Strings in background fields: β functions and vertex operators

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We review the conditions for consistent propagation of closed strings in background fields and discuss the connection between conformal invariance and the vanishing of the renormalization-group β functions for the generalized σ model on a curved world sheet. The β functions with up to four derivative terms are found to be compatible with graviton and dilaton equations of motion provided the former are computed in a nonminimal subtraction scheme. Finally, vertex operators in background fields are discussed and it is shown that the anomalous dimension operator is given by the first variation of the β function to all orders in α'.

I. INTRODUCTION

The covariant functional-integral approach to string theory introduced by Polyakov[1] generalizes naturally to arbitrary (curved) backgrounds, and leads to the investigation of two-dimensional σ models. The relationship between the latter and the string has been used by several authors[2-5] to discuss the effective low-energy field equations generated by the string, for its massless modes. In this paper we will follow closely the work of Callan, Martinec, Perry, and Friedan,[4] to investigate some of the theoretical questions associated with this approach to string theory.

The interpretation of the two-dimensional (2D) generalized σ model with $D-\sigma$ fields as a string depends on the existence of a Virasoro algebra. The latter is generated by moments of the 2D energy-momentum (EM) tensor so that in the first place we must require the quantum σ model be well defined on a curved world sheet, thus permitting the definition of the EM tensor. However (as we shall discuss in detail in Sec. IV) we need (nonconstant) counterterms[5] proportional to $\sqrt{g}R^{(2)}$ and thus it is necessary to add the dilaton term $\Phi\sqrt{g}R^{(2)}$ to the model. In the second place as pointed out by Callan, Martinec, Perry, and Friedan,[4] the Virasoro algebra exists provided that the "β functions," associated with the couplings of the generalized σ model and occurring in formula [Eq. (2.2)] for the trace of the two-dimensional energy-momentum tensor, vanish. The detailed argument for this (which has not so far been published in the literature) is given in Sec. II. The connection between these "β functions" and the renormalization-group β functions (for the case of the σ model on a curved world sheet) has also not been clarified before[6] and we will explore this in Sec. III.

The most important observation in the work of Callan, Martinec, Perry, and Friedan[4] is the relationship between the conditions for consistent string propagation and the equations of motion for the background fields. In that work this connection was established only to lowest order in α'. In Sec. IV we investigate the validity of this connection to the next order in α' and find that it can be established[7] provided the β functions are computed in a nonminimal subtraction scheme. Of course this is again a phenomenological observation [to $O(\alpha'^2)$] and we are unable to offer a proof of this relationship to all orders in α'. Another important observation was the existence of a Bianchi-type identity for β functions which was shown to hold to the lowest order. In Sec. V we show that if the β functions are derivable from an effective action then this identity is a simple consequence of D-dimensional general covariance. Finally in Sec. VI we discuss, following the work of Callan and Gan,[8] the construction of vertex operators in a curved background and show that to all orders in α' the anomalous dimension operator acting on the fluctuations around the background configuration is given by the first variations of the β functions corresponding to these excitation modes. This means that not only the background configuration but also the field configuration defined by background plus fluctuations, gives rise to a two-dimensional conformal field theory.

II. β FUNCTIONS AND CONSISTENT STRING PROPAGATION

The action for the generalized σ model is

$$I=\frac{1}{4\pi\alpha'}\int d^2z[\tfrac{1}{2}\sqrt{g}g^{\alpha\beta}\partial_\alpha x^\mu\partial_\beta x^\nu G_{\mu\nu}(x)+\alpha'\sqrt{g}R^{(2)}\Phi(x)+\cdots]\equiv\int d^2z\,L\ . \tag{2.1}$$

In the above we have coupled the "external" metric field $G_{\mu\nu}(x)$ and the dilaton $\Phi(x)$ and the ellipsis represents all other fields which can be coupled in a 2D reparametrization-invariant and D-dimensional generally covariant manner. $x^\mu=x^\mu(z)$ $(\mu=0,\ldots,D-1)$ defines the embedding of the world sheet in the external manifold Σ, and $g_{\alpha\beta}$ is the metric on the world sheet with $R^{(2)}$ its curvature. Note that the dilaton coupling[2] explicitly breaks the classical Weyl invariance of the first term (under $g_{\alpha\beta}\to e^{2\rho}g_{\alpha\beta}$) and is therefore introduced at $O(\alpha')$. Henceforth until Sec. VI we will ignore all terms represented by the ellipsis in (2.1).

Let us choose (complex) conformal coordinates on the world sheet which is taken to have a Euclidean metric:

$$ds^2=e^{2\rho}dz\,d\bar{z},\ \ g_{z\bar{z}}=\tfrac{1}{2}e^{2\rho},\ \ g_{zz}=g_{\bar{z}\bar{z}}=0\ . \tag{2.2}$$

Covariant derivatives on totally symmetric complex tensors of rank n are given by[9]

$$\nabla^z=g^{z\bar{z}}\partial_{\bar{z}}=g^{z\bar{z}}\nabla_{\bar{z}},\ \ \nabla_z=(g_{z\bar{z}})^n\partial_z g^{z\bar{z}^{\,n}}\ . \tag{2.3}$$

Under traceless variations $\delta g_{zz},\delta g^{zz}$ $[=-(g^{z\bar{z}})^2\delta g_{\bar{z}\bar{z}}]$ we have[9]

$$\delta\nabla^z=\tfrac{1}{2}\delta g^{zz}\nabla_z+\frac{n}{2}\nabla_z(\delta g^{zz})\ , \tag{2.4a}$$

$$\delta\nabla_z=-\tfrac{1}{2}\delta g_{zz}\nabla^z+\frac{n}{2}\nabla^z(\delta g_{zz})\ , \tag{2.4b}$$

$$\delta R^{(2)}=\nabla^z\nabla^z\delta g_{zz}-\nabla_z\nabla_z\delta g^{zz}\ . \tag{2.4c}$$

The quantum effective action for the σ model (2.1) is given by

$$\Gamma[y,g]=\int[d\xi]e^{-I[x]}\,|_{\,\mathrm{1PI}}\ , \tag{2.5}$$

where it is understood that the integrand is defined by the background-field expansion with ξ the quantum field and $y(z)$ the classical field which is usually taken to obey the classical equation of motion. Thus $x^\mu=y^\mu+\xi^\mu+\tfrac{1}{2}\xi^\mu\xi^\lambda\Gamma^\mu_{\nu\lambda}(y)+\cdots$. The instruction on the right in (2.5) tells us to compute only connected one-particle-irreducible (1PI) graphs.

Let us define the renormalized metric and dilaton fields by

$$G_{\mu\nu}=\mu^{n-2}\left[G^R_{\mu\nu}+\frac{T^G_{1\mu\nu}}{n-2}+\frac{T^G_{2\mu\nu}}{(n-2)^2}+\cdots\right]\ , \tag{2.6a}$$

$$\Phi=\mu^{n-2}\left[\Phi^R+\frac{T^\Phi_1}{n-2}+\frac{T^\Phi_2}{(n-2)^2}+\cdots\right]\ , \tag{2.6b}$$

where the counterterms T_i are power series in α' and are functionals of $G^R_{\mu\nu}$ and Φ^R. μ is the renormalization scale. The renormalizability of (2.1) implies that Γ computed in perturbation theory as a functional of $G^R_{\mu\nu},\Phi^R$, is finite in the limit $n\to 2$.

The variational derivatives of Γ with respect to the metric then define a (finite) energy-momentum tensor in (renormalized) perturbation theory:

$$T_{zz}=\frac{1}{\sqrt{g}}\frac{\delta\Gamma}{\delta g^{zz}}=\int[d\xi]\frac{1}{\sqrt{g}}\frac{\delta I}{\delta g^{zz}}e^{-I}\bigg|_{\mathrm{1PI}}$$

$$\equiv\langle\theta_{zz}\rangle_y\ , \tag{2.7a}$$

$$T_{\bar{z}z}=\frac{1}{\sqrt{g}}\frac{\delta\Gamma}{\delta g^{\bar{z}z}}\equiv\langle\theta_{\bar{z}z}\rangle_y\ . \tag{2.7b}$$

$\langle\theta_{zz}\rangle_y$ ($\langle\theta_{\bar{z}z}\rangle_y$) is the expectation value of the energy-momentum tensor (trace) operator in the background $y(z)$. The diffeomorphism invariance of Γ gives

$$\nabla^zT_{zz}+\nabla_zT_{\bar{z}z}-\frac{\delta\Gamma}{\delta y^\mu(z)}\nabla_zy^\mu(z)=0\ . \tag{2.8}$$

Consider this equation at a point z where $\nabla_zy^\mu=0$ [alternatively, choose $y(z)$ to be a solution of $\delta\Gamma/\delta y=0$], and take the variational derivative $\delta/\delta g^{ww}$ at a point w. From the first term we have using (2.7a) and (2.3) the expression

$$\nabla^z\langle\theta_{zz}\theta_{ww}\rangle_y+\langle\theta_{ww}\rangle_y\nabla_z\delta^2(z,w)-\tfrac{1}{2}\nabla_w\langle\theta_{ww}\rangle_y\delta^2(z,w)\ ,$$

where

$$\delta^2(z,w)=\frac{1}{\sqrt{g}}\delta^2(z-w)$$

is the invariant δ function.

On covariance and dimensional grounds we may write

$$\theta_{z\bar{z}}=\beta^\Phi(x)\sqrt{g}R^{(2)}+\beta^G_{\mu\nu}(x)\nabla_zx^\mu\nabla_{\bar{z}}x^\nu\ . \tag{2.9}$$

[Note that if we had kept the additional terms represented by the ellipsis in (2.1) there would be corresponding terms in (2.9).] Substituting this expression we have, from the second term of (2.8),

$$\nabla_z\langle\beta^\Phi(x)[-\nabla_w{}^2\delta^2(z,w)]+\tfrac{1}{2}\beta^G_{\mu\nu}(x)\nabla_zx^\mu\nabla_zx^\nu\delta^2(z,w)\rangle_y+\nabla_z\langle(\beta^G_{\mu\nu}\nabla_zx^\mu\nabla^zx^\nu+\beta^\Phi\sqrt{g}R^{(2)}),\theta_{ww}\rangle_y\ .$$

Let us now take the flat-world-sheet limit $\rho\to 0$. Using the relation

$$\frac{1}{\pi}\partial_{\bar{z}}\frac{1}{z-w}=\delta^2(z-w)$$

we find that the necessary and sufficient condition for the validity of the relations

$$\langle\theta_{zz}\theta_{ww}\rangle_y=\frac{c}{4(z-w)^4}+\frac{\langle\theta_{ww}\rangle_y}{(z-w)^2}+\frac{\langle\nabla_w\theta_{ww}\rangle_y}{2(z-w)}$$

$$+\text{finite}\ ,$$

$$\langle\theta_{zz}\theta_{\bar{w}\bar{w}}\rangle_y=\text{finite}$$

[which are equivalent to the Virasoro algebra (the second of the above conditions, which is equivalent to $[L^+_n,L^-_n]=0$, is obtained by deriving (2.9) with respect to g_{ww}, L^+_n (L^-_n) being the Virasoro generators for right (left) movers), see Ref. 10] is that $\beta^{\mu\nu}=0$ and $\beta_\Phi=c$ number.

The above discussion clarifies two points. The first is that in order to derive the Virasoro algebra (as an operator statement) one has to consider nonconstant background $y^\mu(z)$. Second and more importantly we see the role of world-sheet curvature. In order to construct a Virasoro algebra from (2.1) one has to take variations $\delta/\delta g^{zz}$ (which takes the metric out of the conformal class) and hence the model must be well defined on a curved world sheet. As we shall see in the next section this entails the presence of (nonconstant) counterterms proportional to $\sqrt{g}R^{(2)}$ and hence necessitates the addition of the dilaton coupling $\sqrt{g}R^{(2)}\Phi$.

III. COUNTERTERMS, RENORMALIZATION-GROUP EQUATIONS, AND CONFORMAL INVARIANCE

In this section we discuss the connection between the conformal "β functions" introduced in (2.9) and the renormalization-group β functions. As we shall see this relationship is not quite as straightforward as in the case of ordinary renormalizable theories. Consider (2.6) with Γ taken as a functional of $G^R_{\mu\nu},\Phi^R$, the background field y^μ, and the two-metric g:

$$\Gamma=\Gamma[G^R_{\mu\nu},\Phi^R,g_{\alpha\beta},y^\mu] \; . \tag{3.1}$$

As discussed in Eqs. (2.6a) and (2.6b) we use dimensional regularization $2\to n=2+\epsilon$. Under a conformal variation $\delta g_{\alpha\beta}=2\delta\rho g_{\alpha\beta}$ we have

$$\delta(\sqrt{g}g^{\alpha\beta})=(n-2)\sqrt{g}g^{\alpha\beta}\delta\rho$$

and

$$\delta(\sqrt{g}R^{(2)})=(n-2)\sqrt{g}R\delta\rho-2(n-1)\sqrt{g}g^{\alpha\beta}\nabla_\alpha\nabla_\beta\delta\rho$$

so that we obtain, from (2.1),

$$\frac{\delta I}{\delta\rho(z)}=(n-2)L-\frac{2(n-1)}{4\pi}\sqrt{g}g^{\alpha\beta}\times(\partial_\alpha x^\mu\partial_\beta x^\nu\nabla_\mu\nabla_\nu\Phi+\partial_\mu\Phi D_\alpha\partial_\beta x^\mu)(z), \tag{3.2}$$

where

$$D_\alpha\partial_\beta x^\mu\equiv\nabla_\alpha\partial_\beta x^\mu+\Gamma^\mu_{\nu\sigma}\partial_\alpha x^\nu\partial_\beta x^\sigma \; . \tag{3.3}$$

From (3.2) we see that the trace anomaly has a short-distance quantum piece as well as a classical piece coming from the explicit breaking of conformal invariance, due to the $\sqrt{g}R\Phi$ term in (2.1). In fact using the equation of motion

$$\sqrt{g}g^{\alpha\beta}D_\alpha\partial_\beta x^\mu=\sqrt{g}R^{(2)}\nabla^\mu\Phi(x)$$

inside the functional integral we have, from (3.2),

$$T_\alpha{}^\alpha=\frac{\delta\Gamma}{\delta\rho}=\int[dx]e^{-I}\frac{1}{4\pi\alpha'}\{\tfrac{1}{2}\sqrt{g}g^{\alpha\beta}\partial_\alpha x^\mu\partial_\beta x^\nu[(n-2)G_{\mu\nu}-4\alpha'(n-1)\nabla_\mu\nabla_\nu\Phi]+\alpha'\sqrt{g}R^{(2)}[(n-2)\Phi-2\alpha'(n-1)(\nabla\Phi)^2]\} \; . \tag{3.4a}$$

The short-distance contribution can be reexpressed using renormalization-group methods. We observe from (2.6) that the first term on the right-hand side of (3.2) can be rewritten as $\partial I/\partial\mu(z)$ where $\delta\mu(z)$ is a local variation of the renormalization scale. Hence we have for the conformal variation of the effective action

$$T_\alpha{}^\alpha=\frac{\delta\Gamma}{\delta\rho(z)}=\mu\frac{\partial\Gamma}{\partial\mu(z)}+\frac{\partial\Gamma}{\partial\rho(z)} \; , \tag{3.4b}$$

where

$$\frac{\partial\Gamma}{\partial\rho}\equiv\int[dx]e^{-I}\frac{1}{4\pi\alpha'}\{\sqrt{g}g^{\alpha\beta}\partial_\alpha x^\mu\partial_\beta x^\nu[-2\alpha'(n-1)\nabla_\mu\nabla_\nu\Phi]+\alpha'\sqrt{g}R^{(2)}[-2\alpha'(n-1)(\nabla\Phi)^2]\} \; . \tag{3.4c}$$

Now since the effective action is independent of the renormalization scale we have

$$0=\mu\frac{d\Gamma}{d\mu}=\mu\left.\frac{\partial\Gamma}{\partial\mu(z)}\right|_{\mu(z)=\mu}+\beta^G_{\mu\nu}\frac{\delta\Gamma}{\delta G^R_{\mu\nu}}+\beta^\Phi\frac{\delta\Gamma}{\delta\Phi^R} \; , \tag{3.5}$$

where

$$\beta^G_{\mu\nu}\frac{\delta\Gamma}{\delta G^R_{\mu\nu}}=\int[dy]\mu\frac{\partial G^R_{\mu\nu}[y(z)]}{\partial\mu}\frac{\delta\Gamma}{\delta G^R_{\mu\nu}[y(z)]} \; , \tag{3.6a}$$

$$\beta^\Phi\frac{\delta\Gamma}{\delta\Phi}=\int[dy]\mu\frac{\partial\Phi^R[y(z)]}{\partial\mu}\frac{\delta\Gamma}{\delta\Phi^R[y(z)]} \; . \tag{3.6b}$$

In the above $[dy]$ is defined to be an invariant measure.

Consider now the diffeomorphism

$$\delta y^\mu=\nabla^\mu\Phi(y),\quad \delta G_{\mu\nu}=\nabla_\mu\nabla_\nu\Phi,\quad \delta\Phi=\nabla^\mu\Phi\nabla_\mu\Phi \; . \tag{3.7}$$

Since dimensional regularization preserves the D-dimensional general covariance of the theory, we have (choosing y^μ to be a solution of $\delta\Gamma/\delta y^\mu=0$)

$$\int[dy]\left[2\nabla_\mu\nabla_\nu\Phi(y)\frac{\delta\Gamma}{\delta G_{\mu\nu}(y)}+(\nabla\Phi)^2\frac{\delta\Gamma}{\delta\Phi}\right]=0 \; . \tag{3.8}$$

The left-hand side is precisely $[1/2(n-1)]\partial\Gamma/\partial\rho$ [see Eqs. (2.5), (2.1), and (3.4c)]. Hence from (3.4b), (3.8), and (3.5) we have

$$-T_\alpha{}^\alpha=\beta^G_{\mu\nu}\frac{\delta\Gamma}{\delta G_{\mu\nu}{}^R}+\beta^\Phi\frac{\delta\Gamma}{\delta\Phi^R} \; , \tag{3.9}$$

where the β functions are the renormalization-group β functions. By rewriting (3.8) in terms of $G^R_{\mu\nu},\Phi^R$ we see that one may add a term proportional to $2\nabla_\mu\nabla_\nu\Phi$ $[(\nabla\Phi)^2]$ to $\beta^G_{\mu\nu}$ $[\beta_\Phi]$. One may of course consider more general diffeomorphisms $\delta y^\mu=v^\mu\delta G^R_{\mu\nu}=2\nabla_{(\mu}v_{\nu)}$, $\delta\phi^R=v^\mu\nabla_\mu\Phi^R$ with v^μ and arbitrary vector field on Σ. This would mean that the β functions admit the transformations $\beta^G_{\mu\nu}\to\beta^G_{\mu\nu}+2\nabla_{(\mu}v_{\nu)}$ and $\beta^\Phi\to\beta^\Phi+v^\mu\nabla_\mu\Phi$. We will make use of this in the next section.

IV. β FUNCTIONS AND STRING EFFECTIVE ACTION TO $O(\alpha'^2)$

To leading order in α' the following results have been obtained for the β functions:

$$(2\alpha')^{-1}\beta^G_{\mu\nu} = -R_{\mu\nu} - 2\nabla_\mu\nabla_\nu\Phi \; , \tag{4.1}$$

$$\alpha'^{-1}\beta^\Phi = \frac{R}{2} + 2\nabla^2\Phi - 2(\nabla\Phi)^2 \; . \tag{4.2}$$

In the above we have put $D=26$ and have included the contribution of the reparametrization ghosts to cancel the free field theory conformal anomaly. Also in the above and in what follows we will omit the superscript R on renormalized fields.

Callan, Martinec, Perry, and Friedan[4] showed that linear combinations of these functions are variational derivatives of an action Γ which is in fact the low-energy effective action for the closed string. The vanishing of the β functions was thus shown (to leading order) to be equivalent to a set of equations of motion for the background fields $G_{\mu\nu}, \Phi$. Let us now check this to the next order in α'.

We have to work out the higher-order contributions to $\beta^G_{\mu\nu}$ and β^Φ for the bosonic σ model (2.1). The α' expansion is in effect a low-energy expansion and we will be interested in the four derivative terms in the field $G_{\mu\nu}$ and Φ in the effective action for these background fields. We thus need to find the two-loop contribution to $\beta^G_{\mu\nu}$ and the three-loop contribution to β^Φ. For this calculation we need the normal coordinate expansion of the action I and the one- and two-loop counterterms up to terms which are sixth order in the quantum fields. Fortunately for the three-loop calculation of β^Φ, we need only terms which are independent of $y^\mu(z)$.

The quantum field ξ^μ is the tangent vector to Σ at y^μ (chosen to be classical solution) in the direction of the geodesic joining y^μ to x^μ. It is convenient to define the dimensionless fields by the replacement $\xi^\mu \to \sqrt{4\pi\alpha'}\xi^\mu$. We then have the following expansion:

$$\begin{aligned}
I = {} & \frac{1}{4\pi\alpha'}\int d^nz\left(\tfrac{1}{2}\sqrt{g}g^{\alpha\beta}\partial_\alpha y^\mu\partial_\beta y^\nu G_{\mu\nu} + \alpha'\sqrt{g}R^{(2)}\Phi\right) + \tfrac{1}{2}\int d^nz\sqrt{g}g^{\alpha\beta}D_\alpha\xi^\mu D_\beta\xi^\nu G_{\mu\nu} \\
& + \int d^nz\left(\tfrac{1}{2}\sqrt{g}g^{\alpha\beta}\partial_\alpha y^\mu\partial_\beta y^\nu\xi^\lambda\xi^\sigma R_{\mu\lambda\sigma\nu} + \alpha'\sqrt{g}R^{(2)}\tfrac{1}{2}\nabla_\mu\nabla_\nu\Phi\xi^\mu\xi^\nu\right) \\
& + \sqrt{4\pi\alpha'}\int d^nz\left[\sqrt{g}g^{\alpha\beta}\left(\tfrac{1}{6}\partial_\alpha y^\mu\partial_\beta y^\nu\xi^\lambda\xi^\sigma\xi^\tau R_{\mu\lambda\sigma\nu;\tau} + \tfrac{2}{3}\partial_\alpha y^\mu D_\beta\xi^\nu\xi^\lambda\xi^\sigma R_{\mu\lambda\sigma\nu}\right) + \alpha'\sqrt{g}R^{(2)}\xi^\mu\xi^\nu\xi^\lambda\tfrac{1}{6}\nabla_\mu\nabla_\nu\nabla_\lambda\Phi\right] \\
& + 4\pi\alpha'\int d^nz\Big\{\sqrt{g}g^{\alpha\beta}\Big[\tfrac{1}{24}\partial_\alpha y^\mu\partial_\beta y^\nu\xi^\lambda\xi^\sigma\xi^\tau\xi^\delta\left(R_{\mu\lambda\sigma\nu;\tau\delta} + R^{\zeta}_{\ \lambda\sigma\mu}R_{\zeta\tau\delta\nu}\right) \\
& \qquad\qquad + \tfrac{1}{4}\partial_\alpha y^\mu D_\beta\xi^\nu\xi^\lambda\xi^\sigma\xi^\tau R_{\mu\lambda\sigma\nu;\tau} + \tfrac{1}{6}D_\alpha\xi^\mu D_\beta\xi^\nu\xi^\lambda\xi^\sigma R_{\mu\lambda\sigma\nu}\Big] \\
& \qquad + \alpha'\sqrt{g}R^{(2)}\xi^\mu\xi^\nu\xi^\lambda\xi^\sigma\tfrac{1}{24}\nabla_\mu\nabla_\nu\nabla_\lambda\nabla_\sigma\Phi\Big\} \\
& + (4\pi\alpha')^2\int d^nz\left[\tfrac{1}{40}\sqrt{g}g^{\alpha\beta}D_\alpha\xi^\mu D_\beta\xi^\nu\xi^\lambda\xi^\sigma\xi^\tau\xi^\delta\left(R_{\mu\lambda\sigma\nu;\tau\delta} + \tfrac{8}{9}R^{\zeta}_{\ \lambda\sigma\mu}R_{\zeta\tau\delta\nu}\right)\right] \; .
\end{aligned} \tag{4.3}$$

In the above all background fields and their derivatives are evaluated at the point y on S. We have only kept terms which are relevant for the calculation of two-loop counterterms in $G_{\mu\nu}$ and three-loop counterterms in Φ. In computing β_Φ one has to expand

$$\sqrt{g}g^{\alpha\beta} = \delta_{\alpha\beta} - h_{\alpha\beta} - \tfrac{1}{2}h_{\gamma\gamma}\delta_{\alpha\beta} + \cdots \equiv \delta_{\alpha\beta} + \bar{h}_{\alpha\beta} \; ,$$

with $h_{\alpha\beta}$ treated as a weak external field.

The counterterm action to $O(\alpha')$ computed from (4.3) is[11]

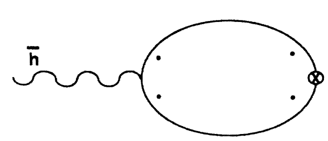
$$I^1_{\rm ct} = -\frac{1}{n-2}\frac{1}{4\pi}\int d^nz\sqrt{g}g^{\alpha\beta}\partial_\alpha x^\mu\partial_\beta x^\nu R_{\mu\nu}(x) \; . \tag{4.4}$$

In order to compute the two- and three-loop graphs we need to expand this up to quartic terms in ξ. We will not write all the terms but it should be remarked that this expansion contains a vertex

$$-\frac{1}{n-2}\frac{1}{4\pi}\int\sqrt{g}g^{\alpha\beta}D_\alpha\xi^i D_\beta\xi^j R_{ij}(y(z))d^nz \tag{4.5}$$

which will give rise to the diagram in Fig. 1.

This diagram is of crucial importance. It is precisely this which gives the leading-order counterterm to the dilaton coupling. The contribution to T^Φ_1 coming from (4.3) is zero as observed by Fradkin and Tseytlin.[3] Including also the dilaton contribution we thus have, after making use of the ambiguity in the counterterm $T_{\mu\nu}$ [i.e., replacing $R_{\mu\nu} \to R_{\mu\nu} + 2\nabla_\mu\nabla_\nu\Phi$ in (4.4) and adding the corresponding term to T^Φ_1—see Sec. III],



⊗ counterterm vertex, • derivative

FIG. 1. Diagram contributing to T^Φ_1.

$$T_1^{\Phi(2)} = \frac{\alpha'}{2} R + 2\alpha' \nabla^2 \Phi + 2(\nabla\Phi)^2 . \quad (4.6)$$

The two-loop counterterm $T_{1\mu\nu}^{G(2)}$ has also been calculated by Friedan[11] and the result is

$$T_{1\mu\nu}^{G(2)} = -\frac{\alpha'^2}{2} R_{\mu\lambda\sigma\tau} R_{\nu}^{\ \lambda\sigma\tau} . \quad (4.7)$$

In order to compute the quartic derivative terms in β^{Φ} a three-loop calculation is necessary and thus apart from interaction terms coming from the expansion of these there are also terms coming from the counterterms $T_{1\mu\nu}^{G(2)}$, $T_{2\mu\nu}^{G(2)}$ (i.e., the two-loop double-pole term) $T_1^{\Phi(2)}$.

It is easy to see that no Ricci scalar squared terms can arise in β_{Φ}. In fact no such terms can contribute to any of the counterterms. This follows from the structure of the normal-coordinate expansions. Counterterms involving ${R_{\mu\nu}}^2$ are indeed present but leave β^{Φ} unaffected since they are not simple pole terms. (This is a scheme-dependent statement valid in the same scheme as that used to compute $\beta_{\mu\nu}$.) This follows from an examination of all diagrams generated by (4.3) and the one- and two-loop counterterms and observing that there are no combinations of double poles which would give simple poles. Thus the only (pure gravity) contribution to β^{Φ} in this order is a ${R_{\mu\nu\lambda\sigma}}^2$ term which comes from a diagram similar to Fig. 1 together with contribution (the direct calculation of these terms may be difficult because of problems associated with infrared singularities[12]) from vertices in (4.3). Thus we may write

$$T_1^{\Phi(2)} = \frac{(\alpha')^2}{2} \frac{A}{2} R_{\mu\nu\lambda\sigma} R^{\mu\nu\lambda\sigma} + O(\nabla\Phi) . \quad (4.8)$$

Now let us try to find an action which yields the β functions obtained from these counterterms,

$$-(2\alpha')^{-1}\beta_{\mu\nu}^{G} = R_{\mu\nu} + 2\nabla_\mu\nabla_\nu\Phi + \frac{\alpha'}{2} R_{\mu\lambda\sigma\tau} R_{\nu}^{\ \lambda\sigma\tau} + O(\nabla^2\Phi R, \nabla^4\Phi) , \quad (4.9a)$$

$$\alpha'^{-1}\beta^{\Phi} = \frac{R}{2} + 2\nabla^2\Phi - 2(\nabla\Phi)^2 + \alpha' A R_{\mu\nu\lambda\sigma} R^{\mu\nu\lambda\sigma} + O(\nabla^2 R, \nabla^2\Phi R, \nabla^4\Phi) , \quad (4.9b)$$

as equations of motion. [Note that

$$\beta = \left. \left[1 + \Lambda\frac{\partial}{\partial\Lambda} + \Delta\frac{\partial}{\partial\Delta} \right] T(\Lambda^{-1} G_{\mu\nu}^{R}, \Delta^{-1}\Phi^{R}) \right|_{\Lambda,\Delta=1}$$

except that we have made use of the ambiguity discussed in Sec. III to add $-4\alpha'\nabla_\mu\nabla_\nu\Phi$ to $\beta_{\mu\nu}$ and $-2\alpha'(\nabla\Phi)^2$ to β^{Φ}.] The (tree-level) string effective action must take the form

$$\begin{aligned} S[\Phi, G_{\mu\nu}] &= \int d^D x\, e^{-2\Phi}\sqrt{G}\, L[\Phi, G] \\ &= (2\alpha')^{-1} \int d^D x\, e^{-2\Phi}\sqrt{G}\, [R + 4(\nabla\Phi)^2 + a R_{\mu\nu\lambda\sigma} R^{\mu\nu\lambda\sigma} + b R_{\mu\nu} R^{\mu\nu} + c R^2 + d\nabla_\mu\nabla_\nu\Phi R^{\mu\nu} \\ &\qquad + e\nabla^2\Phi R + f\nabla^2 R + O(\nabla\Phi)] . \end{aligned} \quad (4.10)$$

The exponential coupling of the dilaton arises[2] from the term $(1/4\pi)\int\sqrt{g}R^{(2)}\Phi$ in the σ model since $\int\sqrt{g}R^{(2)} = 4\pi\chi$ and $\chi = 2$ for the sphere. The first two terms are those found in Refs. 2 and 3. The quartic derivative Lagrangian is the most general possible one when account is taken of the Bianchi identity $\nabla^\mu R_{\mu\nu} = \frac{1}{2}\nabla_\nu R$ and the possibility of relating certain terms by partial integration. We have then the following variational derivatives:

$$\frac{1}{\sqrt{G}}\frac{\delta S}{\delta\Phi} = -2e^{-2\Phi}\{L[\Phi,G] + 4[\nabla^2\Phi - 2(\nabla\Phi)^2]\} + \nabla_\mu\nabla_\nu(d\bar{R}^{\mu\nu} + e\bar{R}G^{\mu\nu}) , \quad (4.11)$$

$$\begin{aligned} \frac{1}{\sqrt{G}}\frac{\delta S}{\delta G^{\lambda\sigma}} &= e^{-2\Phi}[R_{\lambda\sigma} + 4G_{\lambda\sigma}(\nabla\Phi)^2 + 2(\nabla_\lambda\nabla_\sigma\Phi - \nabla^2\Phi G_{\lambda\sigma}) - 2aR^{\mu}_{\ \nu\lambda\tau}R^{\nu\tau}_{\ \ \mu\sigma} + 2bR_{\lambda\tau}R^{\tau}_{\ \sigma} + 2cR_{\lambda\sigma}R \\ &\qquad + 2d\nabla_{(\lambda}\nabla^\mu\Phi R_{\sigma)\mu} + e\nabla_\lambda\nabla_\sigma\Phi R + f(\nabla_\lambda\nabla_\sigma R + \nabla^2 R_{\lambda\sigma})] \\ &\quad - 4a(\nabla_\mu\nabla_{(\lambda}\bar{R}_{\sigma)}^{\ \mu} - \Box\bar{R}_{\lambda\sigma}) + 2b(2\nabla^\mu\nabla_{(\lambda}\bar{R}_{\sigma)\mu} - \Box\bar{R}_{\sigma\lambda} - G_{\lambda\sigma}\nabla^\mu\nabla^\nu R_{\mu\nu}) \\ &\quad + c(2\nabla_\sigma\nabla_\lambda - 2G_{\lambda\sigma}\Box)\bar{R} - \tfrac{1}{2}G_{\lambda\sigma}e^{-2\Phi}L[\Phi,G] . \end{aligned} \quad (4.12)$$

In the above we have again omitted pure dilaton quartic derivative terms and put $\bar{R}_{\mu\nu} = e^{-2\Phi}R_{\mu\nu}$. Now in order that the leading-order calculation of $\beta_{\mu\nu}$, β_Φ agrees with the equations of motion from S we must have

$$-\sqrt{G}e^{-2\Phi}\beta_{\mu\nu} = \frac{\delta S}{\delta G^{\mu\nu}} - \frac{1}{4}G_{\mu\nu}\frac{\delta S}{\delta\Phi} , \quad (4.13)$$

$$\sqrt{G}e^{-2\Phi}\beta_\Phi = -\frac{1}{4}\frac{\delta S}{\delta\Phi} . \quad (4.14)$$

The question is whether these equations remain valid up to quartic derivative terms. Using the expression for $\beta_{\mu\nu}$ we find immediately from (4.13) that $b = c = 0$ and $\alpha' A = a = -\alpha'/4$. This is consistent with (4.14) and the expression for

β_Φ because of the absence of ${R_{\mu\nu}}^2$ and R^2 terms in the latter. The terms involving covariant derivatives of the Ricci tensor in (4.12) would seem to disagree with the β functions but we have to allow for the possibility of adding finite $O(\alpha')$ counterterms in computing the β functions. Thus suppose we make the change $G_{\mu\nu}\to G_{\mu\nu}+\lambda\alpha' R_{\mu\nu}$ and also make use of the ambiguity under diffeomorphism mentioned at the end of Sec. III. Then we have to change the β functions in the following way:

$$-(2\alpha')^{-1}\beta^G_{\mu\nu}\equiv-(2\alpha')^{-1}\mu\frac{\partial}{\partial\mu}G_{\mu\nu}\to R_{\mu\nu}+\frac{\alpha'}{2}\lambda(\nabla_\lambda\nabla_\nu R^\lambda_\mu+\nabla_\lambda\nabla_\mu R^\lambda_\nu-\Box R_{\mu\nu}-\nabla_\mu\nabla_\nu R)+\frac{\alpha'}{2}R_{\mu\lambda\sigma\tau}R_\nu^{\lambda\sigma\tau}+\nabla_\mu v_\nu+\nabla_\nu v_\mu \ ,$$

$$\alpha'^{-1}\beta^\Phi=\alpha'^{-1}\mu\frac{\partial}{\partial\mu}\Phi\to\tfrac{1}{2}R-\frac{\alpha'}{2}R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma}+v^\mu\nabla_\mu\Phi \ .$$

Now choosing $v_\mu=(\alpha'/2)\nabla_\mu R$ we find that (4.13) and (4.14) are satisfied with $\lambda=\frac{1}{2}, v_\mu=\alpha'\nabla_\mu R$.

The effective action that we find [Eq. (4.10) with $a=-\alpha'/4$, $b=0$, $c=0$, $f=\alpha'/4$, $d=3\alpha'/2$] is not of the ghost-free form[13] ${R_{\mu\nu\lambda\sigma}}^2-4{R_{\mu\nu}}^2+R^2$. Indeed this action has a ghost at $p^2=\alpha'^{-1}$ but this is clearly an artifact of our low-energy expansion (which is expected to be valid only for $\alpha' p^2\ll 1$).

Recently Callan, Klebanov, and Perry[7] have also computed this effective action from a somewhat different method. Instead of computing β_Φ directly they use the expression for $\beta_{\mu\nu}$ and the Bianchi identities and the lower-order equation of motion to show that $\nabla^\nu\beta_{\mu\nu}=\nabla_\mu F$ for some scalar functional F. They then identify β_Φ with F. The resulting effective action agrees with ours up to terms which can be obtained by a field redefinition. This also agrees (up to field redefinition) with the direct S-matrix calculation of Neopomechie.[14]

V. THE "BIANCHI IDENTITY" FOR β FUNCTIONS

To lowest order in α' Callan, Martinec, Perry, and Friedan[4] found that the two β functions satisfy an identity of the form

$$\nabla_\mu\beta^\Phi=\nabla^\nu\beta_{\mu\nu}-2\nabla^\mu\Phi\beta_{\mu\nu} \ . \tag{5.1}$$

This identity implies that if $\beta_{\mu\nu}=0$, β^ϕ is a c number as would be required by the Virasoro algebra (see Sec. II). Furthermore it means that if the graviton field equation is satisfied the dilaton field equation is automatically satisfied. It is then important to establish whether such an identity is valid to all orders. In the following we shall show that if the β functions are derivable as gradients of an effective action [as in (4.13) and (4.14)] then the above identity is a simple consequence of the diffeomorphism invariance of the action.

As argued in Refs. 3 and 4 (see also Sec. IV) the string effective action must have the general structure

$$S=\int d^Dy\sqrt{G}e^{-2\Phi}L[G_{\mu\nu},\nabla_\lambda\Phi] \ .$$

This follows from the general form of the string loop expansion where $e^{2\Phi}$ serves as the coupling constant, and from the fact that in the normal-coordinate expansion Φ occurs only through its covariant derivatives. Now we have

$$\frac{1}{\sqrt{G}}\frac{\delta S}{\delta G^{\mu\nu}}=e^{-2\Phi}\left[\frac{\delta L}{\delta G^{\mu\nu}}-\tfrac{1}{2}G_{\mu\nu}L\right] , \tag{5.2a}$$

$$\frac{1}{\sqrt{G}}\frac{\delta S}{\delta\Phi}=-2e^{-2\Phi}L-\sum_{n=1}\nabla^n_\lambda\left[e^{-2\Phi}\frac{\partial L}{\partial\nabla^n_\lambda\Phi}\right] . \tag{5.2b}$$

From the normal-coordinate expansion we can see that $\beta_{\mu\nu}$ will not have any terms proportional to $G_{\mu\nu}$ so that the vanishing of the β functions is equivalent to the equation of motion only if $L\sim\beta_\Phi$ and $\delta L/\delta G^{\mu\nu}\sim\beta_{\mu\nu}$ with $\sum\nabla^n[e^{-2\Phi}(\partial\beta^\Phi/\partial\nabla^n\Phi)]=0$. (The above argument has also been given in Ref. 7.) Thus we may expect to have to all orders equations of the form (4.13) and (4.14), with the β functions being suitably defined by some nonminimal subtraction prescription as discussed in Sec. IV. Now under a diffeomorphism $\delta y^\mu=V^\mu$ (with V^μ an arbitrary vector field). We have $\delta G_{\mu\nu}=2\nabla_{(\mu}V_{\nu)}$ and $\delta\Phi=V^\mu\nabla_\mu\Phi$ and the invariance of the action

$$0=\delta S=\int d^Dy\left[2\nabla_\mu V_\nu\frac{\delta S}{\delta G_{\mu\nu}}+V^\mu\nabla_\mu\Phi\frac{\delta S}{\delta\Phi}\right] \tag{5.3}$$

gives us from (4.17) and (4.18) and integration by parts the identity (5.1). It should also be noted that irrespective of the connection with β functions the equation of motion for the dilaton is automatically satisfied when the graviton (and other fields in the effective action not considered explicitly above) obey their equations of motion.

VI. VERTEX OPERATORS IN BACKGROUND FIELDS

In flat space the S-matrix elements are given by correlation functions of dimension two vertex operators. The spectrum of string fluctuations in flat space is determined by these operators. However a realistic string theory must be formulated in curved space and hence it is necessary to discuss the construction of vertex operators in a general background which allows the consistent propagation of the string. Such vertex operators will determine the mass spectrum of the compactified theory. The conditions that these vertex operators have to satisfy (in a background which has vanishing β functions) have been recently discussed by Callan and Gan.[8] They explicitly calculate in perturbation theory the anomalous dimension operator for the massless fields, and find that this operator is essentially the variational derivative of the corresponding β functions. Furthermore they conjecture that this relationship is valid to all orders in σ-model perturbation theory and for the massive fields as well as for the massless fields. In the following we give a simple argument proving the validity of these conjectures.

Let us consider the fully generalized σ model, where in addition to the couplings explicitly written down in (2.1), we allow all possible terms restricted only by world-sheet reparametrization invariance and D-dimensional general covariance. Such terms would have the general structure

$$\int d^2z\sqrt{g(z)}V[x(z)]=\int d^2z\sqrt{g(z)}H_{\mu_1\cdots\mu_n}[x(z)]V^{\mu_1\cdots\mu_n}[\partial x(z),g(z)]\,, \tag{6.1}$$

where $V^{\mu_1\cdots\mu_n}$ is a function of the derivatives of x and the 2-metric g. For the massless fields $G_{\mu\nu}$, $B_{\mu\nu}$, and Φ, the $V^{\mu_1\cdots\mu_n}$ are, respectively, $g^{\alpha\beta}\partial_\alpha x^\mu\partial_\beta x^\nu$, $\epsilon^{\alpha\beta}\partial_\alpha x^\mu\partial_\beta x^\nu$, and $\sqrt{g}R^{(2)}$. For the tachyon, $V^{\mu_1\cdots\mu_n}$ is just unity. For the first positive (mass)^2 level we have the following possibilities[8] for V:

$$H_{\mu\nu\lambda\sigma}\partial_\alpha x^\mu\partial_\beta x^\nu\partial_\gamma x^\lambda\partial_\delta x^\sigma g^{\alpha\beta}g^{\gamma\delta},\quad H_{\mu\nu\lambda}\nabla_\alpha\partial_\beta x^\mu\partial_\gamma x^\nu\partial_\delta x^\lambda g^{\alpha\beta}g^{\gamma\delta}\,,$$
$$H'_{\mu\nu\lambda}\nabla_\alpha\partial_\beta x^\mu\partial_\gamma x^\nu\partial_\delta x^\lambda g^{\alpha\gamma}g^{\beta\delta},\quad H_{\mu\nu}\nabla_\alpha\partial_\beta x^\mu\nabla_\gamma\partial_\delta x^\nu g^{\alpha\gamma}g^{\beta\delta}\,,$$
$$H'_{\mu\nu}\partial_\alpha\partial_\beta x^\mu\partial_\gamma\partial_\delta x^\nu g^{\alpha\beta}g^{\gamma\delta}\,,$$
$$H''_{\mu\nu}\partial_\alpha x^\mu\partial_\beta x^\nu g^{\alpha\beta}R^{(2)},\quad H_\mu\nabla_\alpha\partial_\beta x^\mu g^{\alpha\beta}R^{(2)},H(R^{(2)})^2\,.$$

As stressed by Callan and Gan[8] (and also by Tseytlin[15]) one should in general expect mixing with lower dimension operators having $R^{(2)}$ factors. In fact in terms of σ-model couplings, arguments similar to the ones given for the dilaton couplings ($\Phi\sqrt{g}R$) given in Sec. IV can be generalized to justify such terms on grounds of renormalizability. Of course such (naive) dimension four (and higher) operators are not superficially renormalizable but the point is that since we add all possible terms allowed by the symmetries of the theory (i.e., including $R^{(2)}$ terms in a curved world sheet) the model is a renormalizable one.

The arguments of Secs. II and III would then apply and a β function may be associated with each vertex V. Thus the trace formula (2.9) and (3.9) is generalized to

$$T_\alpha{}^\alpha=\sum_i\beta^{(i)}_{\mu_1\cdots\mu_{i_n}}\langle V^{(i)\mu_1\cdots\mu_{i_n}}\rangle\,. \tag{6.2}$$

The multiparticle emission vertices corresponding to S-matrix elements in flat space are given by the correlation functions

$$S(i_1,\ldots,i_N)=\int\prod_{r=1}^N d^2z_{i_r}(g_{i_r})^{1/2}\times\langle\delta V(i_1)\cdots\delta V(i_N)\rangle\,, \tag{6.3}$$

where

$$\delta V^i\equiv\delta H^{(i)}_{\mu_1\cdots\mu_n}V^{\mu_1\cdots\mu_n}_{(i)} \tag{6.4}$$

with δH being a fluctuation of the field H^i. The correlation functions (6.3) can then be written as the variational derivatives of the fully generalized σ model:

$$S(i_1,\ldots,i_N)=\prod_i\int_{y_i}\delta H^{(i)}\frac{\delta}{\delta H^{(i)}}\Gamma[\{H^i\},g]\bigg|_{H=H_0}\,, \tag{6.5}$$

where Γ is the 2D effective action for the model and the variational derivatives are to be evaluated in a background which is conformally invariant, i.e., $\beta^i(H_0)=0$. The conformal invariance of the functions $S(i_1,\ldots,i_N)$ then gives us

$$\begin{aligned}0=\frac{\delta S}{\delta\rho}&=\int_{y_i}\delta H^{(i)}\frac{\delta}{\delta H^{(i)}}\frac{\delta\Gamma}{\delta\rho}\bigg|_{H=H_0}\\&=\int_{y_i}\delta H^{(i)}\frac{\delta}{\delta H^i}T_\alpha{}^\alpha\bigg|_{H=H_0}\\&=\sum_i\int\delta\beta^{(i)}_{\mu_1\cdots\mu_N}\langle V^{\mu_1\cdots\mu_{i_N}}_{(i)}\rangle\bigg|_{H=H_0}\end{aligned} \tag{6.6}$$

using (6.2). Thus the condition for the conformal invariance of the particle-emission vertices is that the variations of the β functions around a conformally invariant background should also be zero. These give a set of equations of motion for the fluctuations which reduce in the trivially conformally invariant flat-space background to the usual mass-shell and gauge conditions of conventional string theory, which have been discussed by Weinberg from the functional-integral point of view.[16] In fact it is easily checked from the graviton dilaton sector equations $\delta\beta^G=0,\delta\beta^\Phi=0$ that one gets for this case the usual graviton vertex and the dilaton vertex discussed in Ref. 17.

A question may arise at this point to the consistency of a nontrivial background in which only the massless modes condense. Since as we have just seen the background fields must satisfy an infinite set of coupled equations $\beta^{(i)}_{\mu_1\cdots\mu_N}=0$. However it is easily seen that all these β functions for operators whose (naive) dimension is greater than two are trivially zero when all massive fields are set equal to zero. This follows from a power-counting argument which shows that the diagrams involving *only* vertices with massless fields ($G_{\mu\nu}$, Φ, and $B_{\mu\nu}$) do not contribute to the renormalization of operators whose dimension is greater than two. In other words, the β functions for the couplings of the massive fields are at least linear in such fields and hence vanish when the latter are set equal to zero.

The conditions $\delta\beta^{(i)}=0$ are equivalent to a weak form of the standard Virasoro conditions. The easiest way to see this is to use the fact that in Becchi-Rouet-Stora-Tyutin (BRST) quantization the condition for conformal invariance is equivalent to the nilpotency ($Q_{\rm BRST}{}^2=0$) of the BRST charge.[18]

Let us recall that the BRST charge is given by [using the world-sheet coordinates defined in (2.3)]

$$Q_{\rm BRST}=\oint dz\, c^z(z)[\theta^x_{zz}(z)+\tfrac{1}{2}\theta^g_{zz}(z)]\ .$$

In the above c^z is the reparametrization ghost, θ^x_{zz} is the energy-momentum tensor for x^μ [see (2.7a)] and θ^g_{zz} is the ghost energy-momentum tensor. All that we need to use about the latter is that the ghost system is free so that under a variation of the fields H^i we have

$$\delta Q_{\rm BRST}=\oint dz\, c^z(z)\delta\theta^x_{zz}\ .$$

Now for a conformally invariant background $H=H_0$ $[\beta^i(\{H_0\})=0]$ we have $(Q^0_{\rm BRST})^2=0$ where $Q^0_{\rm BRST}=Q_{\rm BRST}(H_0)$. The condition for conformally invariant vertex functions $\delta\beta^i=0$ implies that the σ model with $H=H_0+\delta H$ is also conformally invariant so that

$$(Q_0+\delta Q)^2=0$$

which implies, to leading order in the fluctuations,

$$\{Q_0,\delta Q\}=0\ .$$

As discussed, for instance, in the second paper of Ref. 10 such a condition is equivalent to a weak form of the Virasoro conditions.

VII. CONCLUSION

There are two major issues which need further discussion. One is the validity of the connection between β functions and string effective actions (4.13) and (4.14) to all orders in α'. We have shown that even to second order in α' it is valid only when finite $O(\alpha')$ counterterms are added to the original action. Presumably this persists to all orders so that the D-dimensional metric $G^{\mu\nu}$ and other fields which enter in the string effective action are related to the couplings of the σ model only after the addition of a series of finite counterterms. There is also the puzzling question of the relationship of this method of calculating the effective action to the direct method proposed by Fradkin and Tseytlin.[3] The latter take the σ-model effective action Γ [Eq. (2.6)] and integrate it over the conformal mode ρ. However there seems to be no clear prescription of how to do this integral and indeed the ρ dependence of Γ (unlike in the case of a free theory) is extremely complicated. In a recent paper[19] Tseytlin has proposed fixing the conformal gauge and dropping the ρ integral. However, it is unclear to us whether this leads to a consistent string picture. On the other hand, if as Tseytlin says[20] (quoting a result due to Zamolodchikov) the relation between β functions and effective actions is more complicated than (4.13), (4.14), i.e., is of the form

$$\sum_j \kappa_{ij}(H)\beta_j[H]=\delta\Gamma/\delta H^i\ , \qquad (7.1)$$

where $[H]$ is the set of background fields and κ_{ij} is a nondegenerate matrix which also has to be computed order by order in α', then the simple connection $S=\int e^{-2\Phi}\beta_\Phi$ is lost. Of course it could be the case that κ_{ij} is simply the differential operator that effects the D-dimensional field redefinition (addition of finite counterterms) discussed in Sec. IV to $O(\alpha'^2)$.

The other major question is the effect of string loops. Naively it might be assumed that β functions being short distance effects are unaffected by change of world-sheet topology. However in a recent very interesting work it has been shown by Fischler and Susskind[21] that small handles do affect the β function. This may resolve the question of how the equations of motion from an effective action, which must certainly contain string loop terms, remain consistent with the condition for conformal invariance.

Note added. After completing this work I received a paper prior to publication from Tseytlin[22] in which issues similar to those considered in Secs. III and IV are discussed.

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