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Action principle and quantization of gauge fields

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It is shown that the action principle solves the quantization problem of gauge fields without the recourse to path integrals, without the use of canonical commutation rules, and without the need of going to the complicated structure of the Hamiltonian. We obtain the expression for the vacuum-to-vacuum transition amplitude directly from the action principle in the celebrated Coulomb gauge, and then finally we write the amplitude in terms of δ functionals. To study gauge transformations, we use a variation of the Faddeev-Popov technique which is quite suitable to deal with the nonlinear transformation character involved with non-Abelian gauge fields.

I. INTRODUCTION

Ever since Faddeev and Popov[1] introduced their quantization scheme of gauge fields via Feynman path integrals[2] it has remained a challenging problem to me on how the quantization problem may be solved directly from the very elegant action principle.[3] Why the ingenuous breakthrough of the quantization problem has occurred via the path-integral approach rather than through the action principle is difficult to understand since it is far simpler to apply the latter, as taking various (functional) differentiations, rather than to deal with complicated (continual) integrals. Of course path integrals are extremely useful in many respects and may be formally derived directly from the action principle (cf. Refs. 4–6). The first time I learned about the action (dynamical) principle, I was convinced that it should be able to handle any complicated field-theory models. The papers dealing with the, by now standard, Faddeev-Popov quantization approach are too numerous to be given here and fortunately many of the basic papers have been collected.[7]

Below we give a direct solution to the quantization problem from the action principle—*no appeal is made to path integrals, no commutation rules are used, and also we do not go into the complicated structure*[8] *of the Hamiltonian.* We work in the celebrated Coulomb gauge, where the physical components are clear at the outset, to derive the expression for the vacuum-to-vacuum transition amplitude. Then finally we write the amplitude from the latter expression in terms of so-called δ functionals[9] which is in the spirit of path integrals. The resulting expression makes it explicitly evident the gauge constraint and the gauge-invariant components in the theory and makes it quite suitable to study gauge transformations. To study gauge transformations we use a variation of the very useful Faddeev-Popov trick which is quite suitable to deal with nonlinear transformation properties of non-Abelian gauge fields. Sections II—IV deal with QED, while Secs. V—VII deal with Yang-Mills fields including matter. We urge the reader not to skip over the QED part since the corresponding analysis given is very general to the extent that many of the results are carried over to the Yang-Mills case with minor modifications and makes the paper easier to read and is certainly more instructive. Some aspects of renormalization, including Becchi-Rouet-Stora (BRS) transformations,[10] and the quantization problem of the gravitational field will be dealt with in a subsequent report.

II. ACTION PRINCIPLE AND QED

We choose the following Lagrangian density for QED:

$$\mathscr{L}=-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}+\frac{1}{2}\left[\left(\frac{\partial_\mu}{i}\bar{\psi}\right)\gamma^\mu\psi-\bar{\psi}\gamma^\mu\frac{\partial_\mu}{i}\psi\right]-m_0\bar{\psi}\psi$$
$$+e_0\bar{\psi}\gamma_\mu\psi A^\mu+\bar{\eta}\psi+\bar{\psi}\eta+A_\mu J^\mu\ , \tag{2.1}$$

where

$$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu\ , \tag{2.2}$$

and $\eta,\bar{\eta},J^\mu$ are external (c-number) sources with $\eta,\bar{\eta}$ anticommuting. We work initially in the Coulomb gauge by imposing the constraint

$$\partial_k A^k=0,\ \ k=1,2,3\ . \tag{2.3}$$

Equation (2.3) allows us to solve for A^3 in terms of A^1,A^2 (see also Ref. 8):

$$A^3=-\partial_3{}^{-1}(\partial_i A^i),\ \ i=1,2\ . \tag{2.4}$$

With A^0, A^1, A^2 (not A^3), and ψ as dynamical variables, the equations of motion are readily obtained to be

$$\left[\gamma^\mu\left(\frac{\partial_\mu}{i}-e_0A_\mu\right)+m_0\right]\psi=\eta\ , \tag{2.5}$$

$$\bar{\psi}\left[\gamma^\mu\left(\frac{\overleftarrow{\partial}_\mu}{i}+e_0A_\mu\right)-m_0\right]=-\bar{\eta}\ , \tag{2.6}$$

$$\partial_k F^{k0}=-(e_0\bar{\psi}\gamma^0\psi+J^0)\ , \tag{2.7}$$

$$\partial_\mu F^{\mu i}-\partial_3{}^{-1}\partial^i(\partial_\mu F^{\mu 3}+e_0\bar{\psi}\gamma^3\psi+J^3)=-(e_0\bar{\psi}\gamma^i\psi+J^i)\ , \tag{2.8}$$

$i=1,2$. We note that (2.8) is trivially true for i replaced by 3.

The canonical momenta are

$$\pi(A^0)=0 , \tag{2.9}$$

$$\pi(A^i)\equiv\pi^i=\partial_3^{-1}(\partial^i F^{03}-\partial^3 F^{0i}) , \tag{2.10}$$

$$\pi(\psi)=i\psi^\dagger , \tag{2.11}$$

$$[\pi(\psi)]^\dagger=-i\psi . \tag{2.12}$$

That is, A^0 is a dependent field so defined since its canonical momentum vanishes. From (2.10) and (2.7) we may write (see also Ref. 8)

$$F^{0k}=-\left[g^{ki}-\frac{\partial^k\partial^i}{\vec{\partial}^{\,2}}\right]\pi_i+\frac{\partial^k}{\vec{\partial}^{\,2}}(e_0\bar{\psi}\gamma^0\psi+J^0) , \tag{2.13}$$

$k=1,2,3$; $i=1,2$, and from the fact that $\partial_k F^{0k}=-\vec{\partial}^{\,2}A^0$, we have

$$A^0=-\frac{1}{\vec{\partial}^{\,2}}(e_0\bar{\psi}\gamma^0\psi+J^0) . \tag{2.14}$$

We note that Eqs. (2.5) and (2.6) lead to

$$\partial_\mu(\bar{\psi}\gamma^\mu\psi)=i(\bar{\psi}\eta-\bar{\eta}\psi) , \tag{2.15}$$

and the current is conserved in the absence of the external Fermi sources. We may rewrite (2.8) by eliminating the expression in parentheses on its left-hand side and combining it with the $\nu=0$ component in (2.7) as ($\nu=0,1,2,3$)

$$\partial_\mu F^{\mu\nu}=-\left[g^{\nu\sigma}-g^{\nu k}\frac{\partial_k\partial^\sigma}{\vec{\partial}^{\,2}}\right](e_0\bar{\psi}\gamma_\sigma\psi+J_\sigma) . \tag{2.16}$$

We note that $\partial_\nu\partial_\mu F^{\mu\nu}=0$ is automatically satisfied for consistency for all J_σ. That is J_σ need not be conserved. (See also Ref. 8.)

Let $\langle 0_+|0_-\rangle$ denote the vacuum-to-vacuum transition amplitude in the presence of the external sources $\eta,\bar{\eta},J^\mu$. The action principle reads[3]

$$\frac{\partial}{\partial e_0}\langle 0_+|0_-\rangle=i\left\langle 0_+\left|\int (dx)(\bar{\psi}\gamma_\mu\psi A^\mu)_+\right|0_-\right\rangle . \tag{2.17}$$

Now although A^0 is a dependent field we may use (see Ref. 5) in (2.17) the functional differentiation expression:

$$(-i)\frac{\delta}{\delta J^0(y)}\langle 0_+|[\bar{\psi}(x)\gamma_\mu\psi(x')]_+|0_-\rangle =\langle 0_+|[\bar{\psi}(x)\gamma_\mu\psi(x')A_0(y)]_+|0_-\rangle \tag{2.18}$$

since ψ and $\bar{\psi}$ are not dependent fields and the functional derivative in (2.18) is defined with the independent fields and their conjugate momenta fixed. Hence

$$\int (dx)\langle 0_+|(\bar{\psi}\gamma_\mu\psi A^\mu)_+|0_-\rangle=\int (dx)(-i)\frac{\delta}{\delta J_\mu(x)}\langle 0_+|[\bar{\psi}(x)\gamma_\mu\psi(x)]_+|0_-\rangle$$
$$=\int (dx)(-i)\frac{\delta}{\delta\eta(x)}\gamma_\mu(-i)\frac{\delta}{\delta\bar{\eta}(x)}(-i)\frac{\delta}{\delta J_\mu(x)}\langle 0_+|0_-\rangle , \tag{2.19}$$

and from (2.17)

$$\langle 0_+|0_-\rangle=\exp\left[ie_0\int (dx)(-i)\frac{\delta}{\delta\eta}\gamma^\mu(-i)\frac{\delta}{\delta\bar{\eta}}(-i)\frac{\delta}{\delta J^\mu}\right]\langle 0_+|0_-\rangle_0 \tag{2.20}$$

(up to a normalization factor) where $\langle 0_+|0_-\rangle_0$ stands for $\langle 0_+|0_-\rangle$ with e_0 set equal to zero. The functional $\langle 0_+|0_-\rangle_0$ is determined in the Appendix and is given by

$$\langle 0_+|0_-\rangle_0=\exp\left[i\int (dx)(dx')\bar{\eta}(x)S_+(x-x')\eta(x')\right]\exp\left[\frac{i}{2}\int (dx)(dx')J^\mu(x)D^C_{\mu\nu}(x-x')J^\nu(x')\right] , \tag{2.21}$$

where

$$S_+(x-x')=\int\frac{dp}{(2\pi)^4}\frac{-\gamma p+m_0}{p^2+m_0{}^2-i\epsilon}e^{ip(x-x')} , \tag{2.22}$$

$$D^C_{\mu\nu}(x-x')=\int\frac{(dq)}{(2\pi)^4}D^C_{\mu\nu}(q)e^{iq(x-x')} , \tag{2.23}$$

$$D^C_{km}(q)=\left[g_{km}-\frac{q_k q_m}{\mathbf{q}^2}\right]\frac{1}{q^2-i\epsilon} , \tag{2.24}$$

$$D^C_{0k}(q)=0=D^C_{k0}(q) , \tag{2.25}$$

$$D^C_{00}(q)=-\frac{1}{\mathbf{q}^2} . \tag{2.26}$$

Needless to say, (2.19) is well known and we do not claim originality; the analysis, however, is given to show how an obvious modification arises when dealing with the Yang-Mills case studied in Sec. V *directly from the action principle* in a simple manner.

III. $\langle 0_+|0_-\rangle$ FOR QED IN TERMS OF DELTA FUNCTIONALS

We derive the expression for $\langle 0_+|0_-\rangle_0$, and hence for $\langle 0_+|0_-\rangle$ from (2.20), in terms of delta functionals.[9] Here we will see *the form of the constraint in (2.3) explicitly in the expression for* $\langle 0_+|0_-\rangle$. To this end we have from (2.5)–(2.8) with e_0 in them set equal to zero and upon taking vacuum expectation values:

$$\left[\gamma^\mu\frac{\partial_\mu}{i}+m_0\right](-i)\frac{\delta}{\delta\bar\eta}\langle 0_+|0_-\rangle_0=\eta\langle 0_+|0_-\rangle_0\ , \quad (3.1)$$

$$\langle 0_+|0_-\rangle_0(-i)\frac{\overleftarrow{\delta}}{\delta\eta}\left[\gamma^\mu\frac{\overleftarrow{\partial}_\mu}{i}-m_0\right]=-\bar\eta\langle 0_+|0_-\rangle_0\ , \quad (3.2)$$

$$\partial_k F'^{k0}\langle 0_+|0_-\rangle_0=-J^0\langle 0_+|0_-\rangle_0\ , \quad (3.3)$$

$$(g^{i\nu}-\partial_3{}^{-1}\partial^i g^{\nu 3})\partial^\mu F'_{\mu\nu}\langle 0_+|0_-\rangle_0 =-(g^{i\nu}-\partial_3{}^{-1}\partial^i g^{\nu3})J_\nu\ , \quad (3.4)$$

where

$$F'_{\mu\nu}=-i\left[\partial_\mu\frac{\delta}{\delta J^\nu}-\partial_\nu\frac{\delta}{\delta J^\mu}\right]. \quad (3.5)$$

Let

$$\mathscr{A}'_{01}=-\frac{1}{2}\int(dx)\left[\left[\frac{\partial_\mu}{i}\frac{\delta}{\delta\eta}\right]\gamma^\mu\frac{\delta}{\delta\bar\eta}-\frac{\delta}{\delta\eta}\gamma^\mu\frac{\partial_\mu}{i}\frac{\delta}{\delta\bar\eta}\right] +m_0\int(dx)\frac{\delta}{\delta\eta}\frac{\delta}{\delta\bar\eta}\ , \quad (3.6)$$

$$\mathscr{A}'_{02}=-\tfrac{1}{4}\int(dx)F'_{\mu\nu}F'^{\mu\nu} \quad (3.7)$$

with $F'_{\mu\nu}$ defined in (3.5). We also set

$$\mathscr{A}'_0=\mathscr{A}'_{01}+\mathscr{A}'_{02}\ . \quad (3.8)$$

Upon using the identities

$$\left[\frac{\delta}{\delta J^\mu(x)},J^\nu(x')\right]=g^\nu_\mu\delta(x-x')\ , \quad (3.9)$$

$$\left\{\frac{\delta}{\delta\eta(x)},\eta(x')\right\}=\delta(x-x')\ , \quad (3.10)$$

$$\left\{\frac{\delta}{\delta\bar\eta(x)},\bar\eta(x')\right\}=\delta(x-x')\ , \quad (3.11)$$

$$\left\{\frac{\delta}{\delta\bar\eta(x)},\eta(x)\right\}=0\ , \quad (3.12)$$

$$\left\{\frac{\delta}{\delta\eta(x)},\bar\eta(x)\right\}=0\ , \quad (3.13)$$

we readily derive the relations

$$[i\mathscr{A}'_{01},\eta(x)]=-\left[\gamma^\mu\frac{\partial_\mu}{i}+m_0\right](-i)\frac{\delta}{\delta\bar\eta(x)}\ , \quad (3.14)$$

$$[i\mathscr{A}'_{01},\bar\eta(x)]=-i\frac{\delta}{\delta\eta(x)}\left[\gamma^\mu\frac{\overleftarrow{\partial}_\mu}{i}-m_0\right], \quad (3.15)$$

$$[i\mathscr{A}'_{02},J^\nu(x)]=\partial_\mu F'^{\mu\nu}\ . \quad (3.16)$$

We may then rewrite (3.1)–(3.4) as

$$\{[i\mathscr{A}'_{01},\eta(x)]+\eta(x)\}\langle 0_+|0_-\rangle_0=0\ , \quad (3.17)$$

$$\{[i\mathscr{A}'_{01},\bar\eta(x)]+\bar\eta(x)\}\langle 0_+|0_-\rangle_0=0\ , \quad (3.18)$$

$$\{[i\mathscr{A}'_{02},J^0(x)]+J^0(x)\}\langle 0_+|0_-\rangle_0=0\ , \quad (3.19)$$

$$\{[i\mathscr{A}'_{02},\Sigma^{i\nu}J_\nu(x)]+\Sigma^{i\nu}J_\nu(x)\}\langle 0_+|0_-\rangle_0=0\ , \quad (3.20)$$

where

$$\Sigma^{i\nu}=(g^{i\nu}-\partial_3{}^{-1}\partial^i g^{\nu3})\ . \quad (3.21)$$

We use the well-known identity (e.g., Ref. 9) if $[A,[A,B]]=0$, then

$$e^{A}Be^{-A}=[A,B]+B\ . \quad (3.22)$$

Using (3.22) in (3.17)–(3.20) we obtain

$$e^{i\mathscr{A}'_{01}}\eta(x)e^{-i\mathscr{A}'_{01}}\langle 0_+|0_-\rangle_0=0 \quad (3.23)$$

or

$$\eta(x)(e^{-i\mathscr{A}'_{01}}\langle 0_+|0_-\rangle_0)=0\ , \quad (3.24)$$

and similarly

$$\bar\eta(x)(e^{-i\mathscr{A}'_{01}}\langle 0_+|0_-\rangle_0)=0\ , \quad (3.25)$$

$$J^0(x)(e^{-i\mathscr{A}'_{02}}\langle 0_+|0_-\rangle_0)=0\ , \quad (3.26)$$

$$\Sigma^{i\nu}J_\nu(x)(e^{-i\mathscr{A}'_{02}}\langle 0_+|0_-\rangle_0)=0\ . \quad (3.27)$$

Hence $\langle 0_+|0_-\rangle_0$ is determined from (3.24)–(3.27) to be

$$\langle 0_+|0_-\rangle_0=\exp(i\mathscr{A}'_0)\delta(J^0)\delta(\Sigma^{i\nu}J_\nu)\delta(\eta)\delta(\bar\eta)\ , \quad (3.28)$$

where the $\delta(f)$ are delta functionals[9] defined as the product of delta functions for each space-time point. A product over i is also understood in (3.28). We may write $\delta(\Sigma^{i\nu}J_\nu)=\delta(J^i-\partial_3{}^{-1}\partial^i J^3)$ and use the elementary identities

$$\delta\left[-i\partial^k\frac{\delta}{\delta J^k}\right]\delta(J^\sigma)=\text{const}\times\delta(J^0)\delta(J^i-\partial_3{}^{-1}\partial^i J^3)\ , \quad (3.29)$$

$$\delta\left[-i\partial^k\frac{\delta}{\delta J^k}\right]\delta(J^\sigma)=\text{const}\times\delta\left[\frac{\partial^k}{\vec\partial^{\,2}}\frac{\delta}{\delta J^k}\right]\delta(J^\sigma)\ , \quad (3.30)$$

where the const are independent of the external source J^σ, for $\sigma=0,1,2,3$, and a product over i and σ in (3.29) and (3.30) is understood.

From (3.28)–(3.30) and (2.20) we may then write the full $\langle 0_+|0_-\rangle$ in the convenient and compact form

$$\langle 0_+|0_-\rangle=\exp(i\mathscr{A}')\delta\left[-ia^\mu\frac{\delta}{\delta J^\mu}\right]\delta(J^\sigma)\ , \quad (3.31)$$

where $\mathscr{A}'$ stands for action with A_μ, ψ, $\bar\psi$ replaced by $-i\delta/\delta J^\mu$, $-i\delta/\delta\bar\eta$, and $-i\delta/\delta\eta$, respectively, and

$$a^\mu=\left[0,\frac{\partial^k}{\partial^2}\right]. \quad (3.32)$$

This expression is interesting for many reasons. It is written in terms of the gauge-invariant part $\mathscr{A}'$ rather than the interaction part, and also makes the gauge constraint, via the $\delta(-ia^\mu\delta/\delta J^\mu)$ term, explicit. Because of the explicit appearance of the $\delta(-ia^\mu\delta/\delta J^\mu)$ term in (3.31), the expression for $\langle 0_+|0_-\rangle$ in (3.31) is quite suitable to study gauge transformations. This is studied in the next section. The corresponding expression to (3.31) for Yang-Mills fields is derived in Sec. VI.

IV. GAUGE TRANSFORMATIONS IN QED

Let A^μ, ψ, and $\bar{\psi}$ denote *classical* fields and consider the gauge transformations

$$A^\mu(x) \to A^\mu(x) + \partial^\mu\theta(x) \equiv A'^\mu(x) , \quad (4.1)$$

$$\psi(x) \to e^{ie_0\theta(x)}\psi(x) \equiv \psi'(x) , \quad (4.2)$$

$$\bar{\psi}(x) \to \bar{\psi}(x)e^{-ie_0\theta(x)} \equiv \bar{\psi}'(x) , \quad (4.3)$$

such that

$$a^\mu A_\mu(x) = \tilde{a}^\mu A'_\mu(x) + \Lambda(x) , \quad (4.4)$$

where $\Lambda(x)$ is an *arbitrary* function of x, and

$$\tilde{a}^\mu = \frac{\partial^\mu}{\Box}, \quad a^\mu = \left[0, \frac{\vec{\partial}}{\vec{\partial}^{\,2}}\right] . \quad (4.5)$$

We may then *explicitly* solve for θ directly from (4.1) and (4.4) to be

$$\theta(x) = (a^\mu A'_\mu - \tilde{a}^\mu A'_\mu) - \Lambda(x) , \quad (4.6)$$

expressed in terms of the field A'_μ. Hence we may write

$$A^\mu(x) = A'^\mu - \partial^\mu[a^\sigma A'_\sigma(x)] + \partial^\mu[\tilde{a}^\sigma A'_\sigma(x) + \Lambda(x)] , \quad (4.7)$$

and

$$\psi(x) = \exp\{-ie_0[a^\mu A'_\mu - (\tilde{a}^\mu A'_\mu + \Lambda)]\}\psi'(x) , \quad (4.8)$$

$$\bar{\psi}(x) = \bar{\psi}'(x)\exp\{ie_0[a^\mu A'_\mu - (\tilde{a}^\mu A'_\mu + \Lambda)]\} . \quad (4.9)$$

We use the elementary identity

$$H_1\left[-ia^\mu\frac{\delta}{\delta J^\mu}\right]H_2\left[(-i)\frac{\delta}{\delta J^\mu},(-i)\frac{\delta}{\delta\bar{\eta}},(-i)\frac{\delta}{\delta\eta}\right]\delta(J^\sigma)\delta(\eta)\delta(\bar{\eta})$$
$$= e^{iW'_\Lambda}H_1\left[-i\tilde{a}^\mu\frac{\delta}{\delta j^\mu} + \Lambda(x)\right]H_2\left[(-i)\frac{\delta}{\delta j^\mu},(-i)\frac{\delta}{\delta\bar{\rho}},(-i)\frac{\delta}{\delta\rho}\right]\delta(j^\sigma)\delta(\rho)\delta(\bar{\rho})\Bigg|_{j^\sigma=0,\rho=0,\bar{\rho}=0} , \quad (4.10)$$

where $H_1[a^\mu A_\mu]$ is a functional of the product $a^\mu A_\mu$, and $H_2[A_\mu,\psi,\bar{\psi}]$ is *invariant* under arbitrary transformations of the sort given in (4.1)–(4.3), and

$$W'_\Lambda = \int (dx)\bar{\eta}(x)\exp\left\{-ie_0\left[a^\mu(-i)\frac{\delta}{\delta j^\mu} - \left[\tilde{a}^\mu(-i)\frac{\delta}{\delta j^\mu} + \Lambda\right]\right]\right\}(-i)\frac{\delta}{\delta\bar{\rho}(x)}$$
$$+ \int (dx)(-i)\frac{\delta}{\delta\rho(x)}\exp\left\{ie_0\left[a^\mu(-i)\frac{\delta}{\delta j^\mu} - \left[\tilde{a}^\mu(-i)\frac{\delta}{\delta j^\mu} + \Lambda\right]\right]\right\}\eta(x)$$
$$+ \int (dx)J_\mu(x)\left[(-i)\frac{\delta}{\delta j_\mu} - \partial^\mu a^\sigma(-i)\frac{\delta}{\delta j^\sigma} + \partial^\mu\left[\tilde{a}^\sigma(-i)\frac{\delta}{\delta j^\sigma} + \Lambda\right]\right] . \quad (4.11)$$

The identity in (4.10) is easily derived by making a functional Fourier transform of $\delta(J^\sigma)\delta(\eta)\delta(\bar{\eta})$ in the external sources J^σ, η, $\bar{\eta}$ and by taking into account the linearity of the transformation in (4.1) and (4.7), and by making use of the identity (4.4).

From (3.31), (4.4), (4.10), and (4.11) we may write

$$\langle 0_+ | 0_- \rangle = e^{iW'_\Lambda}e^{(i\mathcal{A}')}\delta\left[-i\tilde{a}^\mu\frac{\delta}{\delta j^\mu} + \Lambda\right]\delta(j^\sigma)\delta(\rho)\delta(\bar{\rho})\Bigg|_{j^\sigma=0,\rho=0,\bar{\rho}=0} , \quad (4.12)$$

where $\mathcal{A}'$ is written in terms of $(-i)\delta/\delta j^\mu, (-i)\delta/\delta\rho, (-i)\delta/\delta\bar{\rho}$. We note that although different factors on the right-hand side of (4.12) depend on Λ, the final expression for $\langle 0_+ | 0_- \rangle$ is *independent* of Λ. We make use of the identity

$$\left[\int\left[-i\tilde{a}^\mu\frac{\delta}{\delta j^\mu} + \Lambda\right]\right]^n\delta\left[-i\tilde{a}^\mu\frac{\delta}{\delta j^\mu} + \Lambda\right]\delta(j) = 0$$

for $n = 1, 2, \ldots$ to simplify (4.12) to the expression

$$\langle 0_+ | 0_- \rangle = e^{iW'}e^{(i\mathcal{A}')}\delta\left[-i\tilde{a}^\mu\frac{\delta}{\delta j^\mu} + \Lambda\right]\delta(j^\sigma)\delta(\rho)\delta(\bar{\rho})\Bigg|_{j^\sigma=0,\rho=0,\bar{\rho}=0} , \quad (4.13)$$

where

$$W'=\int (dx)\bar{\eta}(x)\exp(-e_0 a^\mu \delta/\delta j^\mu)(-i)\frac{\delta}{\delta\bar{\rho}(x)}+\int (dx)(-i)\frac{\delta}{\delta\rho(x)}\exp(e_0 a^\mu\delta/\delta j^\mu)\eta(x)$$
$$+\int (dx)[(g^{\mu\sigma}-a^\mu\partial^\sigma)J_\sigma](-i)\frac{\delta}{\delta j^\mu}\ . \tag{4.14}$$

Finally we use the identity

$$\delta\left[-i\tilde{a}^\mu\frac{\delta}{\delta j^\mu}+\Lambda\right]\delta(j^\sigma)=e^{-i\partial_\mu j^\mu\Lambda}\delta\left[-i\tilde{a}^\mu\frac{\delta}{\delta j^\mu}\right]\delta(j^\sigma) \tag{4.15}$$

and the fact that

$$[i\mathscr{A}'_0,\partial_\mu j^\mu\Lambda]=0\ , \tag{4.16}$$

to rewrite $\langle 0_+\,|\,0_-\rangle$ in the form

$$\langle 0_+\,|\,0_-\rangle=e^{iW'}\exp\left[i\int\mathscr{L}'_I\right]e^{-i\partial_\mu j^\mu\Lambda}F[j^\sigma,\bar{\rho},\rho]\Big|_{j^\sigma=0,\bar{\rho}=0,\rho=0}\ , \tag{4.17}$$

where

$$F[j^\sigma,\bar{\rho},\rho]=\exp(i\bar{\rho}S_+\rho)\exp\left[\frac{i}{2}j^\mu D^L_{\mu\nu}j^\nu\right]\ , \tag{4.18}$$

$$D^L_{\mu j}(q)=\left[g_{\mu\nu}-\frac{q_\mu g_\nu}{q^2}\right]\frac{1}{q^2-i\epsilon}\ , \tag{4.19}$$

and where we have made use of Eqs. (A9)—(A15) in the Appendix. Equation (4.19) defines the (free) photon propagator in the so-called Landau gauge.

Since $\langle 0_+\,|\,0_-\rangle$ is independent of Λ we may apply to it *any* functional differential operator of the form

$$U=\exp\left[iH\left[\frac{\delta}{\delta\Lambda}\right]\right] \tag{4.20}$$

where

$$H[f]\,|_{f=0}=0 \tag{4.21}$$

without changing $\langle 0_+\,|\,0_-\rangle$. Hence

$$\langle 0_+\,|\,0_-\rangle=U\langle 0_+\,|\,0_-\rangle$$
$$=e^{iW'}\exp\left[i\int\mathscr{L}'_I\right]e^{iH[-i\partial_\mu j^\mu]}F[j^\sigma,\bar{\rho},\rho]\Big|_{j^\sigma=0,\bar{\rho}=0,\rho=0}\ , \tag{4.22}$$

where we have finally set $\Lambda=0$. *In particular* for a bilinear form

$$H[f]=\tfrac{1}{2}\int (dx)(dx')f(x)M(x-x')f(x')\ , \tag{4.23}$$

we have

$$H[-i\partial_\mu j^\mu]=\tfrac{1}{2}\int (dx)(dx')[-i\partial_\mu j^\mu(x)]$$
$$\times M(x-x')[-i\partial'_\mu j^\mu(x')]\ , \tag{4.24}$$

which simply amounts in modifying the photon propagator in (4.19) to

$$D^M_{\mu\nu}(q)=\left[g_{\mu\nu}-\frac{q_\mu q_\nu}{q^2}\right]\frac{1}{q^2-i\epsilon}-q_\mu q_\nu M(q)\ , \tag{4.25}$$

where

$$M(q)=\int (dx)e^{-iqx}M(x)\ , \tag{4.26}$$

defining generalized covariant gauges.

With the choice in (4.23) we obtain

$$\langle 0_+\,|\,0_-\rangle=e^{iW'}\exp\left[i\int\mathscr{L}'_I\right]F_M[j^\sigma,\bar{\rho},\rho]\Big|_{j^\sigma=0,\bar{\rho}=0,\rho=0}\ , \tag{4.27}$$

where

$$F_M[j^\sigma,\bar{\rho},\rho]=\exp(i\bar{\rho}S_+\rho)\exp\left[\frac{i}{2}j^\mu D^M_{\mu\nu}j^\nu\right] \tag{4.28}$$

with W *given* in (4.14).

For a conserved external current $\partial^\sigma J_\sigma=0$ the last term in (4.14) simply becomes

$$\int (dx) J^\mu(x) \left[(-i)\frac{\delta}{\delta j^\mu(x)} \right] .$$

Gauge transformations of Green's functions, in general, may be explicitly carried out from (4.27) and this expression is consistent with earlier deviations[11] by different methods.

The simplicity of the derivation given in (4.27) with basic components in (4.14) and (4.28) are based on a gauge transformation (4.1) $A^\mu \to A^\mu_\theta \equiv A'^\mu$, such that the identity in (4.4) holds, is essentially due to the linearity of the gauge transformation $A^\mu \to A^\mu_\theta \equiv A'^\mu$ under the constraint (4.4) in QED. This linearity property is evident by the explicit solution in (4.7) and is explicitly used in writing down the identity of (4.10). The corresponding situation for non-Abelian gauge theories is much more complicated due to the nonlinearity of the corresponding expression to (4.7) in A'^μ. Accordingly, we quickly rederive (4.27) by an approach which is a variation of the Faddeev-Popov trick[1,12] and generalizes immediately to the non-Abelian case.

Let $F[A^\sigma]=a^\mu A_\mu$ and $T^\theta(A^\mu)=A^\mu(\theta)$, where $A^\mu(\theta)$ is a gauge transformation of A^μ, that is $A^\mu(\theta) = A^\mu + \partial^\mu\theta$. We note that (trivially)

$$\Delta_C[A^\sigma] \equiv \det \left| \frac{\partial F[T^\theta(A^\mu)]}{\partial\theta} \right|_{F=0} = \text{const} . \tag{4.29}$$

The importance of introducing this object will become clear when we study the non-Abelian case in Sec. VII. We also introduce for an arbitrary functional $G[\ ]$ the object

$$\Delta_G[A^\sigma] = \det \left| \frac{\partial G[T^\theta(A^\mu)]}{\partial\theta} \right|_{G=0} , \tag{4.30}$$

and make use of the identity that for an arbitrary functional $H[f]$:

$$\delta\left[H\left[(-i)\frac{\delta}{\delta K} \right] \right] \delta(K) = e^{iK\theta(H)} \det \left| \frac{\partial H(\theta)}{\partial\theta} \right|^{-1}_{H=0} , \tag{4.31}$$

where $\theta(H)$ is a solution of $H(\theta)=0$. Equation (4.31) makes the gauge invariance of $\Delta_G[A^\sigma]$ in (4.30) evident upon setting $K=0$. We may write

$$e^{iJ^\mu[(-i)\delta/\delta j^\mu]} \delta\left[-ia^\mu \frac{\delta}{\delta j^\mu} \right] \delta(j) \bigg|_{j=0} = \text{const} \times e^{iJ^\mu[(-i)\delta/\delta j^\mu]} \delta\left[-ia^\mu \frac{\delta}{\delta j^\mu} \right] \Delta_C\left[(-i)\frac{\delta}{\delta j} \right] \Delta_G\left[(-i)\frac{\delta}{\delta j} \right] \times \Delta_G{}^{-1}\left[(-i)\frac{\delta}{\delta j} \right] \delta(j) \bigg|_{j=0} . \tag{4.32}$$

Upon writing

$$\Delta_G{}^{-1}\left[(-i)\frac{\delta}{\delta j} \right] = \delta(G[\tilde{T}^{-i\delta/\delta K}])\delta(K) \bigg|_{K=0} , \tag{4.33}$$

where $\tilde{T}^\theta_\mu = T^\theta[-i\delta/\delta j^\mu]$, and carrying out the transformation $(-i)\delta/\delta j^\mu \to \tilde{T}^{i\delta/\delta K}_\mu$, we obtain for the right-hand side of (4.32), up to a multiplicative factor:

$$\delta(a^\mu \tilde{T}^{i\delta/\delta K}_\mu) e^{iJ^\mu \tilde{T}^{i\delta/\delta K}_\mu} \delta(K) \,|_{K=0} \Delta_c\left[(-i)\frac{\delta}{\delta j} \right] \Delta_G\left[(-i)\frac{\delta}{\delta j} \right] \delta\left[G\left[(-i)\frac{\delta}{\delta j} \right] \right] \delta(j) \bigg|_{j=0} . \tag{4.34}$$

Also

$$\begin{aligned} \delta\left[G\left[(-i)\frac{\delta}{\delta j} \right] \right] \delta(a^\mu \tilde{T}^{i\delta/\delta K})\delta(K) &= \delta\left[G\left[(-i)\frac{\delta}{\delta j} \right] \right] e^{-iK\theta_0} \det \left| \frac{\partial a^\mu \tilde{T}^\theta_\mu}{\partial\theta} \right|_{a^\mu \tilde{T}^\theta_\mu = 0} \\ &= \delta\left[G\left[(-i)\frac{\delta}{\delta j} \right] \right] e^{-iK\theta_0} \Delta_C{}^{-1}\left[(-i)\frac{\delta}{\delta j} \right] , \end{aligned} \tag{4.35}$$

where θ_0 is such that $a^\mu \tilde{T}^{\theta_0}_\mu = 0$ when

$$G\left[(-i)\frac{\delta}{\delta j} \right] = 0 .$$

All told we have

$$\delta\left[a^\mu (-i)\frac{\delta}{\delta J^\mu} \right] \delta(J^\sigma) = e^{iJ^\mu \tilde{T}^{\theta_0}_\mu} \delta\left[G\left[(-i)\frac{\delta}{\delta j} \right] \right] \Delta_G\left[(-i)\frac{\delta}{\delta j} \right] \delta(j) \bigg|_{j=0} . \tag{4.36}$$

In particular if

$$G[A^\mu]=\tilde{a}^\mu A_\mu+\Lambda \tag{4.37}$$

then $\Delta_G[A^\mu]=\text{const}$, and

$$a^\mu T^{\theta_0}\left[(-i)\frac{\delta}{\delta j^\mu}\right]=0 ,$$

yields

$$\theta_0=ia^\mu\frac{\delta}{\delta j^\mu} \tag{4.38}$$

and

$$J^\mu T^{\theta_0}\left[(-i)\frac{\delta}{\delta j^\mu}\right]=[(g^{\mu\sigma}-a^\mu\partial^\sigma)J_\sigma](-i)\frac{\delta}{\delta j^\mu} \tag{4.39}$$

and coincides with our earlier derivation upon the application of the differential operator (4.20) to (4.22).

V. ACTION PRINCIPLE AND YANG-MILLS FIELDS

We consider the following Lagrangian density:

$$\mathscr{L}=-\tfrac{1}{4}G^a_{\mu\nu}G^{\mu\nu a}+\frac{1}{2}\left[\left[\frac{\partial_\mu}{i}\bar{\psi}\right]\gamma^\mu\psi-\bar{\psi}\gamma^\mu\frac{\partial_\mu}{i}\psi\right]-m_0\bar{\psi}\psi$$
$$+g_0\bar{\psi}\gamma_\mu A^{\mu a}t^a\psi+\bar{\eta}\psi+\bar{\psi}\eta+J^a_\mu A^{\mu a} , \tag{5.1}$$

where

$$G^a_{\mu\nu}=\partial_\mu A^a_\nu-\partial_\nu A^a_\mu+g_0 f^{abc}A^b_\mu A^c_\nu , \tag{5.2}$$

and for generality we have also included a (multicomponent) matter field. A summation over different matter fields may be also considered. The t^a are generators of the underlying Lie algebra, and the f^{abc}, totally antisymmetric, are the structure constants satisfying the Jacobi identity. The Lagrangian density (5.1), without the (external) source terms $(\bar{\eta}\psi+\bar{\psi}\eta+J^a_\mu A^{\mu a})$ is invariant under simultaneous local gauge transformations:

$$\psi\to U(\theta)\psi\equiv\psi(\theta) ,$$
$$\bar{\psi}\to\bar{\psi}U^{-1}(\theta)\equiv\bar{\psi}(\theta) , \tag{5.3}$$
$$A^\mu\to U(\theta)A^\mu U^{-1}(\theta)-\frac{i}{g_0}[\partial_\mu U(\theta)]U^{-1}(\theta)\equiv A^\mu(\theta) ,$$

where $U(\theta)=\exp(ig_0\theta^a t^a)$, $[t^a,t^b]=if^{abc}t^c$, $A^\mu=t^a A^\mu_a$.

We work in the Coulomb gauge by imposing the constraint

$$\partial_k A^{ka}=0,\quad k=1,2,3 \tag{5.4}$$

and we may write[8]

$$A^{3a}=-\partial_3{}^{-1}(\partial_i A^{ia}),\quad i=1,2 . \tag{5.5}$$

With A^0,A^1,A^2, and ψ as dynamical variables, the equations of motion and the canonical momenta are[8]

$$\left[\gamma^\mu\left[\frac{\partial_\mu}{i}-g_0A_\mu\right]+m_0\right]\psi=\eta , \tag{5.6}$$

$$\bar{\psi}\left[\gamma^\mu\left[\frac{\overleftarrow{\partial}_\mu}{i}+g_0A_\mu\right]-m_0\right]=-\bar{\eta} , \tag{5.7}$$

$$\nabla^{ab}_k G^{k0}_b=-g_0\bar{\psi}\gamma^0 t^a\psi-J^{0a} , \tag{5.8}$$

$$\nabla^{ab}_\mu G^{\mu i}_b-\partial_3{}^{-1}\partial^i(\nabla^{ab}_\mu G^{\mu 3}_b+g_0\bar{\psi}\gamma^3 t^a\psi+J^{3a})$$
$$=-g_0\bar{\psi}\gamma^i t^a\psi-J^{ia},\quad i=1,2 , \tag{5.9}$$

where

$$\nabla^{ab}_\mu=(\delta^{ab}\partial_\mu+g_0 f^{acb}A^c_\mu) , \tag{5.10}$$

$$\pi(A^{0a})=0 , \tag{5.11}$$

$$\pi(A^{ia})\equiv\pi^{ia}=\partial_3{}^{-1}(\partial^i G^{03a}-\partial^3 G^{0i}),\quad i=1,2 , \tag{5.12}$$

$$\pi(\psi)=i\psi^\dagger , \tag{5.13}$$

$$[\pi(\psi)]^\dagger=-i\psi , \tag{5.14}$$

and A^{0a} is a dependent field. From (5.12) and (5.8) one may write[8]

$$G^{0ka}=-\left[g^{ki}-\frac{\partial^k\partial^i}{\vec{\partial}^2}\right]\pi^{ia}+\partial^k\Psi^a ,$$
$$k=1,2,3;\ i=1,2 , \tag{5.15}$$

where

$$\nabla^{ab}_k\partial^k\Psi^b=g_0 f^{abc}A^b_k\left[g^{ki}-\frac{\partial^k\partial^i}{\vec{\partial}^2}\right]\pi^{ic}+g_0\bar{\psi}\gamma^0 t^a\psi+J^{0a}$$

or

$$\Psi^b=D^{bc}\left[J^{0c}+g_0\bar{\psi}\gamma^0 t^c\psi+g_0 f^{cde}A^d_k\left[g^{ki}-\frac{\partial^k\partial^i}{\vec{\partial}^2}\right]\pi^{ie}\right] , \tag{5.16}$$

where[13,8]

$$\nabla^{ab}_k\partial^k D^{bc}(x,y)=\delta(x-y)\delta^{ac} . \tag{5.17}$$

By eliminating the expression in parentheses on the left-hand side of (5.9) and combining it with the $\nu=0$ component (5.8) we may also write[8]

$$\nabla^{ab}_\mu G^{\mu\nu}_b=-(g^{\nu\sigma}\delta^{ac}-g^{\nu k}\partial_k D^{ab}\nabla^\sigma_{bc})(g_0\bar{\psi}\gamma_\sigma t^c\psi+J^c_\sigma) , \tag{5.18}$$

consistent[8,14] with the identity $\nabla^{ca}_\nu\nabla^{ab}_\mu G^{\mu\nu}_b=0$ for all J^c_σ.

Let $\langle 0_+|0_-\rangle$ denote the vacuum-to-vacuum transition amplitude in the presence of the external sources. The action principle[3] reads

$$\frac{\partial}{\partial g_0}\langle 0_+|0_-\rangle=i\int(dx)\langle 0_+|\tilde{L}(x)|0_-\rangle , \tag{5.19}$$

where

$$\int(dx)\langle 0_+|\tilde{L}(x)|0_-\rangle$$
$$=-\tfrac{1}{2}\int(dx)f^{abc}\langle 0_+|(A^{\mu b}A^{\nu c}G^a_{\mu\nu})_+|0_-\rangle$$
$$+\int(dx)\langle 0_+|(\bar{\psi}\gamma_\mu A^{\mu a}t^a\psi)_+|0_-\rangle . \tag{5.20}$$

We write

$$-\tfrac{1}{2}f^{abc}A^{\mu b}A^{\nu c}G^{a}_{\mu\nu}=-f^{abc}A^{kb}A^{0c}G^{a}_{k0}-\tfrac{1}{2}f^{abc}A^{kb}A^{mc}G^{a}_{km}\ , \qquad (5.21)$$

where $k,m=1,2,3$. We denote $(-i)\delta/\delta J^{\mu a}$ by A'^{a}_{μ}, $(-i)\delta/\delta\overline{\eta}$ by ψ', $(-i)\delta/\delta\eta$ by $\overline{\psi}'$, and $\langle 0_+|B|0_-\rangle\equiv\langle B\rangle$. Then

$$\langle (A^{kb}A^{mc}G^{a}_{km})_+\rangle = A'^{kb}A'^{mc}G'^{a}_{km}\langle 0_+|0_-\rangle\ . \qquad (5.22)$$

On the other hand, quite generally

$$A'^{0c}\langle (A^{kb}G^{a}_{k0})_+\rangle \neq \langle (A^{0c}A^{kb}G^{a}_{k0})_+\rangle\ . \qquad (5.23)$$

This is due to the fact that G^{a}_{k0} depends on the time derivative $\partial_0 A^{a}_{k}$ and also on the dependent field A^{a}_{0}. The rule for carrying out the functional differentiation on the left-hand side of (5.23) is very simple and the rule was completely worked out[5] over 20 years ago, and gives an extra term on the right-hand side of (5.23) from the dependence of G^{a}_{k0} on Ψ^{a}, as given through (5.15)–(5.17), with the latter in turn depending explicitly on the $\nu=0$ component of the external current $J^{\nu a}$:

$$A'^{0c}\langle (A^{kb}G^{a}_{k0})_+\rangle = \langle (A^{0c}A^{kb}G^{a}_{k0})_+\rangle - i\left\langle \left[\frac{\delta}{\delta J^{c}_{0}}A^{kb}G^{a}_{k0}\right]_+\right\rangle$$

$$=\langle (A^{0c}A^{kb}G^{a}_{k0})_+\rangle - i\left\langle \left[A^{kb}\frac{\delta}{\delta J^{c}_{0}}G^{a}_{k0}\right]_+\right\rangle\ . \qquad (5.24)$$

The functional derivative on the right-hand side of (5.24) is carried out by keeping the independent fields $A^{ia},\psi,(\psi^{\dagger})$ [note A^{a}_{3} may be completely expressed in terms of $\partial_i A^{ia}$—see (5.5)] and their conjugate momenta fixed. Equation (5.24) is derived[5] by an elementary application of a completeness relation followed by a repeated application of the action principle (see also Ref. 9). That is in a matrix notation, we have, *directly* from (5.15)–(5.17),

$$\frac{\delta}{\delta J^{c}_{0}}G^{a}_{k0}=-\partial_k D^{ac}\ . \qquad (5.25)$$

Also

$$\langle (A^{kb}G^{a}_{k0})_+\rangle = A'^{kb}G^{a'}_{k0}\langle 0_+|0_-\rangle\ , \qquad (5.26)$$

since G^{a}_{k0} does not explicitly depend on J^{ka}. Finally

$$\int (dx)\langle (\overline{\psi}\gamma_\mu A^{\mu a}t^{a}\psi)_+\rangle = \int (dx)\overline{\psi}'\gamma_\mu A'^{\mu a}t^{a}\psi'\langle 0_+|0_-\rangle\ . \qquad (5.27)$$

All told we have, from (5.20)–(5.27),

$$\frac{\partial}{\partial g_0}\langle 0_+|0_-\rangle = i\left[\int (dx)\widetilde{L}'(x) - i\int f^{abc}A'^{b}_{k}\partial_k D'^{ca}\right]\langle 0_+|0_-\rangle\ . \qquad (5.28)$$

Upon integration over g_0 we have

$$\langle 0_+|0_-\rangle = \exp\left[i\int (dx)\mathscr{L}'_I(x) + \operatorname{Tr}\ln\left[\delta^{ab}+g_0 f^{acb}\frac{1}{\vec{\partial}^{\,2}}A'^{c}_{k}\partial^{k}\right]\right]\langle 0_+|0_-\rangle_0 \qquad (5.29)$$

(up to a normalization factor) and $\mathscr{L}'_I$ is the interaction Lagrangian density without the external source terms, that is

$$\mathscr{L}'_I = -\frac{g_0}{2}f^{abc}A'^{\mu b}A'^{\nu c}G'^{a}_{\mu\nu} - \frac{{g_0}^2}{4}f^{abc}A'^{\mu b}A'^{\nu c}f^{ade}A'^{d}_{\mu}A'^{e}_{\nu} + g_0\overline{\psi}'\gamma_\mu A'^{\mu a}t^{a}\psi'\ . \qquad (5.30)$$

$\langle 0_+|0_-\rangle_0$ is the (free) vacuum-to-vacuum transition amplitude in the presence of the external sources (see the Appendix), that is

$$\langle 0_+|0_-\rangle_0 = \exp(i\overline{\eta}S_+\eta)\exp\left[\frac{i}{2}J^{\mu a}D^{Cab}_{\mu\nu}J^{\nu b}\right]\ , \qquad (5.31)$$

where $D^{Cab}_{\mu\nu}=\delta^{ab}D^{C}_{\mu\nu}$, with $D^{C}_{\mu\nu}$ given in (2.23)–(2.26). The Faddeev-Popov factor has been obtained, from the action principle, in (5.29) without great effort.

VI. $\langle 0_+|0_-\rangle$ FOR YANG-MILLS FIELDS IN TERMS OF DELTA FUNCTIONALS

Directly from (5.29), (5.31), and (3.28)–(3.30), we may write the full $\langle 0_+|0_-\rangle$ for the Yang-Mills case in terms of delta functionals:

$$\langle 0_+|0_-\rangle = \exp(i\mathscr{A}')\exp\left[\operatorname{Tr}\ln\left[\delta^{ab}+g_0 f^{abc}\frac{1}{\vec{\partial}^{\,2}}(-i)\frac{\delta}{\delta J^{kc}}\partial^{k}\right]\right]\delta\left[-ia^{\mu}\frac{\delta}{\delta J^{\mu b}}\right]\delta(J^{\sigma c})\ , \qquad (6.1)$$

where a^μ is defined in (3.32). Here $\mathscr{A}'$ stands for the full action without the external source terms, with A^a_μ replaced by $(-i)\delta/\delta J^{\mu a}$, ψ replaced by $(-i)\delta/\delta\bar{\eta}$, and $\bar{\psi}$ replaced by $(-i)\delta/\delta\eta$. The Faddeev-Popov factor and the gauge constraint are explicit in (6.1). The $\mathscr{A}'$ term is gauge invariant.

VII. GAUGE TRANSFORMATIONS FOR YANG-MILLS FIELDS

We follow the method applied to QED as given through (4.29)—(4.36) as a variation of the Faddeev-Popov technique[1,12] and is quite suitable since it avoids the problem of nonlinear transformations $A_\mu \to A'_\mu$ in non-Abelian gauge theories as opposed to the QED case in (4.4).

Let $F[A^b_\mu]=a^\mu A^b_\mu$ and set $T^\theta_\mu[A^{\sigma a}]=A_\mu(\theta)$, where $A_\mu(\theta)$ is defined in (5.3). We note the validity of the following important equality:

$$\delta\left[-ia^\mu\frac{\delta}{\delta J^\mu}\right]\exp\left[\operatorname{Tr}\ln\left[\delta^{ab}+g_0 f^{acb}(-i)\frac{\delta}{\delta J^{\mu c}}\partial^\mu\right]\right]\delta(J^\sigma)=\delta\left[-ia^\mu\frac{\delta}{\delta J^\mu}\right]\det\left|\frac{\partial F\left[T^\theta_\mu\left[(-i)\dfrac{\delta}{\delta J^{\sigma a}}\right]\right]}{\partial\theta_a}\right|_{\theta=0}\delta(J^\sigma)$$
$$=\delta\left[-ia^\mu\frac{\delta}{\delta J^\mu}\right]\det\left|\frac{\partial F\left[T^\theta_\mu\left[(-i)\dfrac{\delta}{\delta J^{\sigma a}}\right]\right]}{\partial\theta_a}\right|_{F=0}\delta(J^\sigma)\,, \tag{7.1}$$

where in writing the last equality we have made use of the fact that the presence of the $\delta(-ia^\mu\delta/\delta J^\mu)$ factor singles out the solution $\theta=0$ for

$$a^\mu T^\theta_\mu\left[(-i)\frac{\delta}{\delta J^{\sigma a}}\right]=0\,.$$

We set

$$\Delta_C[A^{\mu a}]=\det\left|\frac{\partial}{\partial\theta_a}F[T^\theta_\mu[A^{\sigma a}]]\right|_{F=0} \tag{7.2}$$

and quite generally define for an arbitrary functional $G[A^b_\mu]$:

$$\Delta_G[A^{\mu a}]=\det\left|\frac{\partial}{\partial\theta_b}G[T^\theta_\mu[A^{\sigma a}]]\right|_{G=0}\,. \tag{7.3}$$

As in (4.31) we make use of the identity that for an arbitrary functional $H[f^a]$

$$\delta\left[H\left[(-i)\frac{\delta}{\delta K^a}\right]\right]\delta(K^b)=e^{iK^a\theta_a(H)}\det\left|\frac{\partial H[\theta^a]}{\partial\theta_b}\right|^{-1}_{H=0}\,. \tag{7.4}$$

The gauge invariance of $\Delta_C[A^{\mu a}]$ and $\Delta_G[A^{\mu a}]$ are evident from (7.4) upon setting $K^a=0$ in the latter. We may then repeat the analysis given through (4.32)—(4.36) to write

$$\delta\left[-ia^\mu\frac{\delta}{\delta J^{\mu b}}\right]\delta(J^{\sigma c})=e^{iJ^\mu_a T^{\theta_0}_\mu[(-i)\delta/\delta j^{\sigma a}]}\delta\left[G\left[(-i)\frac{\delta}{\delta J^{\sigma a}}\right]\right]\Delta_G\left[(-i)\frac{\delta}{\delta j^{\mu b}}\right]\delta(j^{\sigma c})\bigg|_{j^{\sigma c}=0}\,. \tag{7.5}$$

In particular if

$$G[A^{\mu b}]=\tilde{a}^\mu A^b_\mu+\Lambda^b \tag{7.6}$$

then $\delta(G[(-i)\delta/\delta j^{\mu b}])=\delta(-i\tilde{a}^\mu\delta/\delta j^{\mu b}+\Lambda^b)$, and $\Lambda^b(x)$ is an arbitrary function of x, with $\tilde{a}^\mu=\partial^\mu/\Box$, $a^\mu T^{\theta_0}_\mu[(-i)\delta/\delta j^{\mu b}]=0$. From (7.3) we also have, in a standard manner,

$$\delta\left[-i\tilde{a}^\mu\frac{\delta}{\delta j^{\mu b}}+\Lambda^b\right]\Delta_G\left[(-i)\frac{\delta}{\delta j^{\mu a}}\right]=\delta\left[-i\tilde{a}^\mu\frac{\delta}{\delta j^{\mu b}}+\Lambda^b\right]\exp\left[\operatorname{Tr}\ln\left[\delta^{ab}+g_0 f^{acb}\frac{1}{\Box}(-i)\frac{\delta}{\delta j^{\mu c}}\partial^\mu\right]\right]\,. \tag{7.7}$$

We may work out θ_0 as a Taylor-series expansion in $(-i)\delta/\delta j^{\mu a}$:

$$T_\mu^\theta\left[(-i)\frac{\delta}{\delta j^{\sigma a}}\right]=(-i)\frac{\delta}{\delta j^{\mu a}}+g_0 f^{abc}\theta^b(-i)\frac{\delta}{\delta j^{\mu c}}+\partial_\mu\theta^a+O(\theta^2)\ , \quad (7.8)$$

$$a^\mu T_\mu^\theta\left[(-i)\frac{\delta}{\delta j^{\sigma a}}\right]=-ia^\mu\frac{\delta}{\delta j^{\mu a}}+g_0 f^{abc}a^\mu\theta^b(-i)\frac{\delta}{\delta j^{\mu c}}+\theta^a+\cdots \quad (7.9)$$

or

$$\theta_0^b=ia^\mu\frac{\delta}{\delta j^{\mu b}}+O\left[\left[\frac{\delta}{\delta j}\right]^2\right]\ , \quad (7.10)$$

$$T_\mu^{\theta_0}\left[-i\frac{\delta}{\delta j^{\sigma a}}\right]=-i\frac{\delta}{\delta j^{\mu a}}+i\partial_\mu a^\sigma\frac{\delta}{\delta j^{\sigma a}}+O\left[\left[\frac{\delta}{\delta j}\right]^2\right]\ , \quad (7.11)$$

and

$$J^{\mu a}T_\mu^{\theta_0}\left[(-i)\frac{\delta}{\delta j^{\sigma a}}\right]=[J^{\mu a}-(a^\mu\partial_\sigma J^{\sigma a})](-i)\frac{\delta}{\delta j^{\mu a}}+J^{\mu a}O\left[\left[\frac{\delta}{\delta j}\right]^2\right]\ . \quad (7.12)$$

By making use in the process of the gauge invariance property of $\mathscr{A}$ we may then write as in (4.13)

$$\langle 0_+|0_-\rangle=e^{iW'}e^{(i\mathscr{A}')}\exp\left[\mathrm{Tr}\ln\left[\delta^{ab}+g_0 f^{acb}\frac{1}{\Box}(-i)\frac{\delta}{\delta j^{\mu c}}\partial^\mu\right]\right]\delta\left[-i\tilde{a}^\mu\frac{\delta}{\delta j^{\mu b}}+\Lambda^b\right]\langle 0_+|0_-\rangle_0\bigg|_{j^\sigma=0,\bar{\rho}=0,\rho=0}\ , \quad (7.13)$$

where

$$W'=\int(dx)\bar{\eta}(x)\exp\left\{t^b\left[-g_0 a^\mu\frac{\delta}{\delta j^{\mu b}}+O\left[\left[\frac{\delta}{\delta j}\right]^2\right]\right]\right\}(-i)\frac{\delta}{\delta\bar{\rho}(x)}+\int(dx)(-i)\frac{\delta}{\delta\rho(x)}\exp\left\{t^b\left[g_0 a^\mu\frac{\delta}{\delta j^{\mu b}}+O\left[\left[\frac{\delta}{\delta j}\right]^2\right]\right]\right\}\eta(x)+\int(dx)\left[\{J^{\mu a}(x)-[a^\mu\partial_\sigma J^{\sigma b}(x)]\}(-i)\frac{\delta}{\delta j^{\mu a}(x)}+J^{\mu a}O\left[\left[\frac{\delta}{\delta j}\right]^2\right]\right]\ , \quad (7.14)$$

and $\mathscr{A}'$ is written in terms of $A^{\mu a}\to(-i)\delta/\delta j^{\mu a}$, $\psi\to(-i)\delta/\delta\bar{\rho}$, $\bar{\psi}\to(-i)\delta/\delta\rho$. Making use of the identity (4.15), and the relation (4.16), and using the fact that $\langle 0_+|0_-\rangle$ in (7.13) is independent of Λ, we have upon applying any functional differential operator of the form in (4.20) to it, and in particular the bilinear expression in (4.23):

$$\langle 0_+|0_-\rangle=e^{iW'}\exp\left[i\int\mathscr{L}_I'\right]\exp\left[\mathrm{Tr}\ln\left[\delta^{ab}+g_0 f^{acb}\frac{1}{\Box}(-i)\frac{\delta}{\delta j^{\mu c}}\partial^\mu\right]\right]F_M[j^{\mu a},\bar{\rho},\rho]\bigg|_{j^{\sigma a}=0,\bar{\rho}=0,\rho=0}\ , \quad (7.15)$$

where

$$F_M[j^{\mu a},\bar{\rho},\rho]=\exp(i\bar{\rho}S_+\rho)\exp\left[\frac{i}{2}j^{\mu a}D_{\mu\nu M}^{ab}j^{\nu b}\right] \quad (7.16)$$

with

$$D_{\mu\nu M}^{ab}=\left[g_{\mu\nu}-\frac{q_\mu q_\nu}{q^2}\right]\frac{\delta^{ab}}{q^2-i\epsilon}-q_\mu q_\nu M^{ab}(q)\ . \quad (7.17)$$

Even more general gauge transformations may be considered by considering more complicated structures for $H[\delta/\delta\Lambda]$ in (4.20).

Note added in proof. A symmetrization over the product of the fields in the Lagrangian densities (2.1) and (5.1) (and the field equations) is understood as symmetric average limits of time-ordered products at the same space-time point, consistent with the action principle. That is, in particular:

$$\bar{\psi}O\psi\to\tfrac{1}{2}[\bar{\psi}O,\psi]\equiv\tfrac{1}{2}[(\bar{\psi}O)_\alpha\psi_\alpha-\psi_\alpha(\bar{\psi}O)_\alpha]$$

and

$$\bar{\psi}O\psi A^\mu\to\tfrac{1}{4}\{[\bar{\psi}O,\psi],A^\mu\}\ ,$$

where $\{A,B\}\equiv AB+BA$.

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APPENDIX: NOTE ON $\langle 0_+|0_-\rangle_0$

The functional differential equations of the functional $\langle 0_+|0_-\rangle_0$ are given in (3.1)—(3.4). Equation (3.4) may be rewritten in the form in (2.16) as

$$\partial_\mu F'^{\mu k a}\langle 0_+|0_-\rangle_0=-\left[g^{k\sigma}-\frac{\partial^k\partial^\sigma}{\vec{\partial}^2}\right]J_\sigma^a\langle 0_+|0_-\rangle_0\ , \quad (\mathrm{A1})$$

where we have also introduced the index a to deal with the Yang-Mills case. Equation (3.3) together with (A1) then gives

$$\vec{\partial}^{\,2}(-i)\frac{\delta}{\delta J_0^a}\langle 0_+|0_-\rangle=-\langle 0_+|0_-\rangle J^{0a}\,, \tag{A2}$$

$$\Box(-i)\frac{\delta}{\delta J^{ka}}\langle 0_+|0_-\rangle_0=-\frac{\partial^k\partial_0}{\vec{\partial}^{\,2}}J^{0a}\langle 0_+|0_-\rangle_0 - \left[g^{k\sigma}-\frac{\partial^k\partial^\sigma}{\vec{\partial}^{\,2}}\right]J_\sigma^a\langle 0_+|0_-\rangle_0\,. \tag{A3}$$

These together with Eqs. (3.1) and (3.2) may be integrated to give

$$\langle 0_+|0_-\rangle_0=\exp(i\bar{\eta}S_+\eta)+\exp\left[\frac{i}{2}J^{\mu a}D_{\mu\nu}^{Cab}J^{\nu b}\right], \tag{A4}$$

where

$$S_+(p)=\frac{-\gamma p+m_0}{p^2+m_0{}^2-i\epsilon}\,, \tag{A5}$$

$$D_{\mu\nu}^{Cab}=\delta^{ab}D_{\mu\nu}^{C}\,, \tag{A6}$$

$$D_{00}^{C}(q)=-\frac{1}{\mathbf{q}^2},\quad D_{0k}^{C}(q)=0=D_{k0}^{C}(q)\,, \tag{A7}$$

$$D_{km}^{C}(q)=\left[g_{km}-\frac{q_kq_m}{\mathbf{q}^2}\right]\frac{1}{q^2-i\epsilon}\,. \tag{A8}$$

Finally we consider the following functional:

$$Z[J]=\exp(i\mathscr{A}'_{02})\delta\left[\partial^\mu(-i)\frac{\delta}{\delta J^\mu}\right]\delta(J^\sigma)\,, \tag{A9}$$

where $\mathscr{A}'_{02}$ is defined in (3.7). We use the identity

$$\delta\left[\partial^\mu(-i)\frac{\delta}{\delta J^\mu}\right]\delta(J^\sigma)=\text{const}\times\delta(J^\lambda-\partial_3{}^{-1}\partial^\lambda J^3)\,,\quad \lambda=0,1,2\,, \tag{A10}$$

to write the functional differential equation for $Z[J]$:

$$e^{i\mathscr{A}'_{02}}(J^\lambda-\partial_3{}^{-1}\partial^\lambda J^3)e^{-i\mathscr{A}'_{02}}Z[J]=0\,. \tag{A11}$$

Using the equality in (3.16), we may rewrite (A11) as

$$\left[g^{\sigma\lambda}-\frac{g^{\sigma 3}\partial^\lambda}{\partial^3}\right]\partial^\mu F'_{\mu\sigma}Z[J]=-\left[g^{\sigma\lambda}-\frac{g^{\sigma 3}\partial^\lambda}{\partial_3}\right]J_\sigma Z[J]\,. \tag{A12}$$

By using the fact that

$$\partial^\mu(-i)\frac{\delta}{\delta J^\mu}Z[J]=0\,,$$

we obtain, from (A12),

$$\Box(-i)\frac{\delta}{\delta J_\nu}Z[J]=-\left[g^{\nu\mu}-\frac{\partial^\nu\partial^\mu}{\Box}\right]J_\mu Z[J]\,, \tag{A13}$$

whose solution is

$$Z[J]=\exp\left[\frac{i}{2}\int J^\nu\left[g_{\nu\mu}-\frac{\partial_\nu\partial_\mu}{\Box}\right]\frac{1}{-\Box-i\epsilon}J^\mu\right]. \tag{A14}$$

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