)

$$(-1)^{(1/8)\sum_{i=1}^{32}b_i}C_{\mathbf{b}}^{0}=2^{-33}\sum_{\{\mathbf{n}\}}(-1)^{\mathbf{b}\cdot\mathbf{n}}\eta_{\mathbf{n}}^{0} . \tag{3.9}$$

From (3.6b) we know, in particular, that $C_0^0=C_1^0$ which implies that

$$\sum_{\{\mathbf{n}\}}[1-(-1)^{\sum_{i=0}^{32}n_i}]\eta_{\mathbf{n}}^{0}=0 . \tag{3.10}$$

In other words, $\eta_{\mathbf{n}}^0$ vanishes unless

$$\sum_{i=0}^{32}n_i=0 \quad (\text{mod}2) . \tag{3.11}$$

From (3.6a) we require $C_{\mathbf{b}}^0=C_0^{\mathbf{b}}$ which with (3.8) implies

$$\sum_{\{\mathbf{n}\}}(-1)^{\mathbf{b}\cdot\mathbf{n}}\eta_{\mathbf{n}}^{0}=\sum_{\{\mathbf{n}\}}\eta_{\mathbf{n}}^{\mathbf{b}}\geq 0 \text{ for all } \mathbf{b} . \tag{3.12}$$

Equations (3.5), (3.11), and (3.12) determine the allowed choices for $\eta_{\mathbf{n}}^0$. Equation (3.9) then determines those choices for $\mathbf{b}$ which give nonvanishing $C_{\mathbf{b}}^0$. It turns out that these choices for $\mathbf{b}$ form a vector space over the field $Z_2$ (this is shown explicitly in Appendix A). Knowing this we can reconsider our equations, restricting $\mathbf{b}$ to an $r$-dimensional vector space, with basis vectors $\mathbf{V}^l$,

$$\mathbf{b}=\beta_1\mathbf{V}^1+\cdots+\beta_r\mathbf{V}^r\equiv\boldsymbol{\beta}\cdot V . \tag{3.13}$$

Here and throughout the following discussion, boldface greek letters denote $r$-dimensional vectors and boldface roman letters 33-dimensional vectors. The only exception is $V$ which is an $r\times 33$ matrix so that $\mathbf{V}^l$ and $\mathbf{V}_i$ are 33- and $r$-dimensional vectors, respectively $(l=1,\ldots,r;$ $i=0,\ldots,32)$.

Given (3.13), Eqs. (3.12), (3.7), and (3.6) guarantee that $\eta_{\mathbf{n}}^{\mathbf{a}},C_{\mathbf{b}}^{\mathbf{a}},C_{\mathbf{a}}^{\mathbf{b}}$ are zero unless

$$\mathbf{a}=\boldsymbol{\alpha}\cdot V , \quad \mathbf{b}=\boldsymbol{\beta}\cdot V ; \quad \alpha_l,\beta_l\in Z_2 . \tag{3.14}$$

With these restrictions, Eq. (3.7) becomes

$$\sum_{\{\boldsymbol{\beta}\}}(-1)^{(1/8)\sum_{i=1}^{32}(\boldsymbol{\alpha}+\boldsymbol{\beta})\cdot\mathbf{V}_i+\boldsymbol{\beta}\cdot\boldsymbol{\gamma}}C_{\boldsymbol{\beta}\cdot V}^{\boldsymbol{\alpha}\cdot V}=\eta_{\boldsymbol{\gamma}}^{\boldsymbol{\alpha}\cdot V}=0 \text{ or } 1 , \quad \gamma^l\equiv\mathbf{V}^l\cdot\mathbf{n}\in Z_2 , \tag{3.15}$$

and the analog of (3.8) is

$$(-1)^{(1/8)\sum_{i=1}^{32}(\boldsymbol{\alpha}+\boldsymbol{\beta})\cdot\mathbf{V}_i}C_{\boldsymbol{\beta}\cdot V}^{\boldsymbol{\alpha}\cdot V}=2^{-r}\sum_{\{\boldsymbol{\gamma}\}}(-1)^{\boldsymbol{\beta}\cdot\boldsymbol{\gamma}}\eta_{\boldsymbol{\gamma}}^{\boldsymbol{\alpha}\cdot V} . \tag{3.16}$$

Now $|C_0^{\boldsymbol{\alpha}\cdot V}|=2^{-r}$ from Eq. (A12) in Appendix A, so that for each $\boldsymbol{\alpha}$ one and only one of the $\eta_{\boldsymbol{\gamma}}^{\boldsymbol{\alpha}\cdot V}$ is nonzero. With this in mind (3.16) becomes

$$(-1)^{(1/8)\sum_{i=1}^{32}(\boldsymbol{\alpha}+\boldsymbol{\beta})\cdot\mathbf{V}_i}C_{\boldsymbol{\beta}\cdot V}^{\boldsymbol{\alpha}\cdot V}=2^{-r}(-1)^{\boldsymbol{\beta}\cdot\boldsymbol{\gamma}(\boldsymbol{\alpha})} , \tag{3.17}$$

where $\boldsymbol{\gamma}$ now depends on $\boldsymbol{\alpha}$.

Before enforcing modular invariance to restrict $\boldsymbol{\gamma}(\boldsymbol{\alpha})$, let us determine which choices of basis vectors $\mathbf{V}^l$ are consistent with our requirements. From (3.17) it is clear that $C_{\boldsymbol{\beta}\cdot V}^{\boldsymbol{\alpha}\cdot V}$ is nonzero for any choice of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, so (3.5) places a severe constraint on all but the zeroth component of the basis vectors $\mathbf{V}^l$. Let $\{\mathbf{U}^m\}$ be a basis for the 32-dimensional subspace obtained by eliminating the zeroth component from the vector space spanned by $\{\mathbf{V}^l\}$. We must require that any vector in the subspace have 0 (mod 8) nonzero components and furthermore any two vectors must share a multiple of 8 nonzero components. We may now easily catalog all of the allowed choices for $\{\mathbf{U}^m\}$. Since all of the 32 fermions in the internal-symmetry space are equivalent we need not consider bases that differ only by reordering of the components. From the analysis of Appendix A we know that each basis must contain the vector with all components equal to one, $\mathbf{U}^1=\mathbf{1}$; thus this is the only possible one-dimensional basis. For two-dimensional subspaces we have two choices; we can add a vector with 8 components equal to 1 ($\overline{\mathbf{U}}_i^2=1$ for $i\le 8$; $=0$ for $i>8$) or one with 16 nonzero components ($\mathbf{U}_i^2=1$ for $i\le 16$; $=0$ for $i>16$). The vector with 24 nonzero components is contained in the first subspace ($\mathbf{U}^1+\overline{\mathbf{U}}^2$). For three-dimensional subspaces we can have two "8-vectors" with no ones in common or two "16-vectors" with eight ones in common. Any allowed sub-basis with an 8- and 16-vector is equivalent to a sub-basis with two nonoverlapping 8-vectors. Following such considerations one can easily find the nine distinct bases listed in Table II. Any collection of more than three 8-vectors or five 16-vectors with allowed overlaps are not independent. For the higher-dimensional bases one must consider mutual overlaps between three or more vectors, but a careful check shows that only those subspaces spanned by the vectors in Table II meet our requirements.

The allowed 33-dimensional basis vectors, $\mathbf{V}^l$, are obtained by adding a zeroth component to the $\mathbf{U}^m$ vectors with value 0 and/or 1. From the analysis of Appendix A it is clear that $\mathbf{V}^1=\mathbf{1}$ must always appear. Two distinct cases now arise: either the vector space includes the vector with components $\mathbf{V}_i^0=\delta_{i,0}$ in which case all vectors in the space can have their zeroth component equal to both 0 and 1 (i.e., the zeroth component and the 32-dimensional vectors completely factor) or it does not, in which case the basis vectors $\mathbf{V}^l$ ($l>1$) coincide with the $\mathbf{U}^l$ with the zeroth component added with value *either* 0 *or* 1 (if both appeared, the sum of the two vectors would be $\mathbf{V}^0$).

Now we return to Eq. (3.17). Knowing from the above discussion that any two vectors in the vector space spanned by $\mathbf{U}^l$ have overlap=0 (mod 8) we can rewrite (3.17) as

$$C_{\beta\cdot V}^{\alpha\cdot V}=2^{-r}(-1)^{\beta\cdot\gamma(\alpha)+(\alpha+\beta)\cdot\rho}\ ,\qquad \rho\equiv\left[\frac{1}{8}\sum_{i=1}^{32}\mathbf{V}_i\right]\ (\text{mod}2)\ . \tag{3.18}$$

All that remains to be done is to determine which choices of $\gamma(\alpha)$ are consistent with modular invariance, which, in the present notation, is the demand that [cf. (3.6)]

$$C_{\beta\cdot V}^{\alpha\cdot V}=C_{\alpha\cdot V}^{\beta\cdot V}\ , \tag{3.19a}$$

$$C_{\beta\cdot V}^{\alpha\cdot V}=C_{(\alpha+\beta)\cdot V+\mathbf{V}^1}^{\alpha\cdot V}\ . \tag{3.19b}$$

Applying (3.19a) to (3.18) gives

$$\beta\cdot\gamma(\alpha)=\alpha\cdot\gamma(\beta)\ . \tag{3.20}$$

This equation must hold for all values of $\alpha_l$ and $\beta_m$, for example, choosing $\beta_m=\delta_{lm}$ (3.20) becomes

$$\gamma_l(\alpha)=\alpha\cdot\gamma(\beta_l=1)\ . \tag{3.21}$$

In other words, $\gamma_l$ is a linear function of the $\alpha_l$'s and (3.18) may be written

$$C_{\beta\cdot V}^{\alpha\cdot V}=2^{-r}(-1)^{\sum_{l,m}\beta_l k_{lm}\alpha_m+(\alpha+\beta)\cdot\rho}\ , \tag{3.22}$$

where $k_{lm}$ take values in $Z_2$ and are independent of $\alpha_l$ and $\beta_l$. Now (3.19a) simply requires that

$$k_{lm}=k_{ml}\ . \tag{3.23}$$

Equation (3.19b) applied to (3.22) gives

$$\sum_{l,m}\alpha_l k_{lm}\cdot\alpha_m+\sum_l k_{1l}\alpha_l+\alpha\cdot\rho+\rho_1=0\ (\text{mod}2)\ \text{ for all }\alpha\ . \tag{3.24}$$

This reduces to

$$k_{ll}+k_{1l}+\rho_l=0\ (\text{mod}2)\ , \tag{3.25}$$

where we have used the fact that $\alpha_l^2=\alpha_l$, $\mathbf{V}^1=\mathbf{1}$, and $\rho_1=0$. Inserting (3.22) and (3.2) into (3.1) gives finally

$$Z=2^{-r}\sum_{\{\alpha,\beta\}}(-1)^{\alpha\cdot\mathbf{v}_0}\text{Tr}\left[\exp\left[i\tau\sum_{j=1}^{32}\hat{H}_{\alpha\cdot\mathbf{v}_j}-i\bar{\tau}\hat{H}_{\alpha\cdot\mathbf{v}_0}\right](-1)^{\sum_l\beta_l(V_0^l+\sum_m k_{lm}\alpha_m+\sum_i V_i^l\hat{N}_i)}\right]\ . \tag{3.26}$$

For each given choice of the set of basis vectors $\mathbf{V}^l$ and each given set $\{k_{lm}\}$ that is consistent with (3.23) and (3.25), we have a fermionic string model. Its spectrum is generated by the various choices of $\alpha_l$, $\beta_l$, and $n_i$ that are allowed by the constraints.

## IV. CLASSIFICATION OF CLOSED-FERMIONIC-STRING MODELS

Equation (3.26) along with (3.23), (3.25), and Table II represent a complete catalog of physically interpretable, closed-fermionic-string spin structures satisfying (2.8). In this section we will demonstrate how to extract from this information the possibilities for string models, in particular, their spectrum of physical states. A given spin structure serves to project out a particular collection of string states. Explicitly, Eq. (3.26) implies that in the sector with Hamiltonian $\sum_{j=0}^{32}\hat{H}_{\alpha\cdot\mathbf{v}_j}$ the physical spectrum includes those states whose fermion numbers $n_k$ satisfy

$$V_0^l+\sum_m k_{lm}\alpha_m+\sum_i V_i^l n_i=0\ \ (\text{mod}2)\ . \tag{4.1}$$

The (mass)$^2$ of a given state depends on its fermion numbers, and on the (mass)$^2$ of the ground state of the physical sector (represented by $\alpha\cdot V$) which is given by a sum of normal-ordering constants [cf. (2.10)]. Figure 1 gives the ground-state (mass)$^2$ levels for each sector we need to consider, with each $R$ or $N$ representing eight fermions in a Ramond or Neveu-Schwarz sector. In addition to satisfying (4.1) a physical state must have the same mass in both its right- and left-moving modes as a consequence of in-

TABLE II. Allowed subspace bases. All basis vector components not equal to one vanish.

| Basis | Basis vectors $U^m$ |
|---|---|
| 1 | $U^1$ ($U_i^1=1$; $i=1,\ldots,32$) |
| 2 | $U^1$, $U^2$ ($U_i^2=1$; $i=1,\ldots,16$) |
| 3 | $U^1$, $U^2$, $U^3$ ($U_i^3=1$; $i=1,\ldots,8$, $17,\ldots,24$) |
| 4 | $U^1$, $U^2$, $U^3$, $U^4$ ($U_i^4=1$; $i=1,\ldots,4$, $9,\ldots,12$, $17,\ldots,20$, $25,\ldots,28$) |
| 5 | $U^1$, $U^2$, $U^3$, $U^4$, $U^5$, ($U_i^5=1$; $i=4n+1$, $4n+2$; $n=0,\ldots,7$) |
| 6 | $U^1$, $U^2$, $U^3$, $U^4$, $U^5$, $U^6$ ($U_i^6=1$; $i=$odd) |
| 7 | $U^1$, $\overline{U}^2$ ($\overline{U}_i^2=1$; $i=1,\ldots,8$) |
| 8 | $U^1$, $\overline{U}^2$, $\overline{U}^3$ ($\overline{U}_i^3=1$; $i=9,\ldots,16$) |
| 9 | $U^1$, $\overline{U}^2$, $\overline{U}^3$, $\overline{U}^4$ ($\overline{U}_i^4=1$; $i=17,\ldots,24$) |

variance of the theory under shifts in the world-sheet parameter $\sigma$. This is enforced by the integration of (3.26) (multiplied by the corresponding contribution of the bosonic fields to the path integral) over $\tau$ in the fundamental domain; the result vanishes unless the right (mass)$^2$ and left (mass)$^2$ are equal.

*A priori* each possible combination of choices of V from Table II and **k** satisfying (3.23) and (3.25) represents a potential new string model. In practice, however, we have found that out of this large number of models only a few are physically distinct; a given gauge group can have many different fermionic representations.

Let us consider a particular example, basis 7 from Table II, and determine the spectrum of tachyons and gauge bosons for the models it represents. First consider the case where the basis of 33-dimensional vectors does not contain $\mathbf{V}^0$. Then (3.23) and (3.25) give

$$k_{11}=0 \text{ or } 1\,, \qquad k_{12}=k_{21}=1+k_{22}\,. \tag{4.2}$$

Equation (4.1) becomes

$$1+k_{11}\alpha_1+(1+k_{22})\alpha_2=n_0+N_8+N_{24} \pmod 2\,,$$
$$V_0^2+k_{22}(\alpha_2+\alpha_1)+\alpha_1=N_8+V_0^2 n_0 \pmod 2\,, \tag{4.3}$$
$$N_8\equiv\sum_{i=1}^{8} n_i\,,\quad N_{24}\equiv\sum_{i=9}^{32} n_i\,.$$

Notice that in this case the spin structures for the first 8 left-moving fermions are always identified, as are the spin structures for the last 24; the physical states will naturally

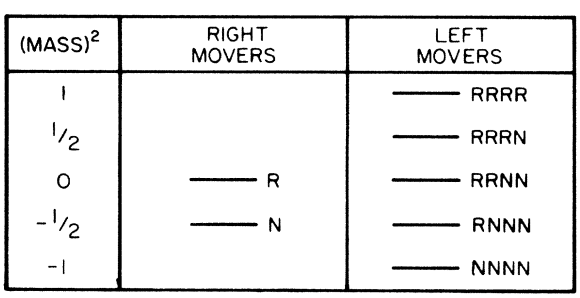


FIG. 1. Ground-state (mass)$^2$ level structure for left- and right-moving modes. $R$, Ramond (periodic); $N$, Neveu-Schwarz (antiperiodic).

be grouped in representations of $SO(8)_R\times SO(8)_L \times SO(24)_L$.

Consider first the $N_R(NNNN)_L$ sector (using the notation of Fig. 1). The only way to build this sector from our vector space is to take $\alpha_1=\alpha_2=0$ so that $\boldsymbol{\alpha}\cdot V$ is the 33-dimensional zero vector. Any tachyon present must have $n_0=0$, $N_8+N_{24}=1$. Equation (4.3) then tells us that the tachyons in this sector fall into two possible representations of $SO(8)_R\times SO(8)_L\times SO(24)_L$ depending on the value of $V_0^2$, $(1,1,24_v)$ or $(1,8_v,1)$ for $V_0^2=0$ or 1, respectively. The massless states in this sector have $n_0=1$, $N_8+N_{24}=2$. From (4.3) the gauge bosons in this sector for both $V_0^2=0$ and 1 are in the representation $(8_v,28,1)+(8_v,1,276)$ since we can have two fermions in either the first 8 or the last 24 modes of the internal space, but not split between the two. The only other bosonic states with (mass)$^2$ less than or equal to zero fall into the $N_R(RNNN)_L$ sector. If $V_0^2=1$ then the vector space does not contain this sector and the only nonpositive mass states are those given above. If $V_0^2=0$ then $\alpha_1=0$, $\alpha_2=1$ are the necessary choice (giving $\boldsymbol{\alpha}\cdot V$ with 8 ones among the 32 left movers, the other components vanishing). Tachyons are present if $n_0=N_{24}=0$ is allowed. $N_8$ can be either 0 or 1. Since the first 8 left-moving fermions fall into a Ramond sector this just determines which of the two eight-dimensional SO(8) spinor representations the ground state transforms as. From (4.3) one indeed finds tachyons, in the representation $(1,8_s,1)$. $N_{24}=n_0=1$ satisfies (4.3) for $\alpha_1=0$, $\alpha_2=1$ so one finds massless bosons in the representation $(8_v,8_s,24_v)$. Taken together with the results from the pure Neveu-Schwarz sector we find for $V_0^2=0$ that the gauge bosons fill out the adjoint representation of SO(32) while the tachyons form an SO(32) vector. $V_0^2=1$ gives the $SO(8)\times SO(24)$ tachyonic model. In similar fashion one can obtain the massless fermionic states of these models by examining the $R_R(NNNN)_L$ and $R_R(RNNN)_L$ sectors (only the latter appears in the present case and only for $V_0^2=1$).

Adding $\mathbf{V}^0$ to the vector space considered above we can, in a completely analogous manner, find the physical spectrum of another set of models. Here we obtain the same vector space for either choice of $V_0^2$, but one still finds two different models distinguished now by the value of $k_{02}$. For $k_{02}=1$ we again find the tachyonic $SO(8)\times SO(24)$ model, though built with a quite different spin structure from the case considered above. If $k_{02}=0$ the model is the tachyon-free SO(32) heterotic string.

Bases 8 and 9 in Table II require somewhat more effort to analyze, but the manipulations are completely straightforward. Cases 1 through 6, those bases built from "16-vectors," are easier to work out. In these cases all $\rho_l=0$ and the value of $V_0^l$, $l>1$, may be chosen to be zero, since $\mathbf{V}^{l'}=\mathbf{V}^1+\mathbf{V}^l$ also appears in the vector space and has all the symmetry properties of $\mathbf{V}^l$ but with $V_0^{l'}=1$. Given this, Eqs. (3.23) and (3.25) become

$$k_{11}=0 \text{ or } 1\ ,\quad k_{1l}=k_{l1}=k_{ll}\ . \tag{4.4}$$

In turn (4.1) reduces to

$$\delta_{l,1}+\delta_{l,0}+\sum_m k_{lm}\alpha_m=\sum_j V_j^l n_j \pmod 2\ . \tag{4.5}$$

The spectra of these models can be considered in detail just as outlined in the example above; here we restrict ourselves to counting the number of tachyons and gauge bosons. It is not difficult to consider bases 1 through 6 simultaneously. For this purpose define $p$ to be the number of basis 16-vectors for the given case. Then $h=2^{5-p}$ is the "block size" in which all the spin structures must be identified, and $q=16/h$ is the number of such blocks in a group of 16. Consider first the case where $\mathbf{V}^0$ is not included in the vector space. The only sector in which tachyons can appear for these models is $N_R(NNNN)_L$ (all $\alpha_l=0$), $n_0=0$ and $\sum_{i=1}^{32} n_i=1$ [$N_R(RNNN)_L$ cannot be obtained with 16-vectors]. With these restrictions, Eq. (4.5) implies that the only $n_i$ which can be nonzero are those such that $V_i^l=0$ for all $l=2,\dots,p+1$. The number of these, and hence the number of tachyons, is easily seen to be $h$.

For the massless gauge bosons we must examine two sectors, $N_R(NNNN)_L$ and $N_R(RRNN)_L$, all others give only massive states or do not appear from our vector space. In the former case we need all $\alpha_l=0$, $n_0=1$, and $\sum_{i=1}^{32} n_i=2$. To satisfy (4.5) two $n_i (i\neq 0)$ can be nonzero only if they appear in the same block, thus the total number of gauge bosons in this sector is $qh(h-1)$.

Any nonvanishing combination of $\alpha_l$, $l\neq 1$, along with $\alpha_1=0$ gives spin structure in the $N_R(RRNN)_L$ sector. Since all such choices are equivalent we may count the number of gauge bosons for the case $\alpha_l=\delta_{l,2}$ and multiply by the number of combinations, $(2q-1)$, to obtain the total number of gauge bosons. For massless bosons we need $n_0=1$ and $\sum_{i=17}^{32} n_i=0$. The first 16 fermions are broken into $q$ blocks of size $h$, each in the Ramond sector. The total fermion number in each block determines the chirality of the ground-state spinor representation in that block. To get the number of gauge bosons we must multiply a factor of the dimension of the spinor representation $(2^{(h/2)-1})$ for each of the $q$ blocks by the number of fermion number combinations allowed by (4.5). Now (4.5) places $p$ constraints on the total of $q$ fermion numbers (one for each block) leaving $2^{q-p}$ allowed combinations; thus the number of gauge bosons for $\alpha_2=1$ is $(2^{(h/2)-1})^q 2^{q-p}$. Putting our results together, the total number of gauge bosons for each model is

$$qh(h-1)+(2q-1)(2^{(h/2)-1})^q 2^{q-p}=240+2^{8-p} \tag{4.6}$$

which gives, in order $p=0, 1,\dots,5$, the last six entries of Table I. Strictly speaking for case 6, where $h$ is one and the dimension of the spinor representation given above is $1/\sqrt{2}$, one should be more careful and directly consider the algebra of the Ramond zero modes, but one finds that (4.6) is in fact the correct result (this is shown in Appendix B).

Massless space-time fermions in these models arise only from the $R_R(RRNN)_L$ sector. The counting of these states is essentially the same as the counting of gauge bosons in this sector given above. The only effective difference is that $n_0$ may be either 0 or 1 so that the number of massless fermions (beyond those in the graviton multiplet) is twice the result given above,

$$(2q-1)(2^{(h/2)-1})^q 2^{q-p+1}=2^{9-p}(2q-1)\ . \tag{4.7}$$

Adding $\mathbf{V}^0$ to the vector spaces considered above gives from (4.5) an additional equation:

$$n_0=1+\sum_l k_{0l}\alpha_l\ . \tag{4.8}$$

For $\alpha_l=0$ this gives $n_0=1$; i.e., there are no tachyons allowed in the physical spectra of these theories. Upon examination one finds two distinct possibilities in these cases depending on whether all of the $k_{0l}$ vanish or some are nonzero. In either case the counting for massless states in the $N_R(NNNN)_L$ sector proceeds exactly as before. The difference arises in the number of allowed combinations of values of $\alpha_l$ which give the $N_R(RRNN)_L$ sector. If $k_{0l}=0$ for all $l$ then the number of such combinations is twice that found above so that the total number of gauge bosons is

$$qh(h-1)+(2q-1)(2^{(h/2)-1})^q 2^{q-p+1}\ , \tag{4.9a}$$

which turns out to be 496 for all five cases. If any of the $k_{0l}$ are nonzero then regardless of the particular choice of $k_{0l}$, there are $(2q-2)$ combinations of $\alpha_l$ which produce the $N_R(RRNN)_L$ sector and so a total of

$$qh(h-1)+(2q-2)(2^{(h/2)-1})^q 2^{q-p} \tag{4.9b}$$

gauge bosons. Remarkably this gives a value of 240 for all the cases at hand, producing many copies of the SO(16)×SO(16) model.[5,6] Note as well that bases 1 through 6 in Table II are sufficient to generate all of the models in Table I (aside from the type-II superstrings). Bases 7, 8, and 9 generate only a subset, not including the last three models.

## V. DISCUSSION AND REMARKS

To maintain manifest covariance in the path-integral formulation of closed-fermionic strings, it is convenient to use the NSR formalism for the space-time fermions and to express the internal-symmetry group in terms of world-sheet fermionic variables. Invariance under world-sheet superdiffeomorphisms, which include both reparametrization and local supersymmetry invariance, is necessary to remove the timelike components of the string superfield $X^\mu$. The measures needed in the definition of the functional integrals, in general, break world-sheet superconformal and Lorentz invariances. This is the origin of superconformal and Lorentz anomalies. The absence of such anomalies impose stringent constraints on the physically acceptable models. Superconformal invariance requires the space-time dimension to be $D=10$, and the number of Majorana-Weyl fermions in the internal-symmetry space (for models with gauge symmetry) to be 32. The absence of global Lorentz anomalies implies that string models must be modular invariant.[9] In fact, modular invariance is the key ingredient to the ultraviolet finiteness property of string theories. With superdiffeomorphism, superconformal, and Lorentz invariances, all potential sources of anomalies are removed. Hence the resulting string models should be completely anomaly-free. In particular, space-time gravitational and gauge anomalies are expected to be absent in these string models.

To obtain a sensible fermionic-string model, the space-time fermions must have Fermi-Dirac statistics while all internal-symmetry degrees of freedom and space-time bosons must have Bose-Einstein statistics. This condition plus one-loop modular invariance impose stringent constraints on physically interesting string models. As pointed out by Seiberg and Witten[4] and others, these conditions may also be sufficient to guarantee multiloop modular invariance. In this case our classification of ten-dimensional closed-fermionic-string models is complete provided we only consider periodic or antiperiodic boundary conditions for the fermionic fields, (2.8) is satisfied, and the world-sheet fermions in the internal symmetry space remain noninteracting. While we have not yet exhausted the possibilities for more complicated boundary conditions for the fermionic fields, our results so far suggest that Table I probably includes all models.[13]

Some of the models in our classification do not have space-time supersymmetry and some have tachyons. The absence of space-time supersymmetry implies that a nonzero cosmological constant $\Lambda$ will be present at the quantum level, as, for example, has been explicitly calculated for the $SO(16)\times SO(16)$ model where $\Lambda$ is finite and positive.[5,6] This clearly indicates that the dilaton tadpole is not zero. The presence of tachyons, as well as the presence of nonzero dilaton one-point functions, are clear signs of vacuum instability at either the classical or quantum level. Since strings that are stable and remain in $D=10$ are physically undesirable, one may go so far as to consider the presence of tachyons and/or a dilaton tadpole as virtues rather than fatal flaws. It will be most interesting to analyze the nonsupersymmetric string models to see if there exist compactifications that result in effectively $D=4$, $N=1$ supersymmetric models with realistic gauge groups and fermion contents.

Dixon and Harvey[5] have given a simple, elegant classification of fermionic string models based on the bosonic formulation of the internal-symmetry space. Their approach, which classifies inequivalent shift vectors on the group lattice, generates all the models except the $E_8$ model. This is not surprising since in our analysis all of the models with the exception of the $E_8$ one have spin structures of pairs of fermions identified so that the fermionic variables can be easily bosonized. The $E_8$ model involves only single fermions and so does not readily emerge from a bosonic formulation. Dixon and Harvey observe, however, that there is one other possible set of models involving the outer automorphism of $E_8\otimes E_8$, which exchanges the two $E_8$'s. The $E_8$ model generated by the spin-structure method may belong to this set. It will be interesting to understand the relation between these two different but complementary approaches.

Finally it should be pointed out that new closed-fermionic-string models in fewer than ten dimensions may be constructed by an extension of the method described in this paper. It should be obvious to the reader that our analysis is a generalization of the method of Gliozzi, Scherk, and Olive.[14] A further generalization allows us to construct, among other models, $N=1$ supersymmetric chiral string models in four space-time dimensions. The results will be presented elsewhere.[13]

*Note added.* Lance Dixon and Jeffrey Harvey informed us that they have also constructed the $E_8$ model from their approach. We thank them for useful discussions.

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## APPENDIX A

Here we demonstrate that in order to satisfy Eqs. (3.9), (3.11), and (3.12) the values of $\mathbf{b}$ for which $C_{\mathbf{b}}^0$ is nonzero must form a linear vector space over the field $Z_2$.

First note that if only one of the $\eta_{\mathbf{n}}^0$ is one then (3.12) implies that $\mathbf{b}\cdot\mathbf{n}=0$ for all $\mathbf{b}$, i.e., $\mathbf{n}=0$. Similarly if only $\eta_{\mathbf{n}_1}^0$ and $\eta_{\mathbf{n}_2}^0$ are nonzero then one of them again must be $\eta_0^0$ and Eq. (3.9) becomes

$$(-1)^{(1/8)\sum_{i=1}^{32} b_i} C_{\mathbf{b}}^0 = 1+(-1)^{\mathbf{b}\cdot\mathbf{n}_1} . \quad \text{(A1)}$$

In other words,

$$C_{\mathbf{b}}^0=0 \quad \text{unless } \mathbf{b}\cdot\mathbf{n}_1=0 \pmod 2 . \quad \text{(A2)}$$

We have a linear equation for the allowed $\mathbf{b}$ over the field $Z_2$.

Now if some number $M$ of the $\eta_{\mathbf{n}_l}^0$ are equal to one then (3.12) becomes

$$\sum_{l=1}^{M} (-1)^{\mathbf{b}\cdot\mathbf{n}_l} \geq 0 \quad \text{for all } \mathbf{b} . \quad \text{(A3)}$$

In general only some number s of the exponents in (A3) will be linearly independent functions of the $b_i$ (mod2); designate these as $B_j$, $j=1,\dots,s$. We wish to show that the left-hand side of (A3) (which we will call $F_s$) must be of the form

$$F_s(B_1,\dots,B_s)=\prod_{j=1}^{s}[1+(-1)^{B_j}] . \quad \text{(A4)}$$

If this is the case then $C_{\mathbf{b}}^0$ will be nonvanishing if and only if $\mathbf{b}$ satisfies

$$B_1=B_2=\cdots=B_s=0 \pmod 2 . \quad \text{(A5)}$$

This is our desired result, the allowed $\mathbf{b}$ being the vector space of solutions of the system (A5) of $s$-independent linear homogeneous equations on $Z_2$.

Our proof of (A4) proceeds by induction. From (A1) it is clear that (A4) holds for $s=1$. Assume that it holds for $s=N-1$. For $s=N$ (A3) can be written

$$\bar{G}(B_1,\ldots,B_{N-1})+(-1)^{B_N}G(B_1,\ldots,B_{N-1})\geq 0 \quad \text{for all } B_j=0 \text{ or } 1 \ . \qquad (A6)$$

Comparing (A6) for $B_N=0$ and 1 we see that $\bar{G}$ satisfies

$$\bar{G}(B_1,\ldots,B_{N-1})\geq 0 \quad \text{for all } B_j,\ j=1,\ldots,N-1, \qquad (A7)$$

So $\bar{G}$ is the solution to (A3) for $S=N-1$ which by assumption takes the form $F_{N-1}$. Now if $B_1=1$ then $F_{N-1}$ vanishes for every choice of $B_j$, $j=2,\ldots,N-1$, and (A6) gives

$$(-1)^{B_N}G(1,B_2,\ldots,B_{N-1})\geq 0 \quad \text{for all } B_j,\ j=2,\ldots,N \ . \qquad (A8)$$

For this to be true for $B_N=0$ or 1 we must have

$$G(1,B_2,\ldots,B_{N-1})=0 \quad \text{for all } B_j,\ j=2,\ldots,N-1 \ . \qquad (A9)$$

But we can always write

$$G=G_0(B_2,\ldots,B_{N-1})+(-1)^{B_1}G_1(B_2,\ldots,B_{N-1}) \ , \qquad (A10)$$

so that (A9) implies $G_0=G_1$ or

$$G=[1+(-1)^{B_1}]G_0(B_2,\ldots,B_{N-1}) \ . \qquad (A11)$$

There was, however, nothing special about $B_1$ so we must in fact have $G=F_{N-1}$ so that the left-hand side of (A6) becomes $F_N$, the desired result.

Finally we note two other results assumed in Sec. III which follow immediately from the present analysis. First from (3.11) if $\eta_{\mathbf{n}}^0$ is nonzero then $\mathbf{n}$ has an even number of nonzero components so that the vector $\mathbf{b}$ with all components equal to one is always present in the vector space of allowed solutions of (A5). Second it follows from (3.9) and (A4) that

$$|C_0^{\mathbf{b}}|=|C_{\mathbf{b}}^0|=2^{-r} \ , \qquad (A12)$$

if $\mathbf{b}$ is in the vector space of solutions of (A5) (and vanishes otherwise).

## APPENDIX B

Here we derive the factor $(2^{(h/2)-1})^q(2^{q-p})=2^{8-p}$ (for $p=1,2,\ldots,5$) that appears in Eq. (4.6) and in the process clarify our use of formal expressions such as (2.11b). For the purpose of constructing the internal-symmetry group with fermionic variables, we use the reducibility property of the Ramond sector by employing Weyl projections of the 32 internal Majorana-Weyl fermions. For the basis vector $U^2$, the Weyl projection is $\frac{1}{2}(1+\Gamma)$, where

$$\Gamma_1=\Gamma(p=1)=\gamma_1\gamma_2\cdots\gamma_{16} \ . \qquad (B1)$$

Here $\gamma_i$ is the zero mode of the Ramond fermion $\chi_i$ [$d_0^i$ in the notation of (2.11)]. This means the $\Gamma(p=1)=1$ states are physical. The dimension of the spinor representation of SO(16) is 128, which is the number of states for the $p=1$ case. For $p=2$,

$$\Gamma_2=\gamma_1\gamma_2\cdots\gamma_8=1 \ . \qquad (B2)$$

Equations (B1) and (B2) together imply

$$\Gamma_2'=\gamma_9\gamma_{10}\cdots\gamma_{16}=1 \ . \qquad (B3)$$

The spinor representation of SO(8)×SO(8) is $(8_s,8_s)$ which has 64 states. For $p=3$, we have

$$\Gamma_3=\gamma_1\gamma_2\gamma_3\gamma_4\gamma_9\gamma_{10}\gamma_{11}\gamma_{12}=1 \ . \qquad (B4)$$

Comparing with (B2), we see that

$$\Gamma_3'=\gamma_1\gamma_2\gamma_3\gamma_4=a=\pm 1 \ . \qquad (B5)$$

Comparing with (B2) and (B3), we find that

$$\gamma_5\gamma_6\gamma_7\gamma_8=\gamma_9\gamma_{10}\gamma_{11}\gamma_{12}=\gamma_{13}\gamma_{14}\gamma_{15}\gamma_{16}=a \ . \qquad (B6)$$

Since the dimension of the spinor representation of SO(4) is 2, the number of states is $2(2^4)=32$ where the first factor comes from the two choices of $a$. For $p=4$, we have

$$\Gamma_4=\gamma_1\gamma_2\gamma_5\gamma_6\gamma_9\gamma_{10}\gamma_{13}\gamma_{14}=1 \ . \qquad (B7)$$

We can freely choose

$$\gamma_1\gamma_2=b \ , \quad \gamma_5\gamma_6=c \ , \quad \gamma_9\gamma_{10}=d \ , \qquad (B8)$$

where $b$, $c$, $d=\pm 1$. Then (B1) to (B7) give

$$\gamma_3\gamma_4=ab \ , \quad \gamma_7\gamma_8=ac \ , \quad \gamma_{11}\gamma_{12}=ad \ , \quad \gamma_{13}\gamma_{14}=bcd \ , \quad \gamma_{15}\gamma_{16}=abcd \ . \qquad (B9)$$

Since the spinor representation of SO(2) is 1, the number of states is given by (choices of $a$, $b$, $c$, and $d$) $2^4=16$. Finally we come to the $p=5$ case, where

$$\Gamma_5=\gamma_1\gamma_3\gamma_5\gamma_7\gamma_9\gamma_{11}\gamma_{13}\gamma_{15}=1 \ . \qquad (B10)$$

In contrast with the earlier cases, $\gamma_{2i-1}\gamma_{2i}$ ($i=1,2,\ldots,8$) from the $p=4$ case anticommutes with $\Gamma_5$:

$$\{\gamma_{2i-1}\gamma_{2i},\Gamma_5\}=0 \qquad (B11)$$

while

$$[\gamma_{4l+1}\gamma_{4l+2}\gamma_{4l+3}\gamma_{4l+4},\ \Gamma_5]=0 \qquad (B12)$$

for $l=0,1,2,3$. For example, let us consider the eigenstates of $\gamma_1\gamma_2$,

$$\gamma_1\gamma_2|\pm\rangle=\pm|\pm\rangle \qquad (B13)$$

then

$$\gamma_1\gamma_2\Gamma_5|\pm\rangle = -\Gamma_5\gamma_1\gamma_2|\pm\rangle = \mp\Gamma_5|\pm\rangle ,$$

that is

$$\Gamma_5|\pm\rangle = |\mp\rangle . \tag{B14}$$

Then $(1/\sqrt{2})(|+\rangle+|-\rangle)$ is an eigenstate of $\Gamma_5$ with eigenvalue 1. In general

$$\frac{1}{\sqrt{2}}(|b,ab,c,ac,d,ad,bcd,abcd\rangle + |-b,-ab,-c,-ac,-d,-ad,-bcd,-abcd\rangle) \tag{B15}$$

is an eigenstate of $\Gamma_5$. A simple counting gives $\frac{1}{2}(2^4)=8$ states. Therefore the $(1/\sqrt{2})$ dimensional spinor representation can be interpreted as this projection of one component eigenstates of $\gamma_{2i-1}\gamma_{2i}$ to an eigenstate of $\Gamma_5$.


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