Repeated measurements in stochastic mechanics

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Stochastic mechanics provides a probabilistic scheme for the description of quantum systems. Grabert, Hänggi, and Talkner, and Nelson have pointed out that its multitime correlations seem to be in disagreement with quantum-mechanical predictions. We show that these difficulties are removed upon a careful analysis of repeated measurements in stochastic mechanics. The wave-packet reduction is naturally described in the stochastic framework, and the predictions for repeated measurements consequently agree in both theories.

I. INTRODUCTION

Since the very beginnings of quantum mechanics a partial formal similarity to statistical phenomena was noticed (see Schrödinger,[1] Fürth,[2] and Jammer[3]). The theory of stochastic mechanics (see Fényes,[4] Nelson,[5–7] and Guerra[8]) is one of such attempts at describing quantum phenomena in terms of stochastic processes. Of course, it has been an interesting and natural question whether the results of stochastic mechanics are consistent with those obtained in the functional analytic approach to quantum mechanics. It was clear from the seminal work of Fürth and Nelson that, in the sense of *average values* of position measurements performed at a fixed instant of time, stochastic mechanics and quantum mechanics make the same predictions. Even more generally, the spatial *probability distributions* coincide in the two theories at any moment.

This correspondence is lost when one is considering more complicated observables because the respective theory may not be adapted to a particular observable. For instance, stochastic mechanics seems not to admit natural representatives of nonconfigurational observables (such as momentum or energy) (see Golin[9]), whereas first hitting times have not been formulated unambiguously in quantum mechanics.

It has been under discussion whether the two theories could be consistent at all (see Mielnik and Tengstrand[10]). In any case, the results of experiments involving momentum, energy, etc., can be found correctly in the stochastic frame. On the other hand, Werner[11] argues that the situation is different when dealing with first hitting times insofar as the stochastically defined hitting times are not equal to any of those defined in conventional quantum mechanics. Grabert, Hänggi, and Talkner,[12] and Nelson[13] looked into the question of multitime correlations and concluded that stochastic mechanics disagrees with quantum-mechanical predictions. It is the aim of our work to explain why this is not so. Given the fact that the stochastic mechanical drifts have a state-dependent content, one realizes that new drifts naturally appear after a measurement on the system at hand has been performed. Thus, by the very nature of stochastic mechanics, one obtains new stochastic processes describing the system after measurement. Our analysis causes the difficulties with repeated measurements in stochastic mechanics to disappear and we obtain conclusions different from Refs. 12 and 13. We will present a thorough discussion in the case of multitime correlations. In a forthcoming paper the question of first hitting times in stochastic mechanics is to be taken up. In any case, we hold the view that the ideas contained in the main body of the present paper indicate that stochastic mechanics is a theory consistent with the predictions of conventional quantum mechanics.

The organization of the paper is as follows. In Sec. II we explain how the apparent contradictions between stochastic mechanics and quantum mechanics come about, and then they are resolved in Sec. III by a proper treatment of the wave-packet reduction in stochastic mechanics. We compare the concepts of measurement and preparation in the two theories in Sec. IV.

II. SOME APPARENT PARADOXES

Stochastic mechanics associates a diffusion process ξ_t in configuration space to the nonrelativistic motion of a quantum system. (Originally a finite number of degrees of freedom was assumed, but later a stochastic field theory was also formulated.[8]) We will present a brief account of stochastic mechanics in Sec. III. The position observable, having a natural representative in this scheme, plays an elevated role. By construction the probability density $\rho(x,t)$ of the position variable ξ_t is equal to the spatial probability density $|\psi(x,t)|^2$ of conventional quantum mechanics. (This is explained in more detail in the next section.) In particular, the expectation value $E(\xi_t)$ of the diffusion coincides with the quantum-mechanical expectation value $\langle X\rangle = \langle \psi(\cdot,t), X\psi(\cdot,t)\rangle$ of the position operator X (X is just multiplication with the spatial coordinate x). In other words, the two theories yield the same predictions for the average position provided one deals with operations at a fixed single instant of time. This correspondence was one of the basic features of the stochastic theory that led to the belief of some people that quantum systems might be correctly described in

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terms of stochastic mechanics.

However, an analysis of multitime correlations (see Grabert, Hänggi, and Talkner,[12] and Nelson[13]) seems to indicate that stochastic mechanics violates quantum mechanics. The common idea behind the construction of these *apparent* paradoxes is to consider at least two quantum-mechanical operators which are supposed to have obvious analogues in stochastic mechanics. In particular, if the two operators commute the expectation value of their product is the unique candidate for the quantum correlation. It turns out, however, that the corresponding stochastic mechanical correlation has a different value. The occurrence of these contradictions will be illustrated in the following by means of two examples.

Example 1. Let us consider a one-dimensional harmonic oscillator. Now suppose that we want to measure its position ξ_0 at time 0, and its position at a later time t. Having done this we might be interested in the corresponding two-time correlation $E(\xi_0\xi_t)$. If, for simplicity, we consider the ground state, the corresponding stochastic differential equation

$$d\xi_t = -\omega_0\xi_t + dw_t \ , \tag{1}$$

where w_t denotes a standard Wiener process with variance $2\nu = \hbar/m$ (cf. Sec. III), is linear and can be solved by quadrature:

$$\xi_t = e^{-\omega_0 t}\left[\xi_0 + \int_0^t e^{\omega_0 z} dw_z\right] \quad (t \geq 0) \ . \tag{2}$$

Here $\omega_0 > 0$ denotes the frequency of the oscillator, $m > 0$ is the particle mass, and $\hbar$ is Planck's constant divided by 2π. Let $\sigma^2 := \hbar/2m\omega_0$. Then the autocorrelation function follows immediately from (2):

$$E(\xi_0\xi_t) = \sigma^2 e^{-\omega_0 t} \quad (t \geq 0) \ . \tag{3}$$

It is to be noted that this correlation, which is just the famous Ornstein-Uhlenbeck correlation, does not exhibit the periodic character of the ground state.

Now let us consider the corresponding quantum-mechanical prediction. The Hamiltonian H of the system is given by

$$H = -\frac{\hbar^2}{2m}\partial_x{}^2 + \frac{m\omega_0{}^2}{2}x^2 \ , \tag{4}$$

and in the Heisenberg picture the position operator at time t is just

$$X_t := e^{i(t/\hbar)H} X e^{-i(t/\hbar)H} \ . \tag{5}$$

For the harmonic oscillator one finds explicitly

$$X_t = \cos\omega_0 t X + \frac{\sin\omega_0 t}{m\omega_0} P \ , \tag{6}$$

where $P := -i\hbar\partial_x$ is the momentum operator. Obviously, the commutator

$$[X, X_t] := XX_t - X_tX = i\hbar\frac{\sin\omega_0 t}{m\omega_0} \tag{7}$$

vanishes whenever $t = n\pi/\omega_0$ for some $n \in Z$. Now let us suppose that t is of this form and that the system is in the ground state

$$\psi(x,t) = (2\pi\sigma^2)^{-1/4}\exp\left[-\frac{1}{2}\left[i\omega_0 t + \frac{x^2}{2\sigma^2}\right]\right] \ . \tag{8}$$

Since X and X_t commute the mathematical correlation

$$\langle XX_t \rangle = (-1)^n\sigma^2 \tag{9}$$

corresponds to the actual, experimentally measurable correlation. Clearly this correlation does show the periodicity of the ground state, and it contradicts the result (3) of stochastic mechanics.

An example along the same lines is due to Nelson.[13] He considers two noninteracting harmonic oscillators with the same frequency. Since the two systems do not interact their position operators (at possibly different times) commute and, in fact, the above analysis carries over to this example. Therefore the same apparent disagreement is noted in Ref. 13.

Example 2. Now let a particle be given moving along the real line. We want to deal with a scattering state and fix the wave function at time 0 to be the Gaussian

$$\psi(x,0) = (2\pi a^2)^{-1/4} e^{-x^2/2a^2} \ , \tag{10}$$

for some $a > 0$. Being interesting in potential scattering we consider the incoming and outgoing Møller operators Ω_{in} and Ω_{out}, respectively. They allow us to define the kinetic energy of the particle in the remote past and the far future, which corresponds (up to the constant $1/2m$) to the operators

$$P_{\text{in}}{}^2 = \Omega_{\text{in}} P^2 \Omega_{\text{in}}^* \ , \tag{11a}$$

$$P_{\text{out}}{}^2 = \Omega_{\text{out}} P^2 \Omega_{\text{out}}^* \ . \tag{11b}$$

The elasticity of the scattering is reflected in the fact that these two operators commute (as a consequence of the intertwining relations):

$$[P_{\text{in}}{}^2, P_{\text{out}}{}^2] = 0 \ . \tag{12}$$

For simplicity we consider the most trivial case of zero potential, where $\Omega_{\text{in}} = \Omega_{\text{out}} = 1$ and $P_{\text{in}} = P_{\text{out}} = P$. One of the experimentally accessible quantities is the momentum and similarly the kinetic energy. The correlation for the kinetic energy at $t = -\infty$ and $t = +\infty$ is easily computed, since we are dealing with a freely evolving Gaussian wave function, viz.,

$$\langle P_{\text{in}}{}^2 P_{\text{out}}{}^2 \rangle = \langle P^4 \rangle = \frac{3}{4}\frac{\hbar^4}{a^4} \ . \tag{13}$$

The corresponding random variables are defined by

$$p_{\text{in}} := \lim_{t\to-\infty} \frac{m\xi_t}{t}, \quad p_{\text{out}} := \lim_{t\to+\infty} \frac{m\xi_t}{t} \ . \tag{14}$$

According to results by Shucker,[14] Biler,[15] Serva,[16] and Carlen[17] these limits exist with probability one and have the correct quantum-mechanical distributions. Since ξ_t is a Gaussian process, p_{in} and p_{out} are jointly Gaussian and their characteristic function

$$\varphi(x,y) := E(e^{i(xp_{\text{in}} + yp_{\text{out}})}) \tag{15}$$

is of the form

$$\exp\left[-\frac{1}{2}\begin{bmatrix}x\\y\end{bmatrix}V\begin{bmatrix}x\\y\end{bmatrix}\right].$$

The covariance matrix

$$V:=\begin{bmatrix}\mathrm{var}(p_{\rm in}) & \mathrm{cov}(p_{\rm in},p_{\rm out})\\ \mathrm{cov}(p_{\rm out},p_{\rm in}) & \mathrm{var}(p_{\rm out})\end{bmatrix} \tag{16}$$

is easily determined since due to the linearity of the stochastic differential equation for ξ_t the autocorrelation $E(\xi_s\xi_t)$ can be explicitly computed. (Details can be found, e.g., in Ref. 7.) It turns out that

$$V=\frac{\hbar^2}{4a^2}\begin{bmatrix}1 & -e^{-\pi}\\ -e^{-\pi} & 1\end{bmatrix}. \tag{17}$$

Therefore one obtains

$$E(p_{\rm in}{}^2p_{\rm out}{}^2)=\partial_x{}^2\partial_y{}^2\varphi(0,0)$$
$$=\left[\frac{\hbar}{2a}\right]^4(1+2e^{-2\pi}). \tag{18}$$

Again, this result differs from the quantum-mechanical correlation (13).

III. RESOLUTION OF THE PARADOXES

As we have seen in the preceding section there seem to be severe difficulties in stochastic mechanics on the level of multitime correlations. Using arguments in the same spirit Grabert, Hänggi, and Talkner,[12] and Nelson[13] arrived at the conclusion that stochastic mechanics being in disagreement with some predictions of quantum mechanics could no longer be viewed as an acceptable model for a description of quantum phenomena. Do we have to discard stochastic mechanics now?

Stochastic mechanics can be viewed as an attempt at describing quantum phenomena in classical terms. In contrast with quantum mechanics a particular theory of measurements should therefore not be needed. Of course, the generic occurrence of quantum-mechanical interference of wave functions must give rise to probabilistic consequences. First accounts of this are due to Shucker[18] and Guerra.[19]

In this section our main objective is to implement repeated measurements in the stochastic framework. Our argument will show that—even on the level of multitime correlations—there is no inconsistency in the measurable predictions of stochastic mechanics and quantum mechanics.

We recall that the description of quantum systems in stochastic mechanics can essentially be divided into two parts. The kinematical content is based on formalizing the quantum motion as the diffusion process ξ_t governed by the stochastic differential equation

$$d\xi_t=b(\xi_t,t)dt+dw_t\ , \tag{19}$$

where w_t denotes a standard Wiener process with variance 2ν (ν being some positive constant). To complete the kinematical picture an initial condition for ξ_0 has to be added. Of course, this distribution is taken to be identical to the quantum-mechanical one at time 0.

On the other hand, the dynamics has to incorporate the influence of the potential. It is required to be some kind of generalization of classical dynamics. This can either be achieved by a Newton law in the mean[5,6] or by a stochastic variational principle.[7,20] As a result of this, the osmotic velocity

$$u(x,t):=\nu\,\mathrm{grad}\ln\rho(x,t) \tag{20}$$

and the current velocity

$$v(x,t):=b(x,t)-u(x,t) \tag{21}$$

satisfy the coupled partial-differential equations

$$\partial_t u=-\nu\,\mathrm{grad}\,\mathrm{div}v-\mathrm{grad}(u\cdot v)\ , \tag{22a}$$

$$\partial_t v=-\frac{1}{m}\mathrm{grad}V-(v\cdot\mathrm{grad})v+(u\cdot\mathrm{grad}u)+\nu\Delta u\ , \tag{22b}$$

and ν is now specified to be equal to $\hbar/2m$. If one adds some initial conditions $u(\cdot,t_0)$ and $v(\cdot,t_0)$ at some time t_0, then Eqs. (22) fully describe the evolution of the stochastic system. The correspondence to the normalized quantum-mechanical wave function

$$\psi(x,t)=e^{R(x,t)+iS(x,t)}\ , \tag{23}$$

where R and S are real-valued functions, follows from

$$u(x,t)=\frac{\hbar}{m}\mathrm{grad}R(x,t)\ , \tag{24}$$

$$v(x,t)=\frac{\hbar}{m}\mathrm{grad}S(x,t)\ , \tag{25}$$

and the probability density $\rho(x,t)$ of the diffusion process ξ_t is given by

$$\rho(x,t)=|\psi(x,t)|^2\ . \tag{26}$$

From the viewpoint of stochastic mechanics the Schrödinger equation

$$i\hbar\partial_t\psi(x,t)=\left[-\frac{\hbar^2}{2m}\Delta+V(x)\right]\psi(x,t) \tag{27}$$

appears as a linear reformulation of (22) on account of hypothesis (25). It follows from (21), (24), and (25) that the drift satisfies

$$b(x,t)=\frac{\hbar}{m}(\mathrm{Re}+\mathrm{Im})\mathrm{grad}\ln\psi(x,t)\ . \tag{28}$$

Formula (28) shows the dependence on the state and the dynamical content of the drift explicitly. In applications, this formula is usually applied to calculate the drift rather than obtaining $b=u+v$ by solving the coupled equations (22).

We now turn to the question of repeated measurements in stochastic mechanics. Suppose we are dealing with a quantum system whose time evolution is given by (19). Moreover let us carry out an ideal position measurement at time 0. The result of this operation is twofold: on one hand, the process is fixed to a single point, say x_0; on the other hand, this new initial condition appears in a new drift $b^{x_0}=u^{x_0}+v^{x_0}$ [which comes from (22) with the corresponding new initial conditions related to the new density]. So the system is appropriately described after the measurement procedure by a family (indexed by x_0) of new stochastic differential equations

$$d\xi_t^{x_0}=b^{x_0}(\xi_t^{x_0},t)dt+dw_t^{x_0} \quad (t>0) , \tag{29a}$$

$$\lim_{t\downarrow 0}\xi_t^{x_0}=x_0 \text{ almost surely} , \tag{29b}$$

where $w_t^{x_0}$, $t\geq 0$, is a possibly new standard Wiener process (again with variance 2ν) and starting from zero, and with increments independent of those of w_t, $t\leq 0$. The probabilistic information about a repeated measurement at an instant $t>0$ is entirely contained in $\xi_t^{x_0}$, whereas in this context ξ_t is of no significance whatsoever.

We want to stress that, taking stochastic mechanics serious, a new stochastic process has to be introduced after measurement. According to our point of view it cannot be argued that such a procedure represents a modification of stochastic mechanics. On the contrary, the stochastic mechanical drifts are functionals of the state (which is not the case in the classical theory of diffusion), and it is this characteristic (though nonclassical) feature which we insist upon when dealing with repeated measurements. It is well known that not all quantum-mechanical aspects can be accounted for by means of a classical probabilistic theory. For instance, the double-slit experiment gives a contradiction to the Bayes rule of composite conditional expectations.

What happens, in comparison, upon a measurement in conventional quantum mechanics? There, an ideal measurement is thought of as an interaction with a classical macroscopic object (the measuring apparatus) instantaneously leading to the so-called *wave-packet reduction.* This means that upon making a record of the position at time 0 the wave function ψ determined by (27) collapses to the point were the system is found. The further time evolution is again subject to the same Schrödinger equation

$$i\hbar\partial_t\phi^{x_0}(x,t)=\left[-\frac{\hbar^2}{2m}\Delta+V(x)\right]\phi^{x_0}(x,t) \quad (t>0) , \tag{30a}$$

but with a new initial condition given by

$$\lim_{t\downarrow 0}\phi^{x_0}(x,t)=\delta(x-x_0) . \tag{30b}$$

In order to construct a process $\xi_t^{x_0}$ one has to determine the corresponding drift b^{x_0}. To achieve this in an accurate way one has to define and solve an initial-value problem in the hydrodynamical picture (22), i.e., in terms of the evolution equations for the osmotic and current velocities. This is probably a hard mathematical problem and has not been tackled yet. At present we have to circumvent this problem by solving the Schrödinger equation (30) for the collapsed wave packet. The drift b^{x_0} is then given by

$$b^{x_0}(x,t)=\frac{\hbar}{m}(\mathrm{Re}+\mathrm{Im})\,\mathrm{grad}\ln\phi^{x_0}(x,t) . \tag{31}$$

Intuitively, the collapse of the wave function means that at the time when the measurement is performed the quantum system has a definite configuration. In quantum mechanics this is expressed by saying that the wave function of the system is a δ distribution, whereas in stochastic mechanics we fixed the density (i.e., the square modulus of the wave function) to be a δ distribution. We hold the view that it is physically more plausible to require the density to have this property. Unfortunately, neither the square nor the square root of a δ distribution is defined in a mathematically rigorous way. Therefore, from the mathematical standpoint, there is still some work to be done in order to relate the wave-packet reduction as described in the functional analytic and the stochastic framework, respectively. The hydrodynamical picture may probably turn out to be the appropriate intermediate step.

Now these considerations will be implemented in the case of two-time correlations. Obviously, $E(\xi_0\xi_t)$ cannot be the right object to have an interpretation as a physical position correlation, since the wave-packet reduction, which enters stochastic mechanics so naturally, has not been taken care of. Assuming that we have obtained the value x_0 upon the first measurement of time 0 we want to record the position at a later time $t>0$ again and the mean correlation for a system conditioned to start from x_0 at time 0 is equal to

$$x_0\int dP(\omega)\xi_t^{x_0}(\omega) , \tag{32}$$

where P is the probability measure on the (fixed) probability space we are working on. The weight of the initial points x_0 is determined by the diffusion process before measurement. Having denoted its density by $\rho(x_0,0)$ we obtain, upon averaging the stochastic mechanical prediction,

$$\int dx_0\rho(x_0,0)x_0\int dP(\omega)\xi_t^{x_0}(\omega) = \int\int dP(\omega')dP(\omega)\xi_0(\omega')\xi_t^{\xi_0(\omega')}(\omega) \tag{33}$$

[as opposed to $E(\xi_0\xi_t)$] for the full correlation.

Similarly, in quantum mechanics multitime correlations cannot in general be associated with a single expectation of some self-adjoint operator. For instance, in the case of repeated measurements XX_t is not self-adjoint since X and X_t do not commute (apart from exceptional cases), and even the symmetric product $\frac{1}{2}(XX_t+X_tX)$ does not give the right expectations. Therefore the two-time correlation has to be given more explicitly, viz., by

$$\int\int dx_0dx_1\,|\psi(x_00)|^2x_0\mathrm{Prob}(x_1,t;x_0,0)x_1 , \tag{34}$$

where $\mathrm{Prob}(x_1t;x_0,0)$ is the quantum-mechanical probability of finding the system in x_1 at time t conditioned to having been in x_0 at time 0. It is at this point that the wave-packet reduction enters the correlation

$$\mathrm{Prob}(x,t;x_0,0)=|\langle\delta(\cdot,-x),\phi^{x_0}(\cdot,t)\rangle|^2 = |\phi^{x_0}(x,t)|^2 . \tag{35}$$

Note that this is only a formal transition probability because the δ function is not in L^2. To be rigorous one should work in terms of L^2 approximations.

In order to prove that the quantum-mechanical correlation (34) coincides with the stochastic mechanical one (33) we only need to show that

$$\int dP(\omega)\xi_t^{x_0}(\omega)=\int dx\,\text{Prob}(x,t;x_0,0)x\ . \tag{36}$$

But this is easily accomplished by checking that $\text{Prob}(x,t;x_0,0)$ is just the probability density $\rho^{x_0}(x,t)$ of $\xi_t^{x_0}$ because [on account of (30) and (35)] it satisfies the Fokker-Planck equation

$$\partial_t \text{Prob}(x,t;x,0)=-\text{div}_x[\text{Prob}(x,t;x_0,0)b^{x_0}(x,t)] + \nu\Delta_x \text{Prob}(x,t;x_0,0) \tag{37a}$$

subject to the initial condition

$$\lim_{t\downarrow 0}\text{Prob}(x,t;x_0,0)=\delta(x-x_0)\ . \tag{37b}$$

Since

$$\int dP(\omega)\xi_t^{x_0}(\omega)=\int dx\,\rho^{x_0}(x,t)x \times \int dx\,\text{Prob}(x,t;x_0,0)x\ , \tag{38}$$

(36) is established.

Next we take up again the examples of Sec. II so as to elucidate our result a little more.

Example 1 (reexamined). In this case the new processes $\xi_t^{x_0}$ can be explicitly determined. The Hamiltonian (4) admits a kernel for the semigroup $e^{-iHt/\hbar}$, viz.,

$$e^{-iHt/\hbar}(x,x')=\left[\frac{m\omega_0}{2\pi\hbar i\sin\omega_0 t}\right]^{1/2} \times\exp\left[-\frac{m\omega_0}{2\hbar}(x^2-x'^2) - \frac{m\omega_0}{\hbar}\frac{(e^{-i\omega_0 t}x-x')^2}{1-e^{-2i\omega_0 t}}\right]. \tag{39}$$

This is just the analytic continuation of Mehler's formula. Therefore the solution of the Schrödinger equation (30a) with initial condition (30b) is just

$$\phi^{x_0}(x,t)=\int dx' e^{-iHt/\hbar}(x,x')\phi^{x_0}(x',0) = e^{-iHt/\hbar}(x,x_0) \tag{40}$$

and the new drift [cf. Eq. (31)] is given by

$$b^{x_0}(x,t)=\omega_0\left[\frac{x}{\tan\omega_0 t}-\frac{x_0}{\sin\omega_0 t}\right]. \tag{41}$$

(In fact, the osmotic velocity vanishes identically.) Consequently the stochastic differential equation (29) is linear and can be solved by quadrature:

$$\xi_t^{x_0}=(\cos\omega_0 t-\sin\omega_0 t\cot\omega_0 s)x_0 + \frac{\sin\omega_0 t}{\sin\omega_0 s}\xi_s^{x_0}+\sin\omega_0 t\int_s^t\frac{dw_z^{x_0}}{\sin\omega_0 z}\ , \tag{42}$$

$0<s\leq t$. For t being a multiple of π/ω_0, i.e., $t=n\pi/\omega_0$, the random variable $\xi_t^{x_0}$ is just the constant $(-1)^n x_0$ (almost surely). Thus the correlation (33) is simply

$$(-1)^n\int dx_0\rho(x_0,0)x_0{}^2=(-1)^n E(\xi_0{}^2)\ , \tag{43}$$

and it coincides with the quantum correlation (9).

It is an easy matter to convince oneself that Nelson's example (two dynamically uncoupled harmonic oscillators) can be treated in an analogous fashion; and the paradox can be removed there, too.

Example 2 (reexamined). The time limiting procedure connected with this example gives rise to some complications, although—as we shall see—the main ideas of this section go through in a straightforward manner. Let us imagine having performed a measurement of momentum in the remote past. (In fact, the outcome of a position measurement divided by the time of measurement gives the corresponding velocity asymptotically.) This disturbing operation originates in a family $\eta_t^{p_0}$ of processes each of which is associated with an incoming momentum p_0 at time $t=-\infty$. Physically speaking, this preparation could not have taken place at $t=-\infty$ exactly, so it is perfectly reasonable to think of an unperturbed process ξ_t having existed before the measurement. In analogy with example 1, the stochastic mechanical correlation is taken to be

$$\int dp_0\tilde{\rho}(p_0)p_0\int dP(\omega)\lim_{t\to\infty}\frac{m\eta_t^{p_0}}{t}\ , \tag{44}$$

where $\tilde{\rho}(p_0)$ is the density of the initial momentum; i.e., if $\tilde{\psi}(p_0,t)$ is the Fourier transform of $\psi(x,t)$,

$$\tilde{\psi}(p_0,t)=(2\pi\hbar)^{-1/2}\int dx\,e^{-i(p_0x/\hbar)}\psi(x,t)\ , \tag{45}$$

then

$$\tilde{\rho}(p_0):=\lim_{t\to-\infty}|\tilde{\psi}(p_0,t)|^2\ . \tag{46}$$

In order to be dealing with proper densities we can approximate the δ distribution by some L^1 function. Correspondingly the improved wave function $\phi^{p_0}(x,t)$ [where $\lim_{t\to-\infty}\widetilde{\phi^{p_0}}(p,t)=\delta(p-p_0)$] may be approximated by L^2 functions. By mimicking the proof of (36) one can show that the quantum correlation $\langle P_{\text{in}}{}^2 P_{\text{out}}{}^2\rangle$ coincides with the stochastic mechanical correlation (44). In fact, this proof is not restricted to the special form (10) of the wave function, nor is it restricted to the case of zero potential (i.e., free dynamics).

As a last remark we note that so far we have only considered correlations for just two different times. The above considerations generalize, however, in an obvious manner, to multitime correlations.

IV. MEASUREMENT AND PREPARATION

To get a better understanding of the implications of the wave-packet reduction let us consider again a wave function ψ and a position measurement at time 0, and suppose that the resulting value is x_0.

As we have seen the measurable predictions about the future coincide in the stochastic and the functional analytic scheme. But if we look to the past we have different degrees of knowledge. In fact, in quantum mechanics we can only make a statement about the probability density

$\rho(x,t)=|\psi(x,t)|^2$, $t<0$, whereas in stochastic mechanics the existence of the sample paths admits a probabilistic statement about a system that was in x at time $t<0$ to reach x_0 at time 0.

To put this in quantitative form we consider the backward drift

$$b_*(x,t):=\frac{\hbar}{m}(\mathrm{Im}-\mathrm{Re})\,\mathrm{grad}\,\ln\psi(x,t) \tag{47}$$

and then solve the associated backward Fokker-Planck equation (Kolmogorov forward equation)

$$\partial_t p_* = -\mathrm{div}_x(p_* b_*) - \nu\Delta p_* , \tag{48a}$$

where $p_* = p_*(x,t;x_0,0)$ is the probability to find a particle in x at time $t<0$ provided it will go to x_0 at time 0. Of course, p_* has to satisfy the condition

$$\lim_{t\uparrow 0} p_*(x,t;x_0,0)=\delta(x-x_0) . \tag{48b}$$

Thus having found the system in x_0 we have a probabilistic knowledge of where it had been before.

It is the classical nature of stochastic mechanics that is reflected here in a distinction between measurement and preparation. Measurement corresponds to a knowledge of past and future inferred from the presence, whereas preparation involves only predictions about future experiments. In the common interpretation of quantum mechanics measurement is identical to state preparation and there is no distinction drawn between the two concepts.

Unfortunately we cannot check the different description of the past in the two theories because if we tried to do so we would have to carry out measurements before time 0 without perturbing the state of the system. And that is, of course, in conflict with the occurrence of the wave-packet reduction. So we are really dealing with a peculiar theory: on one hand, stochastic mechanics claims the existence of trajectories; on the other hand, it provides reasons to make them hidden variables.

V. CONCLUSION

We have presented an exposition of how repeated measurements on quantum systems are properly dealt with in the framework of stochastic mechanics. The phenomenon of wave-packet reduction is thus fully incorporated into this theory without having any recourse to an external quantum-mechanical input. The relation of the notions of measurement and preparation was also discussed, and a comparison was made to the commonly accepted view in conventional quantum mechanics.

Stochastic mechanics lends itself to a description of the collapse of the wave functions inside a purely probabilistic framework. Indeed, we are tempted to maintain that the intrinsically irreversible nature of measuring operations emerges in a theory of diffusions in a more natural way than in conventional quantum mechanics. More precisely, this irreversible character is contained in the state dependence of the stochastic mechanical drift fields.

The dynamics of stochastic mechanics is Markovian by construction, i.e., the trajectories of stochastic mechanics have no memory: given the presence, predictions about the future do not depend on the past. But after an observation this feature is lost because the measuring device carries information about the observed system and thus serves as a memory. The joint system (observed system plus measuring device), however, is still Markovian.

In any case, the paradoxical aspects which appeared in the work of Grabert, Hänggi, and Talkner, and Nelson disappear if the dynamical structure of stochastic mechanics is carefully taken into account in dealing with repeated measurements. The predictions for correlation functions do not differ from those of quantum mechanics. The quantity $E(\xi_0\xi_t)$, although it is a well-defined object in this theory, is of purely mathematical nature but not a physical correlation.

It seems very hard to imagine an experiment that allows us to choose between stochastic mechanics and conventional quantum mechanics, although this statement should by no means imply the epistemological equivalence of these two theories.

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