Bound-state solutions, invariant scalar products, and conserved currents for a class of two-body relativistic systems

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The simplest two-body relativistic system with direct interaction, described by two first-class constraints, is investigated. After a description of the multitime approach (canonical quantization without gauge fixings), the two coupled integro-differential wave equations are solved. The elementary solutions for the bound states are found and are shown to transform as irreducible representations of the Poincaré group. Invariant scalar products are introduced assuring the unitarity of the representations and some of the associated conserved currents are discussed. The initial data problem has been solved by means of a quantum canonical transformation, which transforms the integro-differential equations into differential equations.

I. INTRODUCTION

There has recently been a renewed interest in the relativistic classical and quantum theories of particles in direct interaction.[1-6] Models for systems of two[7-16] interacting particles have been proposed by various authors, and applications to the phenomenology of the elementary particles have been attempted.[17,18] In particular, in Refs. 11 and 12 it has been shown that models of this kind, for two scalar particles or for two spin-$\frac{1}{2}$ particles, recover the quantum relativistic effects up to the α^4 order. Applications of these models to the determination of the masses of the mesons have been also tried, giving a very good fit.[17]

The problem of the foundation of a consistent theory of relativistic interacting particles, in the classical literature on relativity, was unsolved since the discovery of the no-interaction theorem.[19] After that time the main reason of the difficulties encountered was understood, and a correct approach to the problem became possible. As a consequence there has been a new flowering of papers on this subject.

For all these reasons there is now an interest on this kind of problem from a general point of view, but we may underscore other reasons for a thorough study of this matter.

In our opinion these reasons are essentially three. First of all, from a field-theoretical point of view, the two (or more) particle sector of the Fock space on which the field is defined and the corresponding problem connected with the presence of bound states, solutions of the Bethe-Salpeter equation,[20] can receive a clarification from this study.

As we will show, the analysis of the solutions of a simple system of two particles provides for a covariant basis for the bound states in configuration space, which is suited for a central force interaction, but which can be generalized to other kinds of interactions by a suitable generalization of the model.[11-13]

The state of a two-particle system transforms as a reducible representation of the Poincaré group. This basis provides for a reduction of this representation in terms of irreducible representations, each corresponding to a given mass and spin. The mass spectrum and its degeneracy in terms of the spin will of course be determined by the kind of interaction which was assumed. The states of this basis depend in a natural way on two times, and the problem of the physical interpretation of the relative time finds here its natural explanation, as will be more apparent in the next section. This seems to open the way to the solution to the analogue problem in the Bethe-Salpeter amplitude.

A second reason is that these theories provide for an approximation of the kernel of the Bethe-Salpeter equation,[11,21,22] and are related to the quasipotential approach[23] for weak potentials. As shown in Ref. 21 the Bethe-Salpeter equation can be replaced by two coupled integrable equations. Moreover in these models the instantaneous approximation to the Bethe-Salpeter kernel is expressed in a manifestly covariant relativistic form, and it is shown in Ref. 11 that for slow motion there is a canonical equivalence with the Darwin Hamiltonian, i.e., with the Fermi-Breit approximation to the Bethe-Salpeter equation. These models also provide for an instantaneous approximation, in the center-of-mass reference frame, of an action of the Fokker kind, at the classical level. This has an interest because it was shown in Ref. 24 that, at least for the QED field theory, a particle limit exists, and that in this limit the dynamics of the particles is governed by an action of the Fokker kind.

At last it appears that these theories are best studied in the framework of Dirac's theory of constraints.[25] In this formulation they exhibit a gauge invariance, the gauge transformations being generated by the constraints themselves, and this fact put these theories in a very strict analogy with all the gauge theories. They seem to be a good laboratory for the study of the applications of Dirac's theory, and to extend it to more general gauge field theories.

In Sec. II the two-particle system we intend to study is defined and discussed. In Sec. III we briefly review the Klein-Gordon theory for one and two free particles, with a discussion of the possible definitions of a scalar product. A possible model for the coupling of a two-particle system to an external field is given. In Sec. IV we give an analysis of the solutions of the Todorov-Komar model,[7,8] and the definition of various conserved scalar products. In Sec. V the transformation properties of the solutions of the model are given and it is shown that the scalar products defined in the preceding section are invariant under a Poincaré transformation. In Sec. VI the problem of conserved currents is discussed. Finally in Sec. VII the problem of the initial data is discussed, and it is shown that this problem is well defined for a set of transformed equations corresponding to a canonical transformation of the underlying classical constraints. In Appendix A, after the definition of the standard boost and of the polarization vectors for a timelike four-vector, a new set of canonical variables for the two particles is defined. In Appendix B a special set of canonical variables for the free scalar particle is introduced and in Appendix C a singular function connected with the quantization of the previous variables is studied. In Appendix D the variables of Appendix B are extended to the two-particle case to solve the problem discussed in Sec. VII.

II. THE MULTITIME APPROACH

We mentioned in the Introduction the no-interaction theorem,[19] and the possibility of a way out of its consequences.[1-14] Let us now discuss this point in more detail, from the point of view of the constraint theory.

Models with constraints avoid the consequences of this theorem by introducing eight phase-space variables for each particle instead of six: x_i^μ, p_i^μ ($i=1,\ldots,n$; n being the number of particles; $\mu=0,1,2,3$). The dynamics is given in terms of a set of n first-class constraints $\chi_i(x,p)$. That is, the condition $\{\chi_i,\chi_j\}=0$ should be satisfied when the constraint conditions $\chi_i=0$ hold ($\{p_i^\mu,x_j^\nu\}=\delta_{ij}\eta^{\mu\nu}$, $\eta\equiv(+,---)$).

The energies p_i^0 are determined by the constraint equations $\chi_i=0$, while the times x_i^0 are left as gauge variables. Gauge-fixing conditions are needed to determine them, so as to get a reduced phase space of six variables for each particle. Each choice of the gauge-fixing conditions determines a different physical model, or, in other words, a model given in terms of first-class constraints only does correspond to an infinity of physical models. This corresponds to the fact that the constraint equations $\chi_i=0$ by themselves determine an n-dimensional hypersurface on which the motion of the system takes place, and not just a line.

The problem of what are the physical quantities, or "observables," of the system arises.

The usual way to define the observables $A(x,p)$ is to require for them to be gauge invariant; that is, the $A(x,p)$ are the quantities which satisfy the condition $\{A,\chi_i\}=0$, the χ_i being at the same time the constraints and the generators of the gauge transformations.

Clearly, following this definition, the coordinates x_i^μ and p_i^μ in general are not observables. Thus the price we pay to avoid the no-interaction theorem is to lose contact with the quantities which are to be measured, either in classical or in quantum theories.

When a singular Lagrangian is available, giving rise to the constraints χ_i, the previous situation can be seen to correspond to a kind of degeneracy of the Cauchy problem in the configuration space. That is, many gauge equivalent configurations may evolve from the same set of Cauchy data.

To be more definite let us consider the two massive spinless particle system, whose dynamics is given in terms of the Todorov-Komar[7,8] first-class constraints:

$$\chi_1=p_1{}^2-m_1{}^2-V,\quad \chi_2=p_2{}^2-m_2{}^2-V\;, \tag{2.1}$$

where the first-class condition

$$\{\chi_1,\chi_2\}=0 \tag{2.2}$$

is satisfied if

$$V=V(r_\perp{}^2,p^2,q_\perp{}^2,(r_\perp,q_\perp),(p,q))\;, \tag{2.3}$$

where

$$\begin{aligned} &p^\mu=p_1^\mu+p_2^\mu,\quad q^\mu=\tfrac{1}{2}(p_1^\mu-p_2^\mu)\;,\\ &q_\perp^\mu=(\eta_\nu^\mu-p^\mu p_\nu/p^2)q^\nu,\quad r^\mu=x_1^\mu-x_2^\mu\;,\\ &r_\perp^\mu=(\eta_\nu^\mu-p^\mu p_\nu/p^2)r^\nu\;. \end{aligned} \tag{2.4}$$

From now on we will restrict our discussion to this model, in the case of a potential

$$V=V(r_\perp{}^2)\;. \tag{2.5}$$

A singular Lagrangian giving rise to the constraints (2.1) is unfortunately not known (see, however, Ref. 26). In any case we can visualize the gauge freedom of the model by considering the two linear combinations

$$\begin{aligned} \chi_+&=2(\chi_1+\chi_2)\\ &=p^2+4[q^2-V(r_\perp{}^2)]-2(m_1{}^2+m_2{}^2)\\ &=\frac{1}{p^2}\{[p^2-M_+{}^2(q_\perp,r_\perp)][p^2-M_-{}^2(q_\perp,r_\perp)]\\ &\qquad+4\chi_-(\chi_-+m_1{}^2-m_2{}^2)\}\;, \\ \chi_-&=\tfrac{1}{2}(\chi_1-\chi_2)=(p,q)-\tfrac{1}{2}(m_1{}^2-m_2{}^2)\;, \end{aligned} \tag{2.6}$$

where

$$\begin{aligned} &M_\pm(q_\perp,r_\perp)=M_1(q_\perp,r_\perp)\pm M_2(q_\perp,r_\perp)\;,\\ &M_i(q_\perp,r_\perp)=[m_i{}^2-q_\perp{}^2+V(r_\perp{}^2)]^{1/2},\quad i=1,2\;. \end{aligned} \tag{2.7}$$

In the form (2.6) the constraint equation $\chi_+=0$ determines the modification produced by the potential V to the free mass spectrum [which consists in four branches with thresholds $\pm(m_1+m_2)$ and pseudothresholds $\pm(m_1-m_2)$], while $\chi_-=0$ suggests that its conjugated coordinate (p,r) should be a gauge variable.

To each χ_i we now have to associate a gauge-fixing condition. The one associated to χ_- selects one pair of world lines on the world sheet, on which an infinity of gauge-equivalent pairs of world lines exists. This choice can also be interpreted as a statement on the time correlation between the two particles and therefore also as a statement about which Dirac's form of dynamics[27] has to be used (see Ref. 33). For instance, the choice $(p,r)=0$ amounts in stating that in the center-of-mass reference frame an instantaneous interaction takes place whose natural Dirac's form is the one employing spacelike surfaces.

The other choice, associated to χ_+, reflects the arbitrariness in the description of the free motion of the whole system, in the chosen Dirac's form of the dynamics.

Other choices are gauge equivalent; they can be obtained by means of a canonical transformation generated by χ_+ and χ_- (Refs. 5 and 28).

The above discussion shows the necessity, at the classical level of the theory, of a choice of gauge-fixing conditions, essentially of the one associated to χ_-, in order to identify the physical model. For instance, a Bethe-Salpeter approach suggests the choice $(p,r)=0$. On the hypersurface in phase space so identified, the $\mathbf{x}_i$ are the physical positions of the two particles; the corresponding velocities are to be determined from the first set of the Hamilton's equations of motion.

The "observables" $A(x,p)$, defined by the conditions $\{A,\chi_i\}=0$, are constants of motion. They are part of the Cauchy data of the system, and the gauge-fixing conditions will put some restrictions on them.[29]

We have till now discussed the constraint theory approach. Another point of view, which does not make explicit use of gauge-fixing conditions, is that of predictive mechanics,[30,9,5,6] where the physical positions are defined in such a way to get an unambiguous physical interpretation of the model in terms of world lines, for any choice of them.

This last point of view gives rise to the multitime dynamics (one independent time for each particle), which is especially suitable for a relativistic dynamics. Its physical meaning can be however most easily understood in the nonrelativistic case, where many complications, which hide the underlying simplicity of the approach, disappear. For this reason an analysis of the nonrelativistic multitime dynamics, both in the classical and quantum case, will be the subject of a forthcoming paper.[31]

Some results of this analysis,[6,32] in the case of the nonrelativistic limit of the Todorov-Komar constraints (2.1), are the following. It is possible to reformulate the classical mechanics of a system of n particles as an n-time theory and to find a canonical form of the equations of motion; the canonical quantization can be performed, giving rise to n Schrödinger equations for a unique wave function depending on the canonical coordinates $\mathbf{x}_i$ $(i=1,\ldots,n)$ and all the n times $t_1,t_2,\ldots,t_n$. These equations are integrable; this implies that in the space of the times the dynamics does not depend on the path from the initial and final states. A scalar product for the solutions of the wave equations can be defined, which is constant in each time. The restriction to $t_1=t_2=\cdots=t_n$ gives rise to the usual Schrödinger theory. Thus this approach does not require gauge-fixing conditions.

To the physical positions, in the sense of Refs. 9 and 30, are associated Hermitian operators, which coincide with the canonical coordinate operators $\mathbf{x}_i$ when $t_1=t_2=\cdots=t_n$. Outside this surface the physical positions are interaction-dependent functions of the operators $\mathbf{x}_i$ and of their associated velocities $\mathbf{v}_i$, which in general will not commute among themselves.

Having in mind this multitime approach for the nonrelativistic quantum case, we may try to follow the same steps in the two-body relativistic case (2.1).

The first step will be to look for the quantum version of the model, by introducing canonical operators x_i^μ and p_i^μ $(i=1,2)$ satisfying

$$[p_i^\mu,x_j^\nu]=i\delta_{ij}g^{\mu\nu} , \tag{2.8}$$

and besides two constraint operators χ_i such that

$$[\chi_1,\chi_2]=0 . \tag{2.9}$$

This can be done without ordering problems in this case. The second step will be to postulate two wave equations

$$\chi_i\psi(x_1,x_2)=0, \quad i=1,2 , \tag{2.10}$$

which are integrable in consequence of (2.9).

Usually all this is done following the theory of constraints, that is, by defining the extended Hamiltonian $H_D=\lambda_1\chi_1+\lambda_2\chi_2$, which determines the evolution in a scalar unphysical parameter τ, and where λ_1, λ_2 are two arbitrary functions of τ. In this case we will have a unique Schrödinger equation for a wave function $\psi(x_1,x_2,\tau)$.

It is also possible to define a mathematical scalar product by considering ψ as an element of an $L_2(\mathbf{R}^8)$ space; with respect to this scalar product H_D has to be Hermitian. It is then necessary to project out the physical states, which have to be τ independent. H_D is nothing else than the most general generator of the gauge transformations. In this way we recover the two equations (2.10).

In the nonrelativistic theory the two times t_1 and t_2 were the two parameters, in terms of which the dynamical evolution of the wave function $\psi(\mathbf{x}_1,\mathbf{x}_2,t_1,t_2)$ was defined. Here, on the contrary, it is not clear, at least for the moment, which are their natural counterparts. One can use $x_i^0=t_i$, thus breaking the covariance of the theory, but, as is apparent from Eqs. (2.10), by looking at the presence of time derivatives in the argument of the potential V in Eq. (2.5), there are strong reasons to prefer another choice for them.

Indeed as observed, Eqs. (2.10) are not "local" in the two times x_i^0, and the initial data problem for ψ is not well posed. On the other hand, without a clear definition of the initial state, a dynamics in a true sense does not exist. Furthermore any choice in the plane (x_1^0,x_2^0) of a path between an initial and a final state will not clearly correspond to an invariant choice of a gauge-fixing condition, such as, for instance $(p,r)=0$.

This problem of the time parameters and of the initial data will be approached in the last section of this paper. In the following sections we will look for a physical scalar product, constant in both times x_1^0 and x_2^0, and relativistically invariant.

The previously mentioned mathematical scalar product in $L_2(\mathbb{R}^8)$ is indeed divergent for the physical states, for which Eqs. (2.10) hold. So it is not useful for a physical interpretation of the wave function.

We leave aside the problem of the physical positions. From the classical point of view it has been already analyzed by several authors.[9,30,34,5,6]

We close this section by summarizing the problem we intend to study in the following sections. As we have said we will study the two equations (2.10), which explicitly are

$$[\Box_1+m_1{}^2+V(r_\perp{}^2)]\psi(x_1,x_2)=0 \; , \\ [\Box_2+m_2{}^2+V(r_\perp{}^2)]\psi(x_1,x_2)=0 \; , \tag{2.11}$$

and for which the integrability condition (2.9) is satisfied.

We will look for a scalar product for ψ constant in both times x_1^0 and x_2^0, and invariant under the action of the Poincaré group. Since ψ describes a system of two spinless particles we assume it to be a scalar under a Poincaré transformation.

In order to give a meaning to Eqs. (2.11) we shall restrict in this paper to wave functions ψ with support $p^2>0$ (advanced and retarded functions in momentum space), so that the argument of the potential $r_\perp{}^2$ can be well defined as an operator on ψ, by means of its Fourier transform in the collective variable $x=\frac{1}{2}(x_1+x_2)$. Furthermore we assume that

$$V>-\min(m_1{}^2,m_2{}^2) \; , \tag{2.12}$$

in order to preserve the timelike character of the momenta p_1 and p_2 of the two constituent particles. The wave function ψ will be assumed to belong to the Hilbert space of a two-particle system $L_2(\mathbb{R}^6)$, obtained as the closure of the space of test functions $S(\mathbb{R}^6)$, on which the operator potential $V(r_\perp{}^2)$ has to be defined (see, for instance, Ref. 35).

A further comment on the definition of a scalar product is in order. The existence of it has been investigated by Droz-Vincent,[36] by Molina *et al.*,[37] and by Rizov, Sazdjian, and Todorov.[38] These authors define a scalar product in momentum space; the first two authors for the mass branch with threshold m_1+m_2 only, while the third authors give various definitions of a scalar product, trying to match the results known in the case of two scalar particles coupled to an external electromagnetic field. They do not get a conserved tensorial current as a generalization of the current, which can be defined when the two particles are not mutually interacting.

We will review in the next section the Klein-Gordon theory for one and two scalar particles, and the corresponding conserved currents and scalar products which can be defined in that case.

III. KLEIN-GORDON THEORY OF ONE AND TWO SCALAR PARTICLES

In this section we will briefly review the Klein-Gordon theory of one and two massive scalar particles and the corresponding definitions of a scalar product.

The Klein-Gordon action is

$$S=\int d^4x[\partial_\mu\phi^*(x)\partial^\mu\phi(x)-m^2\phi^*(x)\phi(x)] \; , \tag{3.1}$$

from which we get the Klein-Gordon equation (we will denote by $\hat{A}$ the operator corresponding to the classical quantity A):

$$\hat{\chi}\phi(x)=-(\Box+m^2)\phi(x)=0 \; . \tag{3.2}$$

This wave equation can be obtained as the quantum counterpart of a theory based on the constraint $\chi=p^2-m^2$, for a massive scalar particle with canonical variables x^μ and p^μ.

The action (3.1) is invariant under a constant phase transformation $\phi\to\phi+\delta\phi$ with $\delta\phi=i\alpha\phi$, giving rise to a Noether current, which is conserved for the solutions of the wave equation (3.2):

$$j^\mu(x)=\frac{i}{2}\phi^*(x)\overleftrightarrow{\partial}^\mu\phi(x), \quad \overleftrightarrow{\partial}^\mu=\overrightarrow{\partial}^\mu-\overleftarrow{\partial}^\mu \; , \\ \partial_\mu j^\mu(x)=\frac{i}{2}\{\phi^*(x)(\Box+m^2)\phi(x) \\ -[(\Box+m^2)\phi^*(x)]\phi(x)\} \\ =0 \; . \tag{3.3}$$

This current is extended to

$$J^\mu(x)=\frac{i}{2}\phi_A^*(x)\overleftrightarrow{\partial}^\mu\phi_B(x), \quad \partial_\mu J^\mu(x)=0 \; , \tag{3.4}$$

where ϕ_A and ϕ_B are two different solutions of Eq. (3.2).

Equation (3.4) implies the usual non-positive-definite scalar product for smooth solutions of (3.2):

$$(\phi_A,\phi_B)_1=\int d\sigma_\mu(x)\phi_A^*(x)\frac{i}{2}\overleftrightarrow{\partial}^\mu\phi_B(x) \; , \\ \partial^0(\phi_A,\phi_B)_1=0 \; , \tag{3.5}$$

where $d\sigma_\mu(x)$ is a spacelike surface element.

The current (3.4) is interpreted as the electric current, with the sign of the charge corresponding to positive- and negative-energy states, suitably reinterpreted. Consistently there is a minimal coupling to an external electromagnetic potential $A_\mu(x)$, in which case it becomes

$$J^\mu(x)=\phi_A^*(x)\left[\frac{i}{2}\overleftrightarrow{\partial}^\mu+A^\mu(x)\right]\phi_B(x) \; .$$

We can mention Ref. 38 for a general discussion on this point, and for the use of an energy norm[39] in order to get a Hilbert space for the physical states.

On the other hand, it is shown in Ref. 40 that a positive-definite (nonlocal) scalar product can be defined for bosons, on a group-theoretical ground. It is now easily realized that a conserved Noether current can be associated to it also.

Let us introduce the operator $\hat{W}=(m^2-\nabla^2)^{1/2}$ [defined on $S(\mathbb{R}^3)$ and then extended to all $L_2(\mathbb{R}^3)$]. On the space of the Klein-Gordon solutions it transforms as a zero component of a four-vector. The Klein-Gordon equation can be written accordingly as

$$\hat{\chi}\phi=(i\partial^0-\hat{W})(i\partial^0+\hat{W})\phi=0 .$$

Let us then introduce the operator "sign of the energy"

$$\hat{\eta}=\frac{i\partial^0}{\hat{W}}, \quad [\hat{\eta},x^\mu]=[\hat{\eta},i\partial^\mu]=0 , \tag{3.6}$$

which is well defined on the solutions of the Klein-Gordon equation and invariant under proper Lorentz transformations. Under the infinitesimal transformation

$$\phi(x)\to\phi(x)+\delta\phi(x), \quad \delta\phi(x)=-\frac{i\alpha}{4}\hat{\eta}\phi(x) , \tag{3.7}$$

where α is a real constant, the action (3.1) is quasi-invariant, in the sense of the extension of the first Noether theorem given in Ref. 41. This means that the variation of the action consists of a four-divergence plus terms which vanish with the use of the equations of motion:

$$\delta S=-\frac{i\alpha}{4}\int d^4x(\partial_\mu\{\phi^*(x)\partial^\mu\hat{\eta}\phi(x)-[\partial^\mu\hat{\eta}\phi(x)]^*\phi(x)\}+\phi^*(x)\hat{\chi}\,\hat{\eta}\phi(x)-[\hat{\chi}\,\hat{\eta}\phi(x)]^*\phi(x)) . \tag{3.8}$$

By comparing with the formal variation of the action we get the following conserved current:

$$\tilde{j}^\mu(x)=\frac{i}{4}\phi^*(x)(\hat{\eta}-\hat{\eta}^\dagger)\overleftrightarrow{\partial}^\mu\phi(x) , \tag{3.9}$$

where

$$\hat{\eta}^\dagger=-\frac{i\overleftrightarrow{\partial}_0}{\overleftarrow{\hat{W}}}$$

We may check that it is conserved:

$$\partial_\mu\tilde{j}^\mu(x)=-\frac{i}{4}\{\phi^*(x)\hat{\chi}\,\hat{\eta}\phi(x)-[\hat{\chi}\,\hat{\eta}\,\phi(x)]^*\phi(x)+[\hat{\eta}\phi(x)]^*[\hat{\chi}\phi(x)]-[\hat{\chi}\phi(x)]^*[\hat{\eta}\phi(x)]\}=0 .$$

On the solutions the zero component of $j^\mu(x)$ becomes

$$\begin{aligned}\tilde{j}^0(x)&=\rho(x)+\frac{1}{4}\left[\phi^*(x)\frac{\hat{\chi}}{\hat{W}}\phi(x)+\left[\frac{\hat{\chi}}{\hat{W}}\phi(x)\right]^*\phi(x)\right]\\&\equiv\rho(x) ,\\ \rho(x)&=\phi^*(x)\frac{\hat{W}+\hat{W}^\dagger}{2}\frac{1+\hat{\eta}\hat{\eta}^\dagger}{2}\phi(x) ,\end{aligned} \tag{3.10}$$

where $\hat{W}^\dagger=\overleftarrow{\hat{W}}$. With $\rho(x)$ we can define a positive-definite scalar product:

$$(\phi_A,\phi_B)_2=\int d^3x\,\phi_A^*(x)\frac{\hat{W}+\hat{W}^\dagger}{2}\frac{1+\hat{\eta}\hat{\eta}^\dagger}{2}\phi_B(x) , \tag{3.11}$$

which is scalar on the solutions, constant in x^0, and coincides with that defined in Ref. 40.

For plane-wave solutions

$$\phi_{(\eta,\mathbf{p})}(x)=(2\pi)^{-3/2}\exp[-i(\eta\omega(\mathbf{p})x^0-\mathbf{p}\cdot\mathbf{x})]$$

[$\eta=\pm1$, and $\omega(\mathbf{p})=(\mathbf{p}^2+m^2)^{1/2}$], we have

$$(\phi_{(\eta',\mathbf{p}')},\phi_{(\eta,\mathbf{p})})_1=\eta\delta_{\eta\eta'}\omega(\mathbf{p})\delta^3(\mathbf{p}-\mathbf{p}'), \quad (\phi_{(\eta',\mathbf{p}')},\phi_{(\eta,\mathbf{p})})_2=\delta_{\eta\eta'}\omega(\mathbf{p})\delta^3(\mathbf{p}-\mathbf{p}') . \tag{3.12}$$

The generalization of the result (3.11) in the presence of an external electromagnetic field is a difficult problem. For instance, in the case $A^0(x)=0$, $\mathbf{A}(x)=\mathbf{A}(\mathbf{x})$ we have the same expression as in (3.11) but with $\hat{W}$ modified in

$$\hat{W}'=[m^2-(\nabla-i\mathbf{A})^2]^{1/2} , \tag{3.13}$$

where again

$$[i\partial^0,\hat{W}']=0, \quad \hat{\chi}=(i\partial^0-\hat{W}')(i\partial^0+\hat{W}') .$$

Clearly the operator $\hat{W}'$ can only be defined for a magnetic field weak enough, in order not to destroy the mass gap between positive- and negative-energy states.

Let us now consider the case of two free scalar particles. We can describe this system in terms of a bilocal wave function $\phi(x_1,x_2)$, scalar under Poincaré transformations, satisfying a pair of Klein-Gordon equations:

$$\hat{\chi}_1\phi(x_1,x_2)=-(\Box_1+m_1{}^2)\phi(x_1,x_2)=0, \quad \hat{\chi}_2\phi(x_1,x_2)=-(\Box_2+m_2{}^2)\phi(x_1,x_2)=0 . \tag{3.14}$$

These equations are the quantum counterpart of two first-class constraints $\chi_i = p_i^2 - m_i^2$, $i=1,2$, where p_1^μ and p_2^μ are the canonical momenta of the two particles.

If an external electromagnetic field is present Eqs. (3.14) are modified in the minimal way. The new two constraints $\chi_i = [p_i - A(x_i)]^2 - m_i^2$ are again of first-class character.

The analogue of Eqs. (3.3) and (3.4) is

$$J^{\mu\nu}(x_1,x_2) = \phi_A^*(x_1,x_2)\left[\frac{i}{2}\overleftrightarrow{\partial}_1^\mu + A^\mu(x_1)\right]\left[\frac{i}{2}\overleftrightarrow{\partial}_2^\nu + A^\nu(x_2)\right]\phi_B(x_1,x_2)\ , \tag{3.15}$$

with the scalar product

$$(\phi_A,\phi_B)_1 = \int d^3x_1 d^3x_2 \phi_A^*(x_1,x_2)\left[\frac{i}{2}\overleftrightarrow{\partial}_1^0 + A^0(x_1)\right]\left[\frac{i}{2}\overleftrightarrow{\partial}_2^0 + A^0(x_2)\right]\phi_B(x_1,x_2)\ , \tag{3.16}$$

which is constant in x_1^0 and x_2^0 for ϕ_A and ϕ_B solutions of the equations of motion.

At least in the free case we may generalize Eqs. (3.9) and (3.11) too. We get

$$\tilde{J}^{\mu\nu}(x_1,x_2) = -\tfrac{1}{16}\phi_A^*(x_1,x_2)\overleftrightarrow{\hat{\eta}}_1\overleftrightarrow{\partial}_1^\mu\overleftrightarrow{\hat{\eta}}_2\overleftrightarrow{\partial}_2^\nu\phi_B(x_1,x_2)\ , \tag{3.17}$$

where $\overleftrightarrow{\hat{\eta}}_i = \hat{\eta}_i - \hat{\eta}_i^\dagger$ $(i=1,2)$, and

$$\partial_1^\mu \tilde{J}_{\mu\nu} = \partial_2^\nu \tilde{J}_{\mu\nu} = 0\ . \tag{3.17'}$$

The scalar product becomes

$$(\phi_A,\phi_B)_2 = \int d^3x_1 d^3x_2 \phi_A^*(x_1,x_2)\frac{\hat{W}_1+\hat{W}_1^\dagger}{2}\frac{1+\hat{\eta}_1\hat{\eta}_1^\dagger}{2}\frac{\hat{W}_2+\hat{W}_2^\dagger}{2}\frac{1+\hat{\eta}_2\hat{\eta}_2^\dagger}{2}\phi_B(x_1,x_2)\ , \tag{3.18}$$

where $\hat{W}_i = [m_i^2 - \nabla_i^2]^{1/2}$ and $\hat{\eta}_i = i\partial_i^0/\hat{W}_i$. This scalar product is conserved in both times x_1^0, x_2^0 and is positive definite.

In terms of the collective coordinates

$$x^\mu = \tfrac{1}{2}(x_1+x_2)^\mu,\quad r^\mu = (x_1-x_2)^\mu\ , \tag{3.19}$$

it is not possible to define two corresponding conserved currents. In other words it is not possible to define such a thing as a center-of-mass current and a relative current. Indeed the two equations of motion for ϕ (3.14) can be written as in Eqs. (2.6), with

$$\hat{\chi}_+\phi(x_1,x_2) = -[\Box_x + 4\Box_r + 2(m_1^2+m_2^2)]\phi(x_1,x_2) = 0\ ,$$
$$\hat{\chi}_-\phi(x_1,x_2) = -\left[\partial_x^\mu\partial_{r\mu} + \frac{m_1^2-m_2^2}{2}\right]\phi(x_1,x_2) = 0\ , \tag{3.20}$$

from which we get

$$\partial_x^\mu[\phi_A^*(x,r)\overleftrightarrow{\partial}_{x\mu}\phi_B(x,r)] + \partial_r^\mu[\phi_A^*(x,r)4\overleftrightarrow{\partial}_{r\mu}\phi_B(x,r)] = 0\ ,$$
$$\partial_x^\mu[\phi_A^*(x,r)\overleftrightarrow{\partial}_{r\mu}\phi_B(x,r)] + \partial_r^\mu[\phi_A^*(x,r)4\overleftrightarrow{\partial}_{x\mu}\phi_B(x,r)] = 0\ , \tag{3.21}$$

which are not continuity equations.

So it does not exist something like a tensor current for collective coordinates. We should keep this fact in mind when we introduce a mutual interaction in the following section. In order to separate the variables we will be forced to use collective coordinates, but we will not expect to find a vector or tensor conserved current, from which to define a good scalar product. When the mutual interaction is present the scalar product will be defined without recourse to arguments of the kind discussed here.

When a mutual interaction is present, as in the model (2.11), a simple local coupling to an external field is not possible. This is because the first-class character of the constraints can be destroyed, as can be seen by performing the minimal substitution in Eqs. (2.11), and the equations of motion are no longer integrable.

Only external fields preserving the first-class character of the constraints can be interpreted as probes of the constituent particles.

An example of a coupling to an external field, which preserves the first-class character of the constraints, is

$$\tilde{\chi}_+ = \tilde{p}^2 + 4[\tilde{q}^2 - V(r_\perp^2)] - 2(m_1^2+m_2^2)\ ,$$
$$\tilde{\chi}_- = (p,q) - \tfrac{1}{2}(m_1^2-m_2^2)\ , \tag{3.22}$$

where

$$\tilde{p}^\mu = p^\mu - e(\eta_\nu^\mu - q^\mu q_\nu/q^2)A^\nu(x_\perp)\ ,$$
$$\tilde{q}^\mu = q^\mu - e_r(\eta_\nu^\mu - p^\mu p_\nu/p^2)B^\nu(r_\perp)\ , \tag{3.23}$$

and

$$x_\perp^\mu = (\eta_\nu^\mu - q^\mu q_\nu/q^2)x^\nu\ ,$$
$$r_\perp^\mu = (\eta_\nu^\mu - p^\mu p_\nu/p^2)r^\nu\ . \tag{3.24}$$

A^ν is an arbitrary external potential coupled to the "center-of-mass charge" e. Instead B^ν is an example of a mutual vector interaction coupled to a "relative charge" e_r. For the variables $\tilde{p}$ and $\tilde{q}$ it holds

$$\{\tilde{p}^\mu,\tilde{p}^\nu\} = -e(\eta_\alpha^\mu - q^\mu q_\alpha/q^2)(\eta_\beta^\nu - q^\nu q_\beta/q^2)F^{\alpha\beta}(x_\perp)\ ,$$
$$\{\tilde{p}^\mu,\tilde{q}^\nu\} = 0\ , \tag{3.25}$$
$$\{\tilde{q}^\mu,\tilde{q}^\nu\} = -e_r(\eta_\alpha^\mu - p^\mu p_\alpha/p^2)(\eta_\beta^\nu - p^\nu p_\beta/p^2)G^{\alpha\beta}(r_\perp)\ ,$$

where

$$F^{\alpha\beta}(x_\perp)=\frac{\partial A^\beta}{\partial x_\alpha}-\frac{\partial A^\alpha}{\partial x_\beta},\quad G^{\alpha\beta}(r_\perp)=\frac{\partial B^\beta}{\partial r_\alpha}-\frac{\partial B^\alpha}{\partial r_\beta}\ .$$

It can be verified that $\{\tilde{\chi}_+,\tilde{\chi}_-\}=0$.

In the following section we will study in detail the model described by Eqs. (2.11). Following the discussion given here we will abandon the scalar products $(\ ,\ )_1$ and $(\ ,\ )_2$ of Eqs. (3.16) and (3.18), and we will look for another definition of scalar product, consistent with the equations of motion (3.20).

IV. SOLUTIONS OF THE TODOROV-KOMAR MODEL AND SCALAR PRODUCTS

In this section we study the solutions of Eqs. (2.11), which we now write in the representation (p,r), defined by

$$\psi(x,r)=\int e^{-i(p,x)}\phi(p,r)d^4p\ . \tag{4.1}$$

Equations (2.11) become

$$\hat{\chi}_+\phi(p,r)=\left[p^2-4\left[\frac{\partial}{\partial r^\mu}\frac{\partial}{\partial r_\mu}+V(r_\perp{}^2)\right]-2(m_1{}^2+m_2{}^2)\right]\phi(p,r)=0\ ,$$
$$\hat{\chi}_-\phi(p,r)=\left[ip^\mu\frac{\partial}{\partial r^\mu}-\frac{1}{2}(m_1{}^2-m_2{}^2)\right]\phi(p,r)=0\ , \tag{4.2}$$

where

$$r_\perp^\mu=(\eta_\nu^\mu-p^\mu p_\nu/p^2)r^\nu\ .$$

By hypothesis the support of $\phi(p,r)$ is $p^2>0$ ($p^0\gtrless 0$). Observe that the first equation can also be written as

$$\frac{1}{p^2}[(p^2-M_+{}^2)(p^2-M_-{}^2)+4\hat{\chi}_-(\hat{\chi}_-+m_1{}^2-m_2{}^2)]\phi(p,r)=0\ , \tag{4.3}$$

where the mass operators $M_\pm{}^2$ are

$$M_\rho{}^2=(M_1+\rho M_2)^2,\quad \rho=\pm 1\ ,$$

with

$$M_i=\left[m_i{}^2+V(r_\perp{}^2)+\left[\eta_\nu^\mu-\frac{p^\mu p_\nu}{p^2}\right]\frac{\partial}{\partial r^\mu}\frac{\partial}{\partial r_\nu}\right]^{1/2},\quad i=1,2\ .$$

The operators M_i are well defined if the potential satisfies $V>-m_2{}^2$ (from now on we assume $m_1>m_2$). With the second equation (4.2) we have that the first can be written in Todorov's notations:

$$\left[\left[\eta^{\mu\nu}-\frac{p^\mu p^\nu}{p^2}\right]\frac{\partial}{\partial r^\mu}\frac{\partial}{\partial r^\nu}+V(r_\perp{}^2)\right]\phi(p,r)=\beta(p^2)\phi(p,r)\ ,$$
$$\beta(p^2)=\frac{1}{4p^2}[p^4-2(m_1{}^2+m_2{}^2)p^2+(m_1{}^2-m_2{}^2)^2]\ . \tag{4.4}$$

The hypothesis $V>-m_2{}^2$ implies $p_i{}^2>0$, $i=1,2$, but not $p^2>0$. Here we shall study the sector $p^2>0$, to which four branches of the mass spectrum correspond (two with $p^0>0$ and the two antibranches with $p^0<0$). The solutions with $p^2<0$ will be investigated elsewhere.

If one is only interested in the mass spectrum, this equation can be written for $\mathbf{p}=0$. It amounts in the biquadratics $\beta(p^2)=\varepsilon_\alpha$, where ε_α are the eigenvalues of the left-hand side operator.

Let us notice that for a general potential, like the one in Eq. (2.3), the eigenvalues ε_α will depend upon p^2, and that the equation to be solved can be of a degree more than four (or even transcendental).

Coming back to Eqs. (4.2) and (4.4), they become genuine differential equations when $\mathbf{p}=0$:

$$[-\nabla_r{}^2+V(-\mathbf{r}^2)]\phi_{\mathbf{p}=0}(p,r)=\beta(p^2)\phi_{\mathbf{p}=0}(p,r)\ ,$$
$$\left[ip^0\frac{\partial}{\partial r^0}-\frac{1}{2}(m_1{}^2-m_2{}^2)\right]\phi_{\mathbf{p}=0}(p,r)=0\ , \tag{4.5}$$

but it is not obvious how to get from their solutions the solutions of the original equations (4.2).

The point is that $\psi(x_1,x_2)$ belongs to a reducible representation of the Poincaré group; this implies that for each mass eigenstate it requires a different boost $L(p,\overset{\circ}{p})$ (see Appendix A) to transform it from the center-of-mass frame to the actual one. It is therefore necessary to look for an operator $\hat{S}$ which transforms each component of $\psi(x_1,x_2)$ of definite mass in the proper way.

From Appendix A we know that in the classical case the canonical transformation generated by the function ψ [see Eq. (A13)] performs the required operation. In the quantum case, where $[\hat{x}^\mu,\hat{p}^\nu]=[\hat{r}^\mu,\hat{q}^\nu]=i\eta^{\mu\nu}$, the operator

$$\hat{S}=\exp\left[\frac{i}{2}w(\hat{p})I_{\mu\nu}(\hat{p})\hat{S}^{\mu\nu}\right]\ , \tag{4.6}$$

where

$$\hat{S}^{\mu\nu}=\hat{r}^\mu\hat{q}^\nu-\hat{r}^\nu\hat{q}^\mu\ , \tag{4.7}$$

performs the required transformation [observe that in (4.6) no ordering problems arise, due to the fact that $I_{\mu\nu}=-I_{\nu\mu}$].

A comment on the definition of the operator $\hat{S}$ is in order: $\hat{S}$ is defined on the space of test functions $S(\mathbb{R}^8)$ of tempered distributions, where a scalar product can be defined, as observed in Sec. II, where it was called a mathematical scalar product. Then it can be extended by continuity to the closure of $S(\mathbb{R}^8)$ with respect to this scalar product.

In the norm so defined $\hat{S}$ is unitary. The physical scalar products we will look for will be defined in $S(\mathbb{R}^6)$, and, with respect to these, no unitarity property for $\hat{S}$ will hold. So $\hat{S}$ does not represent a quantum canonical transformation; here it is merely used to get a set of solutions of our equations.

The action of $\hat{S}$ is specified by the equations

$$\begin{aligned}\tilde{\psi}(x,r)&=(\hat{S}\psi)(x,r)\\&=\int d^4p\, e^{-i(p,x)}\tilde{\phi}(p,r)\\&=\int d^4p\, e^{-i(p,x)}\phi(p,L^{\mu}_{\cdot\nu}(p,\mathring{p})r^{\nu})\\&=\int d^4x'd^4r'S(x-x',r,r')\psi(x',r')\ ,\end{aligned} \tag{4.8}$$

where

$$S(x-x',r,r')=\int_{(p^2>0)}\frac{d^4p}{(2\pi)^4}e^{-i(p,x-x')}\times\delta^4(r'^{\mu}-L^{\mu}_{\cdot\nu}(p,\mathring{p})r^{\nu})\ . \tag{4.9}$$

The inverse transformation is defined by

$$\begin{aligned}\psi(x,r)&=(\hat{S}^{-1}\tilde{\psi})(x,r)\\&=\int d^4p\, e^{-i(p,x)}\tilde{\phi}(p,L^{\mu}_{\cdot\nu}(\mathring{p},p)r^{\nu})\\&=\int d^4x'd^4r'S^{-1}(x-x',r,r')\tilde{\psi}(x',r')\ ,\end{aligned} \tag{4.10}$$

where

$$S^{-1}(x-x',r,r')=\int_{(p^2>0)}\frac{d^4p}{(2\pi)^4}e^{-i(p,x-x')}\times\delta^4(r'^{\mu}-L^{\mu}_{\cdot\nu}(\mathring{p},p)r^{\nu})\ . \tag{4.11}$$

In these equations we used the fact that $(p,x)=(p,\bar{x})$, see Eq. (A18).

The transformed equations are

$$\begin{gathered}\left[i\eta(p^2)^{1/2}\frac{\partial}{\partial r_0}-\frac{1}{2}(m_1{}^2-m_2{}^2)\right]\tilde{\phi}(p,r)=0\ ,\\ [-\nabla_r{}^2+V(-\mathbf{r}^2)]\tilde{\phi}(p,r)=\beta(p^2)\tilde{\phi}(p,r)\ ,\end{gathered} \tag{4.12}$$

where $\eta=\mathrm{sgn}p_0$.

The last equation can be written

$$\frac{1}{p^2}(p^2-\tilde{M}_+{}^2)(p^2-\tilde{M}_-{}^2)\tilde{\phi}(p,r)=0\ , \tag{4.12'}$$

where

$$\tilde{M}_{\rho}{}^2=(\tilde{M}_1+\rho\tilde{M}_2)^{1/2},\quad \rho=\pm 1\ , \tag{4.12''}$$

and

$$\tilde{M}_i=[m_i{}^2-\nabla_r{}^2+V(-\mathbf{r}^2)]^{1/2},\quad i=1,2\ . \tag{4.12'''}$$

Equations (4.12) are formally the same as (4.5), but now $\mathbf{p}$ is not put to zero.

Let us assume that a complete orthonormal set of eigenfunctions of the operator $-\nabla_r{}^2+V(-\mathbf{r}^2)$ is known. To be definite let us assume that it has a discrete spectrum only, with eigenfunctions $\chi_{nlm}(r)$, defined by

$$\begin{gathered}[-\nabla_r{}^2+V(-\mathbf{r}^2)]\chi_{nlm}(\mathbf{r})=\varepsilon_{nl}\chi_{nlm}(\mathbf{r})\ ,\\ \mathbf{S}^2\chi_{nlm}(\mathbf{r})=l(l+1)\chi_{nlm}(\mathbf{r})\ ,\\ S_z\chi_{nlm}(\mathbf{r})=m\chi_{nlm}(\mathbf{r})\ ,\end{gathered} \tag{4.13}$$

where

$$S_{ij}=r_i\hat{q}_j-r_j\hat{q}_i,\quad S_k=\tfrac{1}{2}\epsilon_{kij}S_{ij},\quad \hat{q}_i=-i\frac{\partial}{\partial r_i}\ ,$$

and with the orthonormality and completeness relations satisfied

$$\begin{gathered}\int d^3r\,\chi^*_{nlm}(\mathbf{r})\chi_{n'l'm'}(\mathbf{r})=\delta_{nn'}\delta_{ll'}\delta_{mm'}\ ,\\ \sum_{nlm}\chi_{nlm}(\mathbf{r})\chi^*_{nlm}(\mathbf{r}')=\delta^3(\mathbf{r}-\mathbf{r}')\ .\end{gathered} \tag{4.14}$$

The assumption $V>-m_2{}^2$ reflects in the condition $(m_1>m_2)$:

$$-m_2{}^2<\varepsilon_{nl}<0\ . \tag{4.15}$$

The mass spectrum turns out to be

$$p^2=M_{\rho nl}{}^2=(M_{1nl}+\rho M_{2nl})^2,\quad \rho=\pm 1\ , \tag{4.16}$$

with

$$M_{inl}=(m_i{}^2+\varepsilon_{nl})^{1/2},\quad i=1,2\ .$$

From Eqs. (4.12) we get the following set of elementary solutions:

$$\tilde{\phi}_{(\eta,\mathbf{k},\rho nl,m)}(p,r)=(2\pi)^{-3/2}\delta^3(\mathbf{p}-\mathbf{k})\delta[p^0-\eta(\mathbf{p}^2+M_{\rho nl}{}^2)^{1/2}]\exp\left[-i\eta\frac{m_1{}^2-m_2{}^2}{2M_{\rho nl}}r_0\right]\chi_{nlm}(\mathbf{r})\ , \tag{4.17}$$

or

$$\tilde{\psi}_{(\eta,\mathbf{k},\rho nl,m)}(x,r)=(2\pi)^{-3/2}e^{-i(k,x)}\exp\left[-i\eta\frac{m_1{}^2-m_2{}^2}{2M_{\rho nl}}r_0\right]\chi_{nlm}(\mathbf{r})\ , \tag{4.17'}$$

where

$$k_0=\eta(\mathbf{k}^2+M_{\rho nl}{}^2)^{1/2}\ . \tag{4.17''}$$

With the formula (4.10) we get immediately

$$\psi_{(\eta,\mathbf{k},\rho nl,m)}(x,r)=(2\pi)^{-3/2}e^{-i(k,x)}\exp\left[-i\frac{m_1{}^2-m_2{}^2}{2M_{\rho nl}{}^2}(k,r)\right]\chi_{nlm}(-\epsilon^{\mu}_a(k)r_{\mu})\quad (a=1,2,3) \tag{4.18}$$

[see Appendix A for the definition of the polarization four-vectors $\epsilon^{\mu}_a(k)$].

The functions (4.18) are eigenfunctions of the total momentum, with eigenvalues k^{μ}, and of the Casimir operators of the Poincaré group, with eigenvalues

$$p^2=M_{\rho nl}^2, \quad \mathrm{sgn}p_0=\eta, \quad W^2=-\tfrac{1}{4}M_{\rho nl}^2 l(l+1)\,, \tag{4.19}$$

where W^2 is the square of the Pauli-Lubanski four-vector.

Therefore, for given values of ρ, n, and l, the set (4.18) is a representation of the Poincaré group, as we will verify in detail in the following section, of mass $M_{\rho nl}$ and spin l. In the following section we will also prove that it is an irreducible and unitary representation, transforming as a Wigner function of the given mass and spin.

In the (M^2,l) plane, the spectrum (4.16) gives two families of Regge trajectories: one of rising trajectories (with daughters corresponding to the values of n for given l), and with an asymptotic value $(m_1+m_2)^2$, the other of lowering trajectories, with an asymptotic value $(m_1-m_2)^2$.

The general solution of Eqs. (4.2), obtained with a linear combination of the set (4.18), is

$$\psi(x,r)=\sum_{\eta\rho nlm}\int d^4p\,\delta(p^2-M_{\rho nl}^2)\theta(\eta p_0) \times f(\eta,\mathbf{p},\rho nl,m)\left[(2\pi)^{-3/2}e^{-i(p,x)}\exp\left[-i\frac{m_1{}^2-m_2{}^2}{2p^2}(p,r)\right]\chi_{nlm}(-\epsilon_a^\mu(p)r_\mu)\right]. \tag{4.20}$$

Let us notice that this solution is different from that given by the model proposed by Feynman and Kislinger,[42] and from that given by Kim and Noz[43] and Takabayasi;[4] see Refs. 44–47 also. This is due to the fact that these authors adopt a Gupta-Bleuler quantization, so they start from different wave equations.

From Eq. (4.20) we see that there are four classes of solutions, corresponding to the four possible choices of $\eta=\pm1$ and $\rho=\pm1$, that is, to the four branches of the dispersion relation between p^0 and $\mathbf{p}$. We may now define four conserved (nonlocal) scalar products, each corresponding to a different choice of signature, associated to the values of η and ρ, in analogy to the situation we have discussed in Sec. III for the Klein-Gordon equation for a single particle (where there were two possible scalar products). In other words we may define four Hilbert spaces, spanned by the four classes of solutions (4.20). These spaces can be collected in a unique space, which will be a Hilbert space with indefinite metrics[48] in three cases, and a true Hilbert space in the fourth case.

In the following section we will show that all these scalar products are Poincaré invariant.

At first we define them for the functions $\tilde\psi(x,r)$ and then they will be translated in terms of the functions $\psi(x,r)$. They are

$$\langle\tilde\psi_A,\tilde\psi_B\rangle^{(k)}=\int d^3x\,d^3r\,\tilde\psi_A^*(x,r)A_0^{(k)}\tilde\psi_B(x,r)\,, \tag{4.21}$$

with $k=1,\ldots,4$, and where $A_0^{(k)}$ are operators acting on the variables x^μ and r^μ of both functions $\tilde\psi_A$ and $\tilde\psi_B$. These operators are best defined in the momentum-space representation [see Eq. (4.1)]

$$\langle\tilde\psi_A,\tilde\psi_B\rangle^{(k)}=\int d^4p\int d^4p'(2\pi)^3\delta^3(\mathbf{p}-\mathbf{p}')\int d^3r\,\tilde\phi_A^*(p',r)A_0^{(k)}(p,p')\tilde\phi_B(p,r)\,, \tag{4.22}$$

and are given by the zero components of the following $A_\mu^{(k)}(p,p')$:

$$\begin{aligned}A_\mu^{(1)}(p,p')&=\tfrac{1}{2}(p+p')_\mu\tfrac{1}{2}[p^2\cdot p'^2-(m_1{}^2-m_2{}^2)^2]\,,\\ A_\mu^{(2)}(p,p')&=\tfrac{1}{2}(p+p')_\mu\tfrac{1}{2}[2p^2\cdot p'^2-(p^2+p'^2)(\tilde M_+{}^2+\tilde M_-{}^2)+\tilde M_+{}^4+\tilde M_-{}^4]\,,\\ A_\mu^{(3)}(p,p')&=\tfrac{1}{2}(\eta+\eta')A_\mu^{(1)}(p,p'),\quad A_\mu^{(4)}(p,p')=\tfrac{1}{2}(\eta+\eta')A_\mu^{(2)}(p,p')\,,\end{aligned} \tag{4.23}$$

where the operators $\tilde M_\pm$ (which are Hermitian operators) are defined in Eqs. (4.12″) and (4.12‴), in terms of the operator

$$\hat\varepsilon=-\nabla_r{}^2+V(-\mathbf{r}^2)\,, \tag{4.24}$$

whose eigenvalues are ε_{nl}. From the same Eq. (4.12″) it follows that $\tilde M_+{}^2\tilde M_-{}^2=(m_1{}^2-m_2{}^2)^2$.

With respect to these scalar products the set of elementary solutions (4.18) are normalized as

$$\tilde\psi_{(\eta,\mathbf{p},\rho nl,m)}(x,r)=\frac{K_{\rho nl}^{(k)}}{(2\pi)^{3/2}}e^{-i(p,x)}\exp\left[-i\eta\frac{m_1{}^2-m_2{}^2}{2M_{\rho nl}}r_0\right]\chi_{nlm}(\mathbf{r})\,, \tag{4.25}$$

where $p^0=\eta\omega_{\rho nl}(\mathbf{p})$, $\omega_{\rho nl}(\mathbf{p})=(\mathbf{p}^2+M_{\rho nl}{}^2)^{1/2}$, and $M_{\rho nl}$ is given by Eq. (4.16), with

$$\begin{aligned}K_{\rho nl}^{(1)}=K_{\rho nl}^{(3)}&=\frac{1}{M_{\rho nl}}[(m_1{}^2+\epsilon_{nl})(m_2{}^2+\epsilon_{nl})]^{-1/4}=\frac{2}{M_{\rho nl}(M_+{}^2-M_-{}^2)^{1/2}}\,,\\ K_{\rho nl}^{(2)}=K_{\rho nl}^{(4)}&=\tfrac{1}{2}[(m_1{}^2+\varepsilon_{nl})(m_2{}^2+\varepsilon_{nl})]^{-1/2}=\frac{2}{M_+{}^2-M_-{}^2}\,.\end{aligned} \tag{4.26}$$

With this normalization factor we have

$$\langle\tilde\psi_{(\eta',\mathbf{p}',\rho'n'l',m')},\tilde\psi_{(\eta,\mathbf{p},\rho nl,m)}\rangle^{(k)}=2\omega_{\rho nl}(\mathbf{p})\delta^3(\mathbf{p}-\mathbf{p}')h_{\rho\eta}^{(k)}\delta_{\eta\eta'}\delta_{\rho\rho'}\delta_{nn'}\delta_{ll'}\delta_{mm'}\,, \tag{4.27}$$

with $h^{(1)}_{\rho\eta}=\rho\eta$; $h^{(2)}_{\rho\eta}=\eta$; $h^{(3)}_{\rho\eta}=\rho$; $h^{(4)}_{\rho\eta}=1$. For the general solution

$$\tilde{\psi}(x,r)=\sum_{\eta\rho nlm}\int d^4p\,\delta(p^2-M_{\rho nl}{}^2)\theta(\eta p^0)\tilde{g}(\eta,\mathbf{p},\rho nl,m)\tilde{\psi}_{(\eta,\mathbf{p},\rho nl,m)}(x,r)\ , \tag{4.28}$$

where, for each given $\tilde{\psi}(x,r)$, the inversion formula is

$$\tilde{g}(\eta,\mathbf{p},\rho nl,m)=h^{(k)}_{\rho\eta}\langle\tilde{\psi}_{(\eta,\mathbf{p},\rho nl,m)},\tilde{\psi}\rangle^{(k)}\ , \tag{4.29}$$

we have the scalar products

$$\langle\tilde{\psi}_A,\tilde{\psi}_B\rangle^{(k)}=\sum_{\eta\rho nlm}\int\frac{d^3p}{2\omega_{\rho nl}(\mathbf{p})}h^{(k)}_{\rho\eta}\cdot\times\tilde{g}^*_A(\eta,\mathbf{p},\rho nl,m)\tilde{g}_B(\eta,\mathbf{p},\rho nl,m)\ . \tag{4.30}$$

These four scalar products are clearly conserved in both x^0 and r^0. For the functions $\psi(x,r)$ we now define

$$(\psi_A,\psi_B)^{(k)}\equiv\langle\tilde{\psi}_A,\tilde{\psi}_B\rangle^{(k)}\ . \tag{4.31}$$

Clearly, when in the right-hand side (RHS) of this equation we substitute $\tilde{\psi}$ for its expression in terms of ψ, the nonlocal operator $\hat{S}$ will appear in both arguments of the scalar products, so Eq. (4.31) defines highly nonlocal scalar products. We will see that they are invariant under Poincaré transformations, thus giving a unitary realization of the Poincaré group.

With the definition (4.31) we may write the analogous relations of (4.27), (4.28), and (4.30) for $\psi(x,r)$. Clearly all these scalar products are conserved in x^0 and r^0.

Let us compare the scalar product defined with Eq. (4.31) to those defined in Ref. 38. In this reference three kinds of scalar products are defined. One, in the total momentum p and relative coordinate r representation, is analogous to our definition restricted to states with the same $\hat{p}^\mu=p^\mu/M$. This can be understood if we observe that the measure d^3r of Eq. (4.21) can be made invariant if substituted with $d^4r\,\delta((r,\hat{p}))$, when $\hat{p}$ is a fixed quantity. The same observation is also essentially true for the second scalar product defined in that reference.

The definition given here seems to be more similar to the third definition of Ref. 38, which is given in configuration space, but for functions depending on a further parameter τ, which is a variable conjugated to the total energy mass M. This does not happen in our case. Here we have defined a scalar product in configuration space x^μ,r^μ, constant in both x^0 and r^0, with an explicit structure, for general solutions (not restricted to be eigenfunctions of the energy p^0 or to a particular branch of it) of the original equations.

It is explicitly given by

$$(\psi_A,\psi_B)^{(k)}=\int d^3x\int d^3r\left[\int S(x-x',r,r')\psi_A(x',r')d^4x'd^4r'\right]^*A_0^{(k)}\left[\int S(x-x'',r,r'')\psi_B(x'',r'')d^4x''d^4r''\right]\ ,$$

where the kernel S is defined in Eq. (4.9).

For $k=4$ this scalar product is positive definite (for all the branches of the energy p^0).

In the next section we will show that this scalar product is invariant under Poincaré transformations, as it should however be clear from the way in which it was defined [see Eq. (4.31)].

It does not reduce to the scalar product (3.16) (when the external field is absent) when the interaction vanishes ($V=0$), but, of course, it remains well defined also in this case.

V. TRANSFORMATION PROPERTIES

In this section we study the transformation properties under the restricted Poincaré group of the set of solutions found in Sec. IV, Eq. (4.18).

We assume that the bilocal wave function $\psi(x_1,x_2)\equiv\psi(x,r)$ transforms as a scalar field; this means that, if $U(a,\Lambda)$ represents the action, as a linear operator, of the Poincaré transformation (a,Λ) on the space of the functions $\psi(x_1,x_2)$, we have (for the notations used in this section, see, for instance, Ref. 35)

$$[U(a,\Lambda)\psi](x_1,x_2)=\psi(\Lambda^{-1}(x_1-a),\Lambda^{-1}(x_2-a))\ , \tag{5.1}$$

or

$$[U(a,\Lambda)\psi](x,r)=\psi(\Lambda^{-1}(x-a),\Lambda^{-1}r)\ . \tag{5.2}$$

In this representation the generators of the Poincaré group are as usual

$$\mathbf{P}_\mu=i\frac{\partial}{\partial x^\mu}\ ,\qquad M_{\mu\nu}=i\left[x_\mu\frac{\partial}{\partial x^\nu}-x_\nu\frac{\partial}{\partial x^\mu}\right]+i\left[r_\mu\frac{\partial}{\partial r^\nu}-r_\nu\frac{\partial}{\partial r^\mu}\right]\ . \tag{5.3}$$

The wave function $\psi(x,r)$ belongs in general to a reducible representation. Our problem is to decompose this representation into irreducible components of given mass and spin.

Equation (4.20) accomplishes this decomposition if we show that each elementary solution of the set (4.18) transforms as an irreducible representation of the Poincaré group.

In order to show this, let us apply Eq. (5.2) to this set:

$$[U(a,\Lambda)\psi_{(\eta,\mathbf{p},\rho nl,m)}](x,r)=(2\pi)^{-3/2}e^{-i(p,\Lambda^{-1}(x-a))}\exp\left[-i\frac{m_1{}^2-m_2{}^2}{2M_{\rho nl}{}^2}(p,\Lambda^{-1}r)\right]\chi_{nlm}(-\epsilon^\mu_a(p)(\Lambda^{-1}r)_\mu)\ . \tag{5.4}$$

The transformation properties of the polarization four-vectors $\epsilon_a^\mu(p)$ $(a=1,2,3)$ have been studied in Ref. 49 and are given by

$$\epsilon_a^\mu(\Lambda p)=(R^{-1})_a^{\cdot b}\Lambda^\mu_{\cdot\nu}\epsilon_b^\nu(p) , \tag{5.5}$$

where $R_a^{\cdot b}$ is the Wigner rotation $R=R(\Lambda,p)=L(\mathring{p},p)\Lambda^{-1}L(\Lambda p,p)$ (see Appendix A). From Eq. (5.5) we get

$$\Lambda^\nu_{\cdot\mu}\epsilon_a^\mu(p)=R_a^{\cdot b}\epsilon_b^\nu(\Lambda p) . \tag{5.5'}$$

This means that the argument of χ_{nlm} in Eq. (5.4) can be written

$$(\Lambda^{-1}r)_\mu\epsilon_a^\mu(p)=R_a^{\cdot b}\epsilon_b^\nu(\Lambda p)r_\nu . \tag{5.6}$$

By hypothesis the basis of functions χ_{nlm} belongs to an irreducible (unitary) representation of the rotation group of spin l. It follows

$$\chi_{nlm}(R\mathbf{p})=\sum_{m'}\chi_{nlm'}(\mathbf{p})D^{(l)}_{m'm}(R^{-1}) , \tag{5.7}$$

where the matrix $D^{(l)}(R)$ is the representation of spin l of the rotation R.

By collecting Eqs. (5.4), (5.6), and (5.7) we finally get

$$[U(a,\Lambda)\psi_{(\eta,\mathbf{p},\rho nl,m)}](x,r)=e^{i(\Lambda p,a)}\sum_{m'}\psi_{(\eta,\Lambda\mathbf{p},\rho nl,m')}(x,r)D^{(l)}_{m'm}[R^{-1}(\Lambda,p)] . \tag{5.8}$$

We recognize the transformation rule of a Wigner representation of spin l (Ref. 40). It follows that the set (4.18) transforms as an irreducible representation of the Poincaré group of mass $M_{\rho nl}$ and spin l.

It can now be easily verified, using this result, the orthonormality relation (4.27), and the unitarity of the representation $D^{(l)}[R(\Lambda,p)]$, that all the scalar products defined in Sec. IV are invariant under a Poincaré transformation:

$$(U(a,\Lambda)\psi_A,U(a,\Lambda)\psi_B)^{(k)}=(\psi_A,\psi_B)^{(k)} . \tag{5.9}$$

Since the elements ψ of the Hilbert space are represented by the series of the kind

$$\psi(x,r)=\sum_{\eta\rho nlm}\int d^4p\,\delta(p^2-M_{\rho nl}{}^2)\theta(\eta p^0)g(\eta,\mathbf{p},\rho nl,m)\psi_{(\eta,\mathbf{p},\rho nl,m)}(x,r) , \tag{5.10}$$

which are convergent in the norm induced by the scalar products defined in Sec. IV, it is sufficient to verify Eq. (5.9) for the basis $\psi_{(\eta,\mathbf{p},\rho nl,m)}(x,r)$. For these functions we have

$$(U(a,\Lambda)\psi_{(\eta,\mathbf{p},\rho nl,m)},U(a,\Lambda)\psi_{(\eta',\mathbf{p}',\rho'n'l',m')})^{(k)}=e^{i(\Lambda(p'-p),a)}\sum_{ss'}(\psi_{(\eta,\Lambda\mathbf{p},\rho nl,s)},\psi_{(\eta',\Lambda\mathbf{p}',\rho'n'l',s')})^{(k)}$$
$$\times D^{(l)*}_{sm}[R^{-1}(\Lambda,p)]D^{(l')}_{s'm'}[R^{-1}(\Lambda,p')] . \tag{5.11}$$

With the normalization (4.27) we have

$$\begin{aligned}(\psi_{(\eta,\Lambda\mathbf{p},\rho nl,s)},\psi_{(\eta',\Lambda\mathbf{p}',\rho'n'l',s')})^{(k)}&=\langle\tilde{\psi}_{(\eta,\Lambda\mathbf{p},\rho nl,s)},\tilde{\psi}_{(\eta',\Lambda\mathbf{p}',\rho'n'l',s')}\rangle^{(k)}\\&=2\omega_{\rho nl}(\Lambda\mathbf{p})\delta^3(\Lambda\mathbf{p}-\Lambda\mathbf{p}')h^{(k)}_{\eta\rho}\delta_{\eta\eta'}\delta_{\rho\rho'}\delta_{nn'}\delta_{ll'}\delta_{ss'}\\&=2\omega_{\rho nl}(\mathbf{p})\delta^3(\mathbf{p}-\mathbf{p}')h^{(k)}_{\eta\rho}\delta_{\eta\eta'}\delta_{\rho\rho'}\delta_{nn'}\delta_{ll'}\delta_{ss'} ,\end{aligned} \tag{5.12}$$

since $\omega(\mathbf{p})\delta^3(\mathbf{p}-\mathbf{p}')$ is Lorentz invariant.

Using (5.12) in (5.11), the unitarity of the representation $D^{(l)}[R(\Lambda,p)]$, and the fact that for $\eta=\eta'$, $\rho=\rho'$, $n=n'$, $l=l'$, $\mathbf{p}=\mathbf{p}'$ we have $p^0=p^{0\prime}$, we get

$$\begin{aligned}(U(a,\Lambda)\psi_{(\eta,\mathbf{p},\rho nl,m)},U(a,\Lambda)\psi_{(\eta',\mathbf{p}',\rho'n'l',m')})^{(k)}&=2\omega_{\rho nl}(\mathbf{p})\delta^3(\mathbf{p}-\mathbf{p}')h^{(k)}_{\rho\eta}\delta_{\eta\eta'}\delta_{\rho\rho'}\delta_{nn'}\delta_{ll'}\delta_{mm'}\\&=(\psi_{(\eta,\mathbf{p},\rho nl,m)},\psi_{(\eta',\mathbf{p}',\rho'\eta'l',m')})^{(k)} .\end{aligned} \tag{5.13}$$

It follows that the scalar product for a general solution

$$(\psi_A,\psi_B)^{(k)}=\sum_{\eta\rho nlm}\int\frac{d^3p}{2\omega_{\rho nl}(\mathbf{p})}h^{(k)}_{\rho\eta}g_A^*(\eta,\mathbf{p},\rho nl,m)g_B(\eta,\mathbf{p},\rho nl,m) \tag{5.14}$$

is invariant, and Eq. (5.9) is verified.

For each element ψ of the Hilbert space, the inversion formula holds:

$$g(\eta,\mathbf{p},\rho nl,m)=h^{(k)}_{\rho\eta}(\psi_{(\eta,\mathbf{p},\rho nl,m)},\psi)^{(k)} , \tag{5.15}$$

where g coincides with the $\tilde{g}$ of Eq. (4.29), because of definition (4.31).

In place of the Wigner functions (4.25) we may define a basis of covariant functions:[40]

$$\chi_{(\eta,\mathbf{p},\rho nl,m)}(x,r) = \sum_{m'} \psi_{(\eta,\mathbf{p},\rho nl,m')}(x,r) D^{(l,0)}_{m'm}[L(\mathring{p},p)] , \quad (5.16)$$

where, as usual, it is necessary to extend the representation $D^{(l)}$ of the rotation group to the representation $D^{(l,0)}$ of the Lorentz group. The new basis has more simple transformation properties under the Poincaré group:

$$[U(a,\Lambda)\chi_{(\eta,\mathbf{p},\rho nl,m)}](x,r) = e^{i(\Lambda p,a)} \sum_{m'} \chi_{(\eta,\Lambda\mathbf{p},\rho nl,m')}(x,r) D^{(l,0)}_{m'm}(\Lambda) . \quad (5.17)$$

VI. CURRENTS

In Sec. IV we have defined the scalar product without any recourse to a current-conservation law. We may now wonder if it exists something like a conserved current.

It is easily recognized that the definition (3.15) of a tensor current for the free case cannot be generalized to the interacting case, as discussed in detail in Ref. 38. A tensor current of this kind should be the natural way for the generalization of the model to the case of the presence of an external electromagnetic field. But in general an external field will destroy the first-class character of the two constraints, if not properly introduced, as shown by the example given in Sec. III, Eq. (3.22). But even in this last case a tensor current does not exist.

One difficulty lies in the reducible character of the bilocal wave function $\psi(x_1,x_2)$ under Poincaré transformations. Each irreducible component of $\psi(x_1,x_2)$, with given mass and spin, transforms in a different way, so we may expect that, at most, a different current for each mass and spin value can exist.

Let us consider the integrand of the scalar product and its transformation properties, for a given energy value. Let us define

$$j^{(\eta\rho nl)}_{\mu(\mathbf{p}_1,m_1;\mathbf{p}_2,m_2)}(x,r) = \tilde{\psi}^*_{(\eta,\mathbf{p}_1,\rho nl,m_1)}(x,r) A^{(k)}_\mu \tilde{\psi}_{(\eta,\mathbf{p}_2,\rho nl,m_2)}(x,r) , \quad (6.1)$$

where $A^{(k)}_\mu$ are the operators (4.23), which operate on the arguments of both functions $\tilde{\psi}$. The functions $\tilde{\psi}$ are defined by Eq. (4.17′).

The objects (6.1) are conserved due to the wave equations (4.12) and (4.12′) in the following sense:

$$\frac{\partial}{\partial x^\mu} j^{\mu(\eta\rho nl)}_{(\mathbf{p}_1,m_1;\mathbf{p}_2,m_2)}(x,r) = 0 ,$$
$$\frac{\partial}{\partial r_0} j^{\mu(\eta\rho nl)}_{(\mathbf{p}_1,m_1;\mathbf{p}_2,m_2)}(x,r) = 0 . \quad (6.2)$$

Their transformation properties are easily deduced from that of the $\tilde{\psi}$. These are the same as that of the basis functions $\psi_{(\eta,\mathbf{p},\rho nl,m)}(x,r)$:

$$(U(a,\Lambda)\tilde{\psi}_{(\eta,\mathbf{p},\rho nl,m)})(x,r) = e^{i(\Lambda p,a)} \sum_{m'} \tilde{\psi}_{(\eta,\Lambda\mathbf{p},\rho nl,m')}(x,r) D^{(l)}_{m'm}[R^{-1}(\Lambda,p)] , \quad (6.3)$$

except that now the RHS is no longer equal to $\tilde{\psi}[\Lambda^{-1}(x-a),\Lambda^{-1}r]$, but instead we have

$$(U(a,\Lambda)\tilde{\psi}_{(\eta,\mathbf{p},\rho nl,m)})(x,r) = \tilde{\psi}_{(\eta,\mathbf{p},\rho nl,m)}[\Lambda^{-1}(x-a),R(\Lambda,p)r] . \quad (6.4)$$

The results (6.3) and (6.4) are easily deduced from Eq. (5.8) and the definition (4.8) of the function $\tilde{\psi}$.

It follows that the integrand defined in Eq. (6.1) has the following transformation rule:

$$j'^{(\eta\rho nl)}_{\mu(\mathbf{p}_1,m_1;\mathbf{p}_2,m_2)}(x,r) = \Lambda^\nu{}_\mu \sum_{m_1'm_2'} \tilde{\psi}^*_{(\eta,\Lambda\mathbf{p}_1,\rho nl,m_1')}(x,r) D^{(l)*}_{m_1'm_1}[R^{-1}(\Lambda,p_1)] A^{(k)}_\nu(\Lambda p_1,\Lambda p_2) D^{(l)}_{m_2'm_2}[R^{-1}(\Lambda,p_2)] \tilde{\psi}_{(\eta,\Lambda p_2,\rho nl,m_2')}(x,r) . \quad (6.5)$$

This is a very complicated transformation rule. Nevertheless, in spite of its complicated transformation properties, when integrated in d^3x and d^3r this kind of current gives an invariant (and conserved) scalar product, as we have seen in Sec. V.

For each mass level, determined by the quantum numbers (ρ,n,l), when observed without looking into the details of its internal structure, the composite system described by the wave function $\psi(x_1,x_2)$ must behave as an elementary particle of the given mass $M_{\rho nl}$ and spin l.

Indeed it is possible to define a wave function, depending solely on x, obeying to a Klein-Gordon equation with mass $M_{\rho nl}$, and transforming as an object of spin l.

Let us define

$$\phi^{(\rho nl)}_m(x) = (2\pi)^{-3/2} \int \frac{d^3p}{2\omega_{\rho nl}(\mathbf{p})} [\alpha(\mathbf{p},\rho nl,m) e^{-i(p,x)} + \beta(\mathbf{p},\rho nl,m) e^{i(px)}] , \quad (6.6)$$

where $p^0 = \omega_{\rho nl}(\mathbf{p}) = (\mathbf{p}^2 + M_{\rho nl}{}^2)^{1/2}$, and

$$\alpha(\mathbf{p},\rho nl,m) = \sum_{m'} D^{(l,0)}_{mm'}(L(p,\mathring{p})) g(+1,\mathbf{p},\rho nl,m') ,$$
$$\beta(\mathbf{p},\rho nl,m) = \sum_{m'm''} D^{(l,0)}_{mm'}(L(p,\mathring{p})) C_{m'm''}{}^{-1} \times g^*(-1,\mathbf{p},\rho nl,m'') . \quad (6.7)$$

The matrix $D^{(l,0)}$ belongs to the representation $(l,0)$ of the Lorentz group, as in Eq. (5.17). The matrix C is such that

$$D^{(l)*}(R) = C D^{(l)}(R) C^{-1} , \quad (6.8)$$

where R is a rotation (in which case $D^{(l,0)}$ reduces to $D^{(l)}$); see Ref. 50 for more details.

The coefficients $g(\eta,\mathbf{p},\rho nl,m)$ are defined in Eq. (5.15); their transformation rules under a Poincaré transformation (a,Λ), induced by the transformation of $\psi(x_1,x_2)$, are

$$g(\eta,\mathbf{p},\rho nl,m)\to e^{i(p,a)}\sum_{m'}D^{(l)}_{mm'}[R(\Lambda^{-1},p)]\times g(\eta,(\Lambda^{-1}\mathbf{p}),\rho nl,m') . \quad (6.9)$$

These in turn determine the following transformation rules for the coefficients α and β:

$$\alpha(\mathbf{p},\rho nl,m)\to e^{i(p,a)}\sum_{m'}D^{(l,0)}_{mm'}(\Lambda)\alpha((\Lambda^{-1}\mathbf{p}),\rho nl,m') ,$$
$$\beta(\mathbf{p},\rho nl,m)\to e^{-i(p,a)}\sum_{m'}D^{(l,0)}_{mm'}(\Lambda)\beta((\Lambda^{-1}\mathbf{p}),\rho nl,m') . \quad (6.10)$$

The functions $\phi^{(\rho nl)}_m(x)$ satisfy a Klein-Gordon equation

$$(\Box_x+M_{\rho nl}{}^2)\phi^{(\rho nl)}_m(x)=0 , \quad (6.11)$$

and transform in the following way:

$$(U(a,\Lambda)\phi^{(\rho nl)}_m)(x)=\sum_{m'}D^{(l,0)}_{mm'}(\Lambda)\phi^{(\rho nl)}_{m'}(\Lambda^{-1}(x-a)) . \quad (6.12)$$

The functions $\phi^{(\rho nl)}_m(x)$ are analogous to the fields defined in Ref. 50; the difference lies in the fact that the former transform as states while the last transform as the corresponding fields.

For the functions $\phi^{(\rho nl)}_m(x)$ it is not possible to define a four-vector conserved current as in the Klein-Gordon case, since the representation $D^{(l,0)}$ is not unitary; but, if we extend the representation $(l,0)$ to the representation $(l,0)\oplus(0,l)$ as in Ref. 50, it is possible to define such a current, and accordingly an invariant and conserved scalar product. As a by-product we get a new set of functions for which it is defined the parity operation.

The procedure is straightforward and we may refer to Ref. 50. We quote here some results.

The new representation is defined by

$$\mathscr{D}^{(l)}(\Lambda)=\begin{bmatrix}D^{(l,0)}(\Lambda) & 0\\ 0 & D^{(0,l)}(\Lambda)\end{bmatrix} . \quad (6.13)$$

For this new matrix it holds

$$\mathscr{D}^{(l)\dagger}(\Lambda)=\beta\mathscr{D}^{(l)}(\Lambda^{-1})\beta , \quad (6.14)$$

where

$$\beta=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} . \quad (6.15)$$

The new set of functions are

$$\psi^{(\rho nl)}(x)=\begin{bmatrix}\phi^{(\rho nl)}(x)\\ \chi^{(\rho nl)}(x)\end{bmatrix} , \quad (6.16)$$

denoting a column vector with $2(2l+1)$ components, where

$$\chi^{(\rho nl)}_m(x)=(2\pi)^{-3/2}\int\frac{d^3p}{2\omega_{\rho nl}(\mathbf{p})}[\tilde{\alpha}(\mathbf{p},\rho nl,m)e^{-i(p,x)}+\tilde{\beta}(\mathbf{p},\rho nl,m)e^{i(p,x)}] \quad (6.17)$$

and

$$\tilde{\alpha}(\mathbf{p},\rho nl,m)=\sum_{m'}D^{(l,0)}_{mm'}[L(\bar{p},\mathring{p})]g(+1,\mathbf{p},\rho nl,m') ,$$
$$\tilde{\beta}(\mathbf{p},\rho nl,m)=\sum_{m'}D^{(l,0)}_{mm'}[L(\bar{p},\mathring{p})]C_{m'm''}{}^{-1}\times g^*(-1,\mathbf{p},\rho nl,m'') , \quad (6.18)$$

with $\bar{p}^{\mu}\equiv(p^0,-\mathbf{p})$.

For the particular case of a boost, $L(p,\mathring{p})$, we have the relation between the representations $(l,0)$ and $(0,l)$ (see Ref. 50):

$$D^{(l,0)}[L(\bar{p},\mathring{p})]=D^{(0,l)}[L(p,\mathring{p})] , \quad (6.19)$$

so $\tilde{\alpha}$ and $\tilde{\beta}$ are the analogues of α and β for the $(0,l)$ representation. Since $D^{(l)}(R)$ can be equated to $D^{(l,0)}(R)$, as done for the demonstration of Eqs. (6.10), as well as to $D^{(0,l)}(R)$, we have the following transformation rules for $\tilde{\alpha}$ and $\tilde{\beta}$:

$$\tilde{\alpha}(\mathbf{p},\rho nl,m)\to e^{i(p,a)}\sum_{m'}D^{(0,l)}_{mm'}(\Lambda)\tilde{\alpha}((\Lambda^{-1}\mathbf{p}),\rho nl,m') ,$$
$$\tilde{\beta}(\mathbf{p},\rho nl,m)\to e^{-i(p,a)}\sum_{m'}D^{(0,l)}_{mm'}(\Lambda)\tilde{\beta}((\Lambda^{-1}\mathbf{p}),\rho nl,m') , \quad (6.20)$$

from which we easily get the transformation rule for $\chi^{(\rho nl)}_m(x)$:

$$[U(a,\Lambda)\chi^{(\rho nl)}_m](x)=\sum_{m'}D^{(0,l)}_{mm'}(\Lambda)\chi^{(\rho nl)}_{m'}(\Lambda^{-1}(x-a)) . \quad (6.21)$$

The adjoint of the function $\psi^{(\rho nl)}(x)$ is defined by

$$\bar{\psi}^{(\rho nl)}(x)=\psi^{\dagger(\rho nl)}(x)\beta . \quad (6.22)$$

The transformation properties of ψ and $\bar{\psi}$ are

$$(U(a,\Lambda)\psi^{(\rho nl)}_m)(x)=\sum_{m'}\mathscr{D}^{(l)}_{mm'}(\Lambda)\psi^{(\rho nl)}_{m'}(\Lambda^{-1}(x-a)) ,$$
$$(U(a,\Lambda)\bar{\psi}^{(\rho nl)}_m)(x)=\sum_{m'}\bar{\psi}^{(\rho nl)}_{m'}(\Lambda^{-1}(x-a))\mathscr{D}^{(l)}_{m'm}(\Lambda^{-1}) . \quad (6.23)$$

Finally we have that $\psi^{(\rho nl)}(x)$ satisfy the Klein-Gordon equation

$$(\Box_x+M_{\rho nl}{}^2)\psi^{(\rho nl)}_m(x)=0 , \quad (6.24)$$

and the current

$$j^{(\rho nl)}_\mu(x)=\sum_m\bar{\psi}^{(\rho nl)}_m(x)i\frac{\overleftrightarrow{\partial}}{\partial x^\mu}\psi^{(\rho nl)}_m(x) \quad (6.25)$$

transforms as a four-vector

$$j'^{(\rho nl)}_\mu(x')=\Lambda_\mu^{\cdot\nu}j^{(\rho nl)}_\nu(x) , \quad (6.26)$$

and is conserved

$$\frac{\partial}{\partial x^\mu} j^{\mu(\rho n l)}(x)=0 \; . \tag{6.27}$$

From Eqs. (6.26) and (6.27) we may define an invariant and conserved (in x^0) scalar product as for the Klein-Gordon case (see Sec. III for a positive-definite scalar product).

For the states $\psi^{(\rho nl)}(x)$ the operations of time reversal (T), charge conjugation (C), and space inversion or parity (P) are well defined [see Ref. 50 for all the details and for the other equations satisfied by $\psi^{(\rho nl)}(x)$].

We have here followed a particular construction, based on the representation $(j,0)\oplus(0,j)$. For a more general choice see again Ref. 50 and Refs. 51 and 39.

The current $j_\mu^{(\rho nl)}(x)$ defined in Eq. (6.25) is a bilinear functional of the original wave functions $\psi(x_1,x_2)$. It satisfies the conservation law (6.27) only when $\psi(x_1,x_2)$ is a solution of the original wave equations (4.2), and in this case it does not depend on r^0.

So we see that $\psi^{(\rho nl)}(x)$ describes an object of mass $M_{\rho nl}$ and spin l, without the details of the original internal structure.

The existence of the wave functions $\psi^{(\rho nl)}(x)$ was assured from general physical principles, so the construction given in the second part of this section is an *a posteriori* verification of the physical reliability of the model under study.

VII. THE CAUCHY PROBLEM

As we have seen in Sec. IV, only for eigenfunctions of the total momentum p with $\mathbf{p}=0$ the wave equations (2.11) become true differential equations. In this case the nonlocal equations (2.11) become Eqs. (4.5). The same equations are satisfied by the functions of the basis $\tilde{\psi}_{(\eta,0,\rho nl,m)}(x,r)$.

We can boost to a general momentum each solution found at $\mathbf{p}=0$, in order to get an arbitrary solution such as (4.20), but the boost needed for this operation will depend on the quantum numbers determining the mass and the sign of the energy, that is (η,ρ,n,l). This was the reason for the use of the operator $\hat{S}$ in Sec. IV.

This fact suggests that for this class of interactions, where the potential depends on $r_\perp{}^2$, the Cauchy problem can be formulated in the center-of-mass reference frame only.

In other words, the extended nature of the system requires a special frame for any statement about the initial data. In a general frame an infinite number of initial data is needed.

Let us notice that the transformation from $\psi(x,r)$ to $\tilde{\psi}(x,r)$ is not sufficient for our purpose. Indeed the $\tilde{\psi}(x,r)$ also satisfy a set of nonlocal equations (4.12).

Let us further notice that the expansion (4.20), where the coefficients $g(\eta,\mathbf{p},\rho nl,m)$ are constant in both times x^0 and r^0, does not allow the usual procedure of giving the function $\psi(x,r)$ at a fixed value of x^0 and r^0, in order to determine the general solution. The coefficients $g(\eta,\mathbf{p},\rho nl,m)$ are in fact scalar products involving $\psi(x,r)$ at all times x^0,r^0, due to the nonlocal nature of the operator $\hat{S}$.

This state of affairs suggests that x^0 and r^0 (or x_1^0 and x_2^0) are not the most convenient variables for specifying the two degrees of freedom for the canonical quantization without gauge fixing, along the lines discussed in Ref. 33. They have a natural interpretation as "time variables" in theories with local interactions, but the interactions we are studying are not local. Moreover, the natural gauge fixing, over which the Gupta-Bleuler quantization of the relativistic harmonic oscillator is based,[4,42,47] is $(p,r)=0$. This is a nonlocal concept in configuration space, and has no meaning in the plane (x^0,r^0).

We will therefore look for two new "time variables" T_1 and T_2, with the following properties: (i) they must be Lorentz scalars (like the proper time); (ii) the line $T_1=T_2$ in the (T_1,T_2) plane must be equivalent to the condition $(p,r)=0$; (iii) in the nonrelativistic limit the plane (T_1,T_2) must become the (t_1,t_2) plane, with the Newton equations of motion defined on the line $t_1=t_2$ [which is the nonrelativistic limit of the condition $(p,r)=0$]; (iv) they must be the canonical conjugated variables of the center-of-mass energies (p,p_i) $(i=1,2)$, which are constants of motion for this class of interactions, which privilege the center-of-mass reference frame.

In this research we will take advantage of the works of Aaberge[52] on the free relativistic scalar particle. In his work Aaberge finds a canonical transformation to a new set of variables, which, after quantization, give the Wigner theory of the free spinless particle; in particular the Newton-Wigner position operator belongs to this new basis. We will modify his procedure in that the new time variable will be chosen as $T=\eta(p,x)/(p^2)^{1/2}$, that is the "time" of the center of mass. In some sense this is a nonlocal description of a scalar particle; in terms of this new description it is possible to define new scalar products.

In Appendix B we give this new set of variables for the free particle, and in Appendix D we will extend the set to the two-particle case.

The new times are $T_i=\eta(p,x_i)/M$, and their conjugated variables $\varepsilon_i=\eta(p,p_i)/M$, where $M=(p^2)^{1/2}$. They are the times and the energies in the center-of-mass frame.

With these variables the classical constraints (2.1), (2.6) become

$$\begin{aligned}
&\chi_i=\varepsilon_i{}^2-M_i{}^2(\bar{\mathbf{q}},\bar{\mathbf{r}}) \quad (i=1,2) \; ,\\
&\chi_+=\frac{1}{\varepsilon^2}\{[\varepsilon^2-M_+{}^2(\bar{\mathbf{q}},\bar{\mathbf{r}})][\varepsilon^2-M_-{}^2(\bar{\mathbf{q}},\bar{\mathbf{r}})]\\
&\qquad\qquad +4\chi_-(\chi_-+m_1{}^2-m_2{}^2)\} \; ,\\
&\chi_-=\varepsilon\varepsilon_R-\tfrac{1}{2}(m_1{}^2-m_2{}^2) \; ,
\end{aligned} \tag{7.1}$$

where $M_i{}^2(\bar{\mathbf{q}},\bar{\mathbf{r}})=m_i{}^2+\bar{\mathbf{q}}^2+V(-\bar{\mathbf{r}}^2)$.

The wave equations become ($\bar{\mathbf{q}}\to -i\nabla_{\bar{r}}$)

$$\left[\frac{\partial^2}{\partial T_i{}^2}+\hat{M}_i{}^2\right]\Psi(\mathbf{z},\bar{\mathbf{r}},T_1,T_2)=0, \quad i=1,2 \; , \tag{7.2}$$

where Ψ is the wave function in the new representation (see below).

Equation (4.4) becomes ($\varepsilon=\varepsilon_1+\varepsilon_2$)

$$[-\nabla_{\bar{r}}{}^2+V(-\bar{r}^2)]\Psi(\mathbf{z},\bar{\mathbf{r}},\varepsilon_1,\varepsilon_2)=\beta(\varepsilon^2)\Psi(\mathbf{z},\bar{\mathbf{r}},\varepsilon_1,\varepsilon_2) \; , \tag{7.3}$$

where

$$\Psi(\mathbf{z},\overline{\mathbf{r}},\varepsilon_1,\varepsilon_2)=\int \frac{dT_1dT_2}{(2\pi)^2}e^{i(\varepsilon_1T_1+\varepsilon_2T_2)}\Psi(\mathbf{z},\overline{\mathbf{r}},T_1,T_2)\ . \tag{7.4}$$

The wave equations (7.2) do not determine the dependence on $\mathbf{z}$ of the wave function Ψ; this is in agreement with the fact that $\mathbf{z}$ is essentially an initial constant (see Appendix B). We can choose Ψ to be an eigenfunction of $\hat{\mathbf{k}}$.

From Eqs. (7.2) we see that the evolution of Ψ in T_1 and T_2 is completely determined once the following initial data are given:

$$\Psi(\mathbf{z},\overline{\mathbf{r}},0,0),\quad \frac{\partial\Psi}{\partial T_i}(\mathbf{z},\overline{\mathbf{r}},0,0),\quad \frac{\partial^2\Psi}{\partial T_1\partial T_2}(\mathbf{z},\overline{\mathbf{r}},0,0)\ ,$$

that is, in the new description, the Cauchy problem is well defined.

The choice of the center-of-mass reference frame, that is, the choice of Ψ being an eigenstate of $\hat{\mathbf{k}}$, with eigenvalues $\mathbf{k}=0$ ($\mathbf{p}=\varepsilon\mathbf{k}$), does not now change the situation. We have in any case a well-defined set of initial data.

As before we can classify the elementary solutions with the quantum numbers $(\eta,\mathbf{k},\rho nl,m)$; the basis of the elementary solutions is

$$\Psi_{(\eta,\mathbf{k},\rho nl,m)}(\mathbf{z},\overline{\mathbf{r}},T,T_R)=(2\pi)^{-4}\exp\left\{-i\left[\eta\left[M_{\rho nl}T+\frac{m_1{}^2-m_2{}^2}{2M_{\rho nl}}T_R\right]-\mathbf{k}\cdot\mathbf{z}\right]\right\}\chi_{nlm}(\overline{\mathbf{r}})\ , \tag{7.5}$$

where

$$T=\eta\frac{(p,x)}{M}=\tfrac{1}{2}(T_1+T_2),\quad T_R=\eta\frac{(p,r)}{M}=T_1-T_2\ ,$$

and where the exponent in the RHS can also be written in terms of the individual times T_1 and T_2:

$$\eta\left[M_{\rho nl}T+\frac{m_1{}^2-m_2{}^2}{2M_{\rho nl}}T_R\right]=\eta_1M_{1\rho nl}T_1+\eta_2M_{2\rho nl}T_2\ ,$$

with $\eta_1=\eta$, $\eta_2=\eta\rho$, $M_{\rho nl}=M_{1nl}+\rho M_{2nl}$ (remember that once and for all we have chosen $m_1>m_2$ and that $M_{1nl}{}^2-M_{2nl}{}^2=m_1{}^2-m_2{}^2$).

We can define the following scalar products. We write only two choices of the possible signatures, corresponding to the signatures with $k=3$ and $k=4$ of Sec. IV:

$$\langle\Psi_A,\Psi_B\rangle^{(3)}=\int d^3z\,d^3\overline{r}\,\Psi_A^*(\mathbf{z},\overline{\mathbf{r}},T_1,T_2)\tfrac{1}{2}[(1-\overleftarrow{\nabla}_z{}^2)^{1/2}+(1-\overrightarrow{\nabla}_z{}^2)^{1/2}]\tfrac{1}{2}\overleftrightarrow{\hat{\eta}}_1\tfrac{1}{2}\overleftrightarrow{\hat{\eta}}_2\Psi_B(\mathbf{z},\overline{\mathbf{r}},T_1,T_2)\ , \tag{7.6}$$

$$\langle\Psi_A,\Psi_B\rangle^{(4)}=\int d^3z\,d^3\overline{r}\,\Psi_A^*(\mathbf{z},\overline{\mathbf{r}},T_1,T_2)\tfrac{1}{2}[(1-\overleftarrow{\nabla}_z{}^2)^{1/2}+(1-\overrightarrow{\nabla}_z{}^2)^{1/2}]\tfrac{1}{2}(1+\overleftarrow{\hat{\eta}}_1\overrightarrow{\hat{\eta}}_1)\tfrac{1}{2}(1+\overleftarrow{\hat{\eta}}_2\overrightarrow{\hat{\eta}}_2)\Psi_B(\mathbf{z},\overline{\mathbf{r}},T_1,T_2)\ , \tag{7.6'}$$

where $\hat{\eta}_i=(i\partial/\partial T_i)/\hat{M}_i$.

These scalar products are constant in T_1 and T_2, due to Eq. (7.2).

For the elementary solutions we get

$$\begin{aligned}\langle\Psi_{(\eta_1\eta_2,\mathbf{k},nlm)},\Psi_{(\eta_1'\eta_2',\mathbf{k}',n'l'm')}\rangle^{(3)}&=\eta_1\eta_2(1+\mathbf{k}^2)^{1/2}\delta^3(\mathbf{k}-\mathbf{k}')\delta_{\eta_1\eta_1'}\delta_{\eta_2\eta_2'}\delta_{nn'}\delta_{ll'}\delta_{mm'}\ ,\\ \langle\Psi_{(\eta_1\eta_2,\mathbf{k},nlm)},\Psi_{(\eta_1'\eta_2',\mathbf{k}',n'l'm')}\rangle^{(4)}&=(1+\mathbf{k}^2)^{1/2}\delta^3(\mathbf{k}-\mathbf{k}')\delta_{\eta_1\eta_1'}\delta_{\eta_2\eta_2'}\delta_{nn'}\delta_{ll'}\delta_{mm'}\ .\end{aligned} \tag{7.7}$$

These results can be easily extended to wave packets.

Moreover by using the transformation properties of the wave functions $\Psi_{(\eta_1\eta_2,\mathbf{k},nlm)}(\mathbf{z},\overline{\mathbf{r}},T_1,T_2)$ given in Eq. (D16), we can check the unitarity of these irreducible representations as in Sec. V, Eqs. (5.9)—(5.13), with respect to the previous scalar products.

From Eqs. (D9) and (D13) we get the following relation between $\psi(x,r)$ and $\Psi(\mathbf{z},\overline{\mathbf{r}},T,T_R)$:

$$\psi(x,r)=(2\pi)^{-3}\int d^3z\,d^3k\,e^{-i\mathbf{k}\cdot\mathbf{z}}\Psi\left[\mathbf{z},\mathbf{r}-\mathbf{k}\left[r^0-\frac{\mathbf{k}\cdot\mathbf{r}}{1+(1+\mathbf{k}^2)^{1/2}}\right],(1+\mathbf{k}^2)^{1/2}x^0-\mathbf{k}\cdot\mathbf{x},(1+\mathbf{k}^2)^{1/2}r^0-\mathbf{k}\cdot\mathbf{r}\right]\ , \tag{7.8}$$

$$\Psi(\mathbf{z},\overline{\mathbf{r}},T,T_R)=(2\pi)^{-4}\int d^4x\int\frac{|\varepsilon|^3d\varepsilon\,d^3k}{(1+\mathbf{k}^2)^{1/2}}e^{i\{\varepsilon[(1+\mathbf{k}^2)^{1/2}x^0-\mathbf{k}\cdot\mathbf{x}-T]+\mathbf{k}\cdot\mathbf{z}\}}\times\Psi\left[x;r^0=(1+\mathbf{k}^2)^{1/2}T_R+\mathbf{k}\cdot\overline{\mathbf{r}};\mathbf{r}=\overline{\mathbf{r}}+\mathbf{k}\left[T_R+\frac{\mathbf{k}\cdot\overline{\mathbf{r}}}{1+(1+\mathbf{k}^2)^{1/2}}\right]\right]\ . \tag{7.8'}$$

With these formulas we may check that the elementary solutions (7.5) become those given in Eq. (4.18).

VIII. CONCLUSIONS AND OPEN PROBLEMS

In the previous sections we have carried out a detailed analysis of one of the simplest models for two interacting spinless relativistic particles, namely, the Todorov-Komar model, which is based on a pair of integrable wave equations for a wave function, which depends on the position four-vectors of both particles.

This analysis shows that it is possible to give a description of a two-body system, which is in agreement with the requirements of relativity. We have shown that the general solution can be decomposed into irreducible components describing bound states of given mass and spin. The mass-spin relation being dictated by the particular potential chosen.

In our opinion the most important result of this analysis is the demonstration of the existence of a positive-definite scalar product (choice $k=4$ of Sec. IV), which is constant in both times, x_1^0 and x_2^0, and is invariant under Poincaré transformations. This is the needed ingredient for a consistent quantum theory of the model. The scalar product turns out to be nonlocal, in the sense that its kernel in the momentum space representation is not a polynomial in p. This nonlocality does not seem to be related to the presence of a nonlocal interaction. Indeed the definition of the scalar product given in Sec. IV holds as well in the free case, and, as shown in Sec. III, also in the case of a single Klein-Gordon particle, the positive-definite scalar product is not local. Here the nonlocality has two origins. One, as in the Klein-Gordon case, due to the requirement of positive definiteness (in the non-positive-definite cases only the choice $k=1$ is free from this nonlocality); the other due to the operator $\hat{S}$ defined in Sec. IV, which is essentially a boost for the wave functions. This second origin of the nonlocality is peculiar of the present analysis.

The existence of such a scalar product allows us to represent the action of the Poincaré group on the space of the wave functions with unitary linear operators.

Each mass and spin state of the system can be described as an elementary system in agreement with the usual requirement of relativity (as shown in Sec. VI), in terms of a wave function satisfying a Klein-Gordon equation with the corresponding mass, and admitting a conserved four-vector current. All these features were expected, if the model had to correctly reproduce the physical characteristics of a compound system.

The initial data problem has been analyzed in the last section, where it was shown that the Cauchy problem is well defined in the center-of-mass frame. A canonical transformation was then performed in order to put the equation of motion in a form allowing a two-times (invariant times) description, in the sense of Refs. 6 and 32.

To this end a nonlinear realization of the Poincaré group was introduced. The two-times description so given is suitable for a choice of a gauge-fixing such as the one usually used in this kind of model, namely, $(p,r)=0$. The use of a gauge fixing is not in principle necessary, as explained in Ref. 33, but it can be done in order to go down to a more usual one-time description of the model.

The nonrelativistic limit is in agreement with the expected Newtonian mechanics, once reduced to the equal-time situation.

The equal-mass case was not considered explicitly. It can be easily recovered as a limit, apart from the restrictions due to the statistics.

We now list some of the problems which remain open.

First of all we assumed $p^2>0$. However, when $p^2\leq 0$ other solutions are possible, but they need a physical interpretation, since they are outside of the interpretation of the solutions of the wave equations, as bound states of a two-body system. In the case $p^2<0$, the solutions seem to correspond to an analytic continuation to complex values of l; from this point of view they seem to be related to a crossed channel. This is only a possible suggestion for a future research.

As a second open problem we may mention the external field problem. In Sec. III we have shown an example of a two-body system with mutual interaction, in the presence of an external field. In this example the first class character of the constraints is preserved; this is clearly a necessary requirement in order to have a quantization without gauge-fixing choices.

Another open problem is the connection of this kind of theory with the quantum field theory. This is the most important problem for the physical interpretation of these models. A first answer is given by the work of Refs. 11, 13, and 21: they are an approximation to the two-particle sector of quantum field theory, based upon the replacement of the Bethe-Salpeter equation with two coupled integrable relativistic wave equations. Therefore the bilocal wave function $\psi(x_1,x_2)$ is some kind of approximation of the amplitude $\langle 0 | T(\phi(x_1)\phi(x_2)) | 2\rangle$. The problem of the relative time has now a twofold interpretation. From the point of view of the constraint theory it is a gauge variable, conjugated to the constraint χ_- of Eqs. (2.6). This implies the choice of gauge fixing for the physical identification of the model: with a potential $V(r_\perp{}^2)$ the natural gauge fixing is $(p,r)=0$, i.e., an instantaneous interaction in the center-of-mass frame in accord with the Bethe-Salpeter equation. On the other side, as the coupled wave equations have been obtained with a canonical quantization without gauge fixings, the physical identification associated to the choice $(p,r)=0$ must be implemented with a probabilistic interpretation of the bilocal wave function $\psi(x_1,x_2)$. However, from the results of Sec. VII about the Cauchy problem, it turns out that the wave function $\Psi(\mathbf{z},\bar{\mathbf{r}},T_1,T_2)$ of Eq. (7.2) is the natural object to be interpreted, because the diagonal in the (T_1,T_2) plane is just $(p,r)=0$. $\mathbf{z},\bar{\mathbf{r}}$ are by construction a maximal set of observables for every value of T_1 and T_2, and not only for the natural "one-time" theory, $(p,r)=0$, of this class of models. These problems will be studied in more detail in Ref. 31. A second connection with quantum field theory is given by the effective Fokker action of Ref. 24 deduced from QED in its particle limit: this class of models can be viewed as a short transverse distance $(r_\perp{}^2\to 0)$ approximation[5] to the Fokker action. This point will be further investigated elsewhere.

Connected to these problems there is another important point, that is what has to be understood for causality (or acausality) of these models. When gauge fixing is imposed [like $(p,r)=0$], the resulting one-time dynamics seems to violate the causality, since the two particles interact at a spacelike separation. This is just the situation in the model studied in this paper. This aspect was for a long time questioned and answered in the literature, see, for instance, Ref. 53. However, let us make the following remark. As the choice of a different gauge fixing can be realized through a canonical transformation generated by the constraints,[28] the kind of causality satisfied by the model should not depend upon the chosen gauge fixing, at least in the case of potentials like $V(r_\perp{}^2,p^2)$. For the analogous situation in electrodynamics see Ref. 54.

In any case with potentials such as $V(r_\perp{}^2,p^2)$, the gauge fixing $(p,r)=0$ seems to be the "natural" choice: indeed, as shown in the last section, the Cauchy problem is well defined in the plane (T_1,T_2), and the choice $(p,r)=0$, that is of the diagonal in that plane, is dictated by the instantaneous approximations of the Bethe-Salpeter equation. From this point of view, more general models with a potential such as $V(r_\perp{}^2,p^2,(q,r_\perp),(p,q))$ selecting as "natural" a gauge fixing such as $r^2=0$, would describe a situation in which the two particles interact along the light cone. This line of research requires further investigation.

The natural prosecution of the present work should be the extension of this analysis to the case of two spin-$\frac{1}{2}$ particles.[12,13]

Let us finally remark that this model, as shown in Ref. 33, and its nonrelativistic limit,[6,32] are the first nontrivial examples of gauge theories, whose canonical quantization can be achieved without a gauge-fixing choice, as there are no ordering problems generating anomalies.

APPENDIX A: STANDARD BOOST AND NEW CANONICAL VARIABLES

Let us define the standard boost which carries the timelike four-vector p^μ from its rest frame, where it becomes $\mathring{p}^\mu \equiv (\eta M,0)$, with $\eta=\text{sgn}\,p^0$, $M=(p^2)^{1/2}$, to a general frame:

$$p^\mu = L^\mu{}_{\cdot\nu}(p,\mathring{p})\mathring{p}^\nu \, . \tag{A1}$$

Our choice of $L(p,\mathring{p})$ is

$$L^\mu{}_{\cdot\nu}(p,\mathring{p}) = \eta^\mu_\nu + 2\frac{p^\mu \mathring{p}_\nu}{M^2} - \frac{(p^\mu+\mathring{p}^\mu)(p_\nu+\mathring{p}_\nu)}{(p+\mathring{p},\mathring{p})} \, , \tag{A2}$$

or

$$||L^\mu{}_{\cdot\nu}(p,\mathring{p})|| = \begin{array}{c|cc|} & (\nu=0) & (\nu=j) \\ (\mu=0) & \eta\dfrac{p_0}{M} & -\eta\dfrac{p_j}{M} \\ (\mu=i) & \eta\dfrac{p^i}{M} & \delta^i_j - \dfrac{p^i p_j}{M(M+\eta p_0)} \end{array} \, , \tag{A3}$$

where $\mu\equiv(0,i)$, $i=1,2,3$, is a row index and $\nu\equiv(0,j)$, $j=1,2,3$, is a column index; $\mathbf{p}\equiv(p^i)$, and the same convention will be used for any spatial vector.

The parameter $\boldsymbol{\beta}$ of the Lorentz transformation is

$$\boldsymbol{\beta} = \eta\frac{\mathbf{p}}{p_0} \, , \tag{A4}$$

or, in terms of a complex rotation $\omega(p)$,

$$\gamma = (1-\beta^2)^{-1/2} = \eta\frac{p_0}{M} = \cosh\omega(p) \, , \qquad \eta\beta\gamma = \eta\frac{|\mathbf{p}|}{M} = \sin\omega(p) \, . \tag{A4'}$$

$L(p,\mathring{p})$ in exponential form is

$$L(p,\mathring{p}) = e^{\omega(p)I(p)} \, , \tag{A5}$$

where

$$||I^\mu{}_{\cdot\nu}(p)|| = \begin{vmatrix} 0 & -\dfrac{p_j}{|\mathbf{p}|} \\ \dfrac{p^i}{|\mathbf{p}|} & 0 \end{vmatrix} \tag{A5'}$$

[indices as in (A3)]. $I^\mu{}_{\cdot\nu}(p)$ has the properties

$$I_{\mu\nu}(p) = -I_{\nu\mu}(p), \quad I^3(p) = I(p) \, . \tag{A6}$$

Clearly we have $\det||L(p,\mathring{p})|| = +1$ for both signs of the energy p^0.

The inverse of $L(p,\mathring{p})$ is $L(\mathring{p},p)$

$$L^\mu{}_{\cdot\nu}(\mathring{p},p) = L_\nu{}^{\cdot\mu}(p,\mathring{p}) = L^\mu{}_{\cdot\nu}(p,\mathring{p})\,|_{\mathbf{p}\to-\mathbf{p}} = (e^{-\omega(p)I})^\mu{}_{\cdot\nu} \, . \tag{A7}$$

We define the following vierbeins:

$$\epsilon^\mu_A(p) = L^\mu{}_{\cdot A}(p,\mathring{p}) \, , \qquad \epsilon^A_\mu(p) = L^A{}_{\cdot\mu}(\mathring{p},p) = \eta^{AB}\eta_{\mu\nu}\epsilon^\nu_B(p) \quad (\mu,A=0,1,2,3) \, , \tag{A8}$$

which satisfy

$$\epsilon^A_\mu(p)\epsilon^\nu_A(p) = \eta^\nu_\mu, \quad \epsilon^A_\mu(p)\epsilon^\mu_B(p) = \eta^A_B \, , \tag{A9}$$

and

$$\eta^{\mu\nu} = \epsilon^\mu_A(p)\eta^{AB}\epsilon^\nu_B(p) = \frac{p^\mu p^\nu}{p^2} - \sum_{a=1}^{3}\epsilon^\mu_a(p)\epsilon^\nu_a(p) \, , \qquad \eta_{AB} = \epsilon^\mu_A(p)\eta_{\mu\nu}\epsilon^\nu_B(p), \quad \epsilon^\mu_a(p)\epsilon^\nu_{a'}(p)\eta_{\mu\nu} = -\delta_{aa'} \quad (a,a'=1,2,3) \, , \tag{A10}$$

where we used $\epsilon^0_\mu(p) = \epsilon^\mu_0(p) = \eta p^\mu/M$.

The $\epsilon^\mu_a(p)$ are the polarization four-vectors defined in Ref. 49.

With the boost $L(p,\mathring{p})$ we can define the new relative variables:

$$\bar{r}^A = L^A{}_{\cdot\mu}(\mathring{p},p)r^\mu = \epsilon^A_\mu(p)r^\mu \, , \qquad \bar{q}^A = L^A{}_{\cdot\mu}(\mathring{p},p)q^\mu = \epsilon^A_\mu(p)q^\mu \, . \tag{A11}$$

More explicitly

$$\bar{r}_0 = \eta\frac{(p,r)}{M}, \quad \bar{r}_a = \epsilon^\mu_a(p)r_\mu \quad (a=1,2,3) \, , \tag{A12}$$

and it is shown in Ref. 49 that $\bar{\mathbf{r}}\equiv(\bar{r}^a)$ is a Wigner vector of spin 1, while $\bar{r}^0$ is a scalar. The inverse relation is

$$r^\mu=\epsilon^\mu_A(p)\bar{r}^A=\eta\frac{p^\mu}{M}\bar{r}^0-\epsilon^\mu_a(p)\bar{r}_a \;; \tag{A12'}$$

the same relations hold for q^μ and $\bar{q}^A$.

The transformation from r,q to $\bar{r},\bar{q}$ is a canonical transformation generated by the function

$$\psi=\tfrac{1}{2}\omega(p)I_{\mu\nu}(p)S^{\mu\nu} , \tag{A13}$$

with

$$S^{\mu\nu}=r^\mu q^\nu-r^\nu q^\mu , \tag{A14}$$

That is

$$\bar{r}_\mu=e^{\{\psi,\cdot\}}r_\mu,\quad \bar{q}_\mu=e^{\{\psi,\cdot\}}q_\mu , \tag{A15}$$

and where

$$e^{\{\psi,\cdot\}}A=A+\{\psi,A\}+\tfrac{1}{2}\{\psi,\{\psi,A\}\}+\cdots .$$

If we apply the canonical transformation generated by ψ to all the variables x^μ,r^μ,p^μ,q^μ we get the new variables $\bar{x}^\mu,p^\mu,\bar{r}^A,\bar{q}^A$ (we will also use the notation $\bar{r}^0=\tau_R,\bar{q}^0=\varepsilon_R$), satisfying

$$\begin{aligned}&\{\bar{x}^\mu,p^\nu\}=-\eta^{\mu\nu} ,\\ &\{\tau_R,\varepsilon_R\}=-1 ,\\ &\{\bar{r}_a,\bar{q}_b\}=\delta_{ab} .\end{aligned} \tag{A16}$$

Explicitly

$$\begin{aligned}\bar{x}^\mu&=x^\mu+\frac{1}{2}\epsilon^\nu_A(p)\eta^{AB}\frac{\partial\epsilon^\rho_B(p)}{\partial p_\mu}S_{\rho\nu}\\ &=x^\mu-\frac{1}{M(M+\eta p_0)}\left[p_\nu S^{\nu\mu}+\eta M\left[S^{0\mu}-S^{0\nu}\frac{p_\nu p^\mu}{M^2}\right]\right],\\ \bar{r}^0&\equiv\tau_R=\eta\frac{(p,r)}{M} ,\quad \bar{q}^0\equiv\varepsilon_R=\eta\frac{(p,q)}{M} ,\\ \bar{\mathbf{r}}&=\mathbf{r}-\frac{\mathbf{p}}{M}\left[\eta r^0-\frac{\mathbf{p}\cdot\mathbf{r}}{M+\eta p_0}\right] ,\\ \bar{\mathbf{q}}&=\mathbf{q}-\frac{\mathbf{p}}{M}\left[\eta q^0-\frac{\mathbf{p}\cdot\mathbf{q}}{M+\eta p_0}\right] .\end{aligned} \tag{A17}$$

It can be verified that

$$(p,\bar{x})=(p,x) . \tag{A18}$$

This canonical transformation is a point transformation in p^μ, that is, is linear in x^μ, r^μ, and q^μ.

Its inverse is

$$\begin{aligned}x^0&=\bar{x}^0+\frac{1}{M^2}(\tau_R\mathbf{p}\cdot\bar{\mathbf{q}}-\varepsilon_R\mathbf{p}\cdot\bar{\mathbf{r}}) ,\\ \mathbf{x}&=\bar{\mathbf{x}}+\frac{\eta}{M}(\tau_R\bar{\mathbf{q}}-\varepsilon_R\bar{\mathbf{r}})+\frac{\bar{\mathbf{r}}\cdot\mathbf{p}\bar{\mathbf{q}}-\bar{\mathbf{q}}\cdot\mathbf{p}\bar{\mathbf{r}}}{M(M+\eta p_0)}\\ &\quad+\eta\frac{\tau_R\bar{\mathbf{q}}\cdot\mathbf{p}-\varepsilon_R\bar{\mathbf{r}}\cdot\mathbf{p}}{M^2(M+\eta p_0)}\mathbf{p} ,\end{aligned}$$

$$\begin{aligned}r^0&=\frac{\eta}{M}(\tau_R p_0+\mathbf{p}\cdot\bar{\mathbf{r}}) ,\\ \mathbf{r}&=\bar{\mathbf{r}}+\frac{\mathbf{p}}{M}\left[\eta\tau_R+\frac{\mathbf{p}\cdot\bar{\mathbf{r}}}{M+\eta p_0}\right] ,\\ q^0&=\frac{\eta}{M}(\varepsilon_R p^0+\mathbf{p}\cdot\bar{\mathbf{q}}) ,\\ \mathbf{q}&=\bar{\mathbf{q}}+\frac{\mathbf{p}}{M}\left[\eta\varepsilon_R+\frac{\mathbf{p}\cdot\bar{\mathbf{q}}}{M+\eta p_0}\right] .\end{aligned} \tag{A19}$$

The Wigner rotation corresponding to the Lorentz transformation Λ is

$$\begin{aligned}R^\mu_{\cdot\nu}(\Lambda,p)&=[L(\mathring{p},p)\Lambda^{-1}L(\Lambda p,\mathring{p})]^\mu_{\cdot\nu}\\ &=\begin{bmatrix}1&0\\0&R^a_{\cdot b}\end{bmatrix}\quad(a,b=1,2,3) ,\end{aligned} \tag{A20}$$

and the new variables transform under the Poincaré transformation (a,Λ) in the following way:

$$\begin{aligned}&p'^\mu=\Lambda^\mu_{\cdot\nu}p^\nu ,\\ &\bar{x}'^\mu=\Lambda^\mu_{\cdot\nu}\left[\bar{x}^\nu+\frac{1}{2}\bar{S}_{ab}R^a_{\cdot c}(\Lambda,p)\frac{\partial}{\partial p_\nu}R^b_{\cdot c}(\Lambda,p)\right]+a^\mu ,\\ &\tau'_R=\tau_R,\quad \varepsilon'_R=\varepsilon_R ,\\ &\bar{r}'_a=\bar{r}_bR^b_{\cdot a}(\Lambda,p),\quad \bar{q}'_a=\bar{q}_bR^b_{\cdot a}(\Lambda,p) ,\end{aligned} \tag{A21}$$

where $\bar{S}_{ab}=\bar{r}_a\bar{q}_b-\bar{r}_b\bar{q}_a$. That is, $\bar{\mathbf{r}}$ and $\bar{\mathbf{q}}$ are Wigner vectors.

The generators of the Poincaré group in terms of the new variables are

$$\begin{aligned}&p^\mu,M_{i0}=\bar{x}_ip_0-\bar{x}_0p_i+\eta\frac{\bar{S}_{ij}p_j}{M+\eta p_0} ,\\ &M_{ij}=\bar{x}_ip_j-\bar{x}_jp_i+\bar{S}_{ij} .\end{aligned} \tag{A22}$$

The transformation used in Sec. IV is instead that generated by $-\psi$. The new relative variables are obtained using in (A11) the inverse boost $L(p,\mathring{p})$. They are

$$\tilde{r}^\mu=L^\mu_{\cdot\nu}(p,\mathring{p})r^\nu,\quad \tilde{q}^\mu=L^\mu_{\cdot\nu}(p,\mathring{p})q^\nu , \tag{A23}$$

from which we get, in particular,

$$\begin{aligned}&\tilde{r}^2=r^2,\quad \tilde{r}_\perp{}^2=-\mathbf{r}^2,\quad (p,\tilde{r})=\eta Mr^0 ,\\ &\tilde{q}^2=q^2,\quad \tilde{q}_\perp{}^2=-\mathbf{q}^2,\quad (p,\tilde{q})=\eta Mq^0 .\end{aligned} \tag{A24}$$

To find the final set $\tilde{x},\tilde{p},\tilde{r},\tilde{q}$ from the set x,p,r,q we only have to replace $\mathbf{p}$ with $-\mathbf{p}$ in Eqs. (A17).

APPENDIX B: CANONICAL TRANSFORMATION TO THE VARIABLES ε, τ, $\mathbf{k}$, AND $\mathbf{z}$

In this appendix we consider a free scalar particle, whose dynamics is described in an eight-dimensional phase space ($\{x^\mu,p^\nu\}=-\eta^{\mu\nu}$) by the constraint $\chi=p^2-m^2$.

Let us define the following point transformation (not connected with the identity), restricted to $p^2>0$:

$$\varepsilon=\eta M,\quad \tau=\eta\frac{(p,x)}{M},$$
$$\mathbf{k}=\eta\frac{\mathbf{p}}{M}, \tag{B1}$$
$$\mathbf{z}=\eta M\left[\mathbf{x}-\frac{\mathbf{p}}{p_0}x^0\right],$$

where $M=(p^2)^{1/2}$, $\eta=\mathrm{sgn}p^0$; the inverse transformation is

$$p^0=\varepsilon(1+\mathbf{k}^2)^{1/2},\quad \mathbf{p}=\varepsilon\mathbf{k},$$
$$x^0=(1+\mathbf{k}^2)^{1/2}\left[\tau+\frac{1}{\varepsilon}\mathbf{k}\cdot\mathbf{z}\right],\quad \mathbf{x}=\frac{1}{\varepsilon}\mathbf{z}+\left[\tau+\frac{1}{\varepsilon}\mathbf{k}\cdot\mathbf{z}\right]\mathbf{k}. \tag{B1$'$}$$

The new variables τ, ε, $\mathbf{k}$, and $\mathbf{z}$ satisfy

$$\{\tau,\varepsilon\}=-1,\quad \{z^i,k^j\}=\delta^{ij}, \tag{B2}$$

and the constraint becomes

$$\chi=\varepsilon^2-m^2. \tag{B3}$$

τ coincides with x^0 in the center-of-mass frame, $\mathbf{k}$ is the space part of the four-velocity $k^\mu=\eta p^\mu/M$ $[k^0=(1+\mathbf{k}^2)^{1/2}]$, with $k^2=+1$, and $\mathbf{z}$ is a static coordinate corresponding to the initial position at $x^0=0$, apart from a mass factor.

The canonical transformation (B1) is a point transformation in p^μ, linear in x^μ. It is known[55] that a point transformation belongs to one of the classes of canonical transformations, which can be implemented as a quantum transformation without troubles.

In terms of these new variables we get a nonlinear realization of the Poincaré group. Its generators become

$$p^\mu=\varepsilon k^\mu,\quad M^{i0}=k^0z^i,\quad M^{ij}=z^ik^j-z^jk^i \tag{B4}$$

(only the rotations are linearly realized).

Under the Poincaré transformation (a,Λ) we get

$$\tau\to\tau'+k_\mu(\Lambda^{-1}a)^\mu,\quad \varepsilon\to\varepsilon'=\varepsilon,$$
$$z^i\to z^{i'}=\left[\Lambda^i_{\cdot j}-\frac{\Lambda^i_{\cdot\mu}k^\mu}{\Lambda^0_{\cdot\nu}k^\nu}\Lambda^0_{\cdot j}\right]z^j$$
$$+\varepsilon\left[\Lambda^i_{\cdot\mu}-\frac{\Lambda^i_{\cdot\nu}k^\nu}{\Lambda^0_{\cdot\rho}k^\rho}\Lambda^0_{\cdot\mu}\right](\Lambda^{-1}a)^\mu, \tag{B5}$$
$$k^\mu\to k'^\mu=\Lambda^\mu_{\cdot\nu}k^\nu.$$

It can be verified that this transformation satisfies the group composition law of the Poincaré group $(a_1,\Lambda_1)(a_2,\Lambda_2)=(a_1+\Lambda_1a_2,\Lambda_1\Lambda_2)$.

From (B5) we see that τ and ε are Lorentz scalars, and that $\mathbf{z}$ has an O(3) covariance, such as the Newton-Wigner position variable.

The integration measure on the whole phase space is of course invariant: $d^4p\,d^4x=d\varepsilon\,d\tau\,d^3k\,d^3z$; since for $p^2>0$ we have $d^4p=|\varepsilon|^3d\varepsilon\,d^3k/(1+\mathbf{k}^2)^{1/2}$, we get $d^4x=(1+\mathbf{k}^2)^{1/2}d\tau\,d^3z/|\varepsilon|^3$. From this we see that $(1+\mathbf{k}^2)^{1/2}d^3z$ is a Lorentz scalar.

By quantization we get for the operators x^μ and p^μ, which satisfy $[x^\mu,p^\nu]=-i\eta^{\mu\nu}$, the following transformation:

$$\varepsilon=\eta(p^2)^{1/2},\quad \tau=\frac{\eta(p,x)}{(p^2)^{1/2}}-\frac{3i}{2}\frac{\eta}{(p^2)^{1/2}},$$
$$\mathbf{k}=\eta\frac{\mathbf{p}}{(p^2)^{1/2}}, \tag{B6}$$
$$\mathbf{z}=\eta(p^2)^{1/2}\left[\mathbf{x}-\frac{\mathbf{p}}{p_0}x_0\right]-\frac{i}{2}\frac{(p^2)^{1/2}}{p_0{}^2}\mathbf{p},$$

which satisfies

$$[\tau,\varepsilon]=-i,\quad [z^i,k^j]=i\delta^{ij}. \tag{B7}$$

The new operators are formally Hermitian (with respect to the mathematical scalar product referred to in Sec. II).

The inverse transformation is $[k^0=(1+\mathbf{k}^2)^{1/2}]$

$$p^0=\varepsilon(1+\mathbf{k}^2)^{1/2},\quad \mathbf{p}=\varepsilon\mathbf{k},$$
$$x^0=k^0\tau+\frac{k^0}{\varepsilon}\mathbf{k}\cdot\varepsilon+\frac{i}{2}\frac{3+4\mathbf{k}^2}{k^0}, \tag{B8}$$
$$\mathbf{x}=\mathbf{k}\tau+\frac{\mathbf{z}}{\varepsilon}+\frac{\mathbf{k}}{\varepsilon}(\mathbf{k}\cdot\mathbf{z})+\frac{2i\mathbf{k}}{\varepsilon}.$$

The ordering in (B6) and (B8) corresponds to a symmetrization. These results can be easily obtained by the use of a general formula for quantum point canonical transformations as can be found in Ref. 56.

The Poincaré generators are

$$p^\mu=\varepsilon k^\mu,$$
$$M^{i0}=k^0z^i+\frac{i}{2}\frac{k^i}{k_0}, \tag{B9}$$
$$M^{ij}=z^ik^j-z^jk^i.$$

We verify that ε is a scalar and k^μ a four-vector. For the other variables we have

$$[p^\mu,\tau]=ik^\mu,\quad [M^{\mu\nu},\tau]=0,$$
$$[p^0,\mathbf{z}]=-i\varepsilon\mathbf{k}/k_0,\quad [p^i,z^j]=-i\varepsilon\delta^{ij},$$
$$[M^{i0},z^j]=-i\left[\frac{k^jz^i}{k_0}+\frac{i}{2k_0}\left[\delta^{ij}-\frac{k^ik^j}{k^{02}}\right]\right], \tag{B10}$$
$$[M^{ij},z^l]=i(\delta^{il}z^j-\delta^{jl}z^i).$$

Under a finite Poincaré transformation the Eqs. (B5) give the correct rule of transformation for τ, ε, and k^μ. Instead $\mathbf{z}$ transforms in a much more complicated way.

The wave equation corresponding to the classical constraint p^2-m^2 is, as well known, the Klein-Gordon equation for the wave function $\phi(x^\mu)=\langle x^\mu|\phi\rangle$. In order to formulate this wave equation in terms of the new coordinates, we introduce the new basis $|\varepsilon,\mathbf{k}\rangle$ and its dual $|\tau,\mathbf{z}\rangle$, eigenvectors of the corresponding operators.

Since it is useful to have an invariant normalization, remembering that $d^3k/(1+\mathbf{k}^2)^{1/2}$ is a scalar, we put

$$\langle\varepsilon',\mathbf{k}'|\varepsilon,\mathbf{k}\rangle=\frac{(1+\mathbf{k}^2)^{1/2}}{|\varepsilon|^3}\delta(\varepsilon-\varepsilon')\delta^3(\mathbf{k}-\mathbf{k}'),$$
$$\int|\varepsilon,\mathbf{k}\rangle\frac{|\varepsilon|^3d\varepsilon\,d^3k}{(1+\mathbf{k}^2)^{1/2}}\langle\varepsilon,\mathbf{k}|=\mathbb{1}. \tag{B11}$$

For the dual basis $|\tau,\mathbf{z}\rangle$ we have equivalently an invariant normalization if we take into account that $(1+\mathbf{k}^2)^{1/2}d^3z$ is a scalar. This means a more complicated definition:

$$\langle\tau',\mathbf{z}'\,|\,\tau,\mathbf{z}\rangle=\frac{|\hat{\varepsilon}|^3}{(1+\hat{\mathbf{k}}^2)^{1/2}}\delta(\tau-\tau')\delta^3(\mathbf{z}-\mathbf{z}')\ ,$$
$$\int |\tau,\mathbf{z}\rangle d\tau\, d^3z\frac{(1+\hat{\mathbf{k}}^2)^{1/2}}{|\hat{\varepsilon}|^3}\langle\tau,\mathbf{z}\,|=1\ , \qquad (B12)$$

where $\hat{\varepsilon}=i\partial/\partial\tau$ and $\hat{\mathbf{k}}=-i\nabla_z$.

The RHS of the first equation is a well-defined distribution. In the second equation the factor $(1+\mathbf{k}^2)^{1/2}/|\varepsilon|^3$ is to be interpreted as the nonlocal operator $(1-\nabla_z{}^2)^{1/2}/|\partial/\partial\tau|^3$ applied on a function of τ and $\mathbf{z}$, $f(\tau,\mathbf{z})=\langle\tau,\mathbf{z}\,|\,f\rangle$.

We have of course

$$\langle\tau,\mathbf{z}\,|\,\varepsilon,\mathbf{k}\rangle=(2\pi)^{-2}e^{-i(\varepsilon\tau-\mathbf{k}\cdot\mathbf{z})}\ . \qquad (B13)$$

Observe that though $\varepsilon\tau$ is a scalar $\mathbf{k}\cdot\mathbf{z}$ is not.

The transformation from the old to the new description is determined by the formula

$$\langle p^\mu\,|\,\varepsilon,\mathbf{k}\rangle=\frac{(1+\mathbf{k}^2)^{1/2}}{|\varepsilon|^3}\delta(\varepsilon-\eta(p^2)^{1/2})\delta^3\left[\mathbf{k}-\eta\frac{\mathbf{p}}{(p^2)^{1/2}}\right]\ ,$$
$$\langle x^\mu\,|\,\varepsilon,\mathbf{k}\rangle=(2\pi)^{-2}e^{-i\varepsilon k_\mu x^\mu}\ , \qquad (B14)$$

and

$$\langle p^\mu\,|\,\tau,\mathbf{z}\rangle=(2\pi)^{-2}\exp\left[i\eta\left[(p^2)^{1/2}\tau-\frac{\mathbf{p}\cdot\mathbf{z}}{(p^2)^{1/2}}\right]\right]\ ,$$
$$\langle x^\mu\,|\,\tau,\mathbf{z}\rangle =\int_{(p^2>0)}\frac{d^4p}{(2\pi)^4}e^{-i(p,x)}\exp\left[i\eta\left[(p^2)^{1/2}\tau-\frac{\mathbf{p}\cdot\mathbf{z}}{(p^2)^{1/2}}\right]\right]\ . \qquad (B15)$$

Finally the relation between a function of $(\tau,\mathbf{z})$ and its transformed form, function of $(\varepsilon,\mathbf{k})$, is given by

$$\langle\tau,\mathbf{z}\,|\,\phi\rangle=\phi(\tau,\mathbf{z}) =\int\frac{|\varepsilon|^3 d\varepsilon\, d^3k}{(2\pi)^2(1+\mathbf{k}^2)^{1/2}}e^{-i(\varepsilon\tau-\mathbf{k}\cdot\mathbf{z})}\phi(\varepsilon,\mathbf{k})\ ,$$
$$\langle\varepsilon,\mathbf{k}\,|\,\phi\rangle=\phi(\varepsilon,\mathbf{k}) =\frac{(1+\mathbf{k}^2)^{1/2}}{|\varepsilon|^3}\int\frac{d\tau\, d^3z}{(2\pi)^2}e^{i(\varepsilon\tau-\mathbf{k}\cdot\mathbf{z})}\phi(\tau,\mathbf{z})\ . \qquad (B16)$$

The Klein-Gordon equation becomes

$$\left[-\frac{\partial^2}{\partial\tau^2}-m^2\right]\phi(\tau,\mathbf{z})=0 \qquad (B17)$$

or

$$(\varepsilon^2-m^2)\phi(\varepsilon,\mathbf{k})=0\ . \qquad (B17')$$

The transformation properties of $\phi(\varepsilon,\mathbf{k})$ and $\phi(\tau,\mathbf{z})$ are as follows. From

$$(U(a,\Lambda)\phi)(x)=\phi(\Lambda^{-1}(x-a))\ ,$$
$$(U(a,\Lambda)\phi)(p)=e^{i(p,a)}\phi(\Lambda^{-1}p)\ ,$$

we get

$$(U(a,\Lambda)\phi)(\varepsilon,\mathbf{k})=e^{i\varepsilon(k,a)}\phi(\varepsilon,(\mathbf{\Lambda}^{-1}\mathbf{k}))\ , \qquad (B18)$$

and, after some calculation using Eqs. (B16), we get

$$(U(a,\Lambda)\phi)(\tau,\mathbf{z}) =\int\frac{d^3z'd^3k}{(2\pi)^3}e^{i[(\mathbf{\Lambda k})\cdot\mathbf{z}-\mathbf{k}\cdot\mathbf{z}']}\phi(\tau-(\Lambda k,a),\mathbf{z}')\ . \qquad (B19)$$

Equation (B17) does not determine the $\mathbf{z}$ dependence of $\phi(\tau,\mathbf{z})$, in agreement with the static meaning of $\mathbf{z}$. There are only two eigenvalues of ε, that is, $\pm m$, which are infinitely degenerate. The elementary solutions can be chosen as eigenfunctions of $\hat{\mathbf{k}}$

$$\phi_{(\eta,\mathbf{k})}(\tau,\mathbf{z})=(2\pi)^{-3/2}e^{-i(\eta m\tau-\mathbf{k}\cdot\mathbf{z})}\ . \qquad (B20)$$

We can define two scalar products, which are the analogies of those defined in Sec. III, which are conserved in τ and Poincaré invariant:

$$(\phi_A,\phi_B)_1=\int d^3z\,\phi_A^*(\tau,\mathbf{z})\tfrac{1}{2}[(1-\overleftarrow{\nabla}_z{}^2)^{1/2}+(1-\nabla_z{}^2)^{1/2}] \times\frac{i}{2m}\frac{\overleftrightarrow{\partial}}{\partial\tau}\phi_B(\tau,\mathbf{z})\ ,$$
$$(\phi_A,\phi_B)_2=\int d^3z\,\phi_A^*(\tau,\mathbf{z})\tfrac{1}{2}[(1-\overleftarrow{\nabla}_z{}^2)^{1/2}+(1-\nabla_z{}^2)^{1/2}] \times\frac{1}{2}\left[1+\frac{1}{m^2}\frac{\overleftarrow{\partial}}{\partial\tau}\frac{\partial}{\partial\tau}\right]\phi_B(\tau,\mathbf{z})\ . \qquad (B21)$$

On the set of solutions (B20) they give

$$(\phi_{(\eta',\mathbf{k}')},\phi_{(\eta,\mathbf{k})})_1=\eta\delta_{\eta\eta'}(1+\mathbf{k}^2)^{1/2}\delta^3(\mathbf{k}-\mathbf{k}')\ ,$$
$$(\phi_{(\eta',\mathbf{k}')},\phi_{(\eta,\mathbf{k})})_2=\delta_{\eta\eta'}(1+\mathbf{k}^2)^{1/2}\delta^3(\mathbf{k}-\mathbf{k}')\ . \qquad (B22)$$

The initial data in τ are simply $\phi(0,\mathbf{z})$ and $\partial\phi/\partial\tau(0,\mathbf{z})$.

We close this appendix by observing that the minimal coupling for Eq. (B17) is obtained by requiring an invariance under a local (in τ) phase transformation. This means the minimal substitution $\partial/\partial\tau\to\partial/\partial\tau-iV(\tau,\mathbf{z})$, with the associated gauge transformation for the scalar field V: $V(\tau,\mathbf{z})\to V(\tau,\mathbf{z})+(\partial/\partial\tau)\alpha(\tau)$. A pure gauge is given by $V=V(\tau)$.

Of course when this coupling is translated in the configuration space we get a highly nonlocal coupling.

APPENDIX C: THE SINGULAR FUNCTION

In Appendix B we have found the expression of the transition matrix element from the representation x^μ to the new representation $(\tau,\mathbf{z})$. They are given by

$$\langle x^\mu\,|\,\tau,\mathbf{z}\rangle=\int_{(p^2>0)}\frac{d^4p}{(2\pi)^4}e^{-i(p,x)} \times\exp\left[i\eta\left[(p^2)^{1/2}\tau-\frac{\mathbf{p}\cdot\mathbf{z}}{(p^2)^{1/2}}\right]\right]\ . \qquad (B15)$$

In this appendix we give the evaluation of this distribution. Using the results quoted in Ref. 35 we have

$$\langle x^\mu | \tau,\mathbf{z}\rangle = \frac{i}{2\pi}\int_0^\infty dm^2 \Bigg[-e^{im\tau}\left[\frac{\epsilon(x^0)\delta(x'^2)}{4\pi} + \frac{mi}{8\pi(x'^2)^{1/2}}\theta(x'^2)[N_1(m(x'^2)^{1/2}) + i\epsilon(x^0)J_1(m(x'^2)^{1/2})] - \frac{mi}{4\pi^2(-x'^2)^{1/2}}\theta(-x'^2)\mathscr{K}_1(m(-x'^2)^{1/2})\right] + e^{-im\tau}\left[\frac{\epsilon(x^0)\delta(x''^2)}{4\pi} - \frac{mi}{8\pi(x''^2)^{1/2}}\theta(x''^2)[N_1(m(x''^2)^{1/2}) - i\epsilon(x^0)J_1(m(x''^2)^{1/2})] + \frac{mi}{4\pi^2(-x''^2)^{1/2}}\theta(-x''^2)K_1(m(-x''^2)^{1/2})\right]\Bigg] , \quad \text{(C1)}$$

where

$$x'^2 = x_0{}^2 - \left[\mathbf{x} - \frac{\mathbf{z}}{m}\right]^2 , \quad x''^2 = x_0{}^2 - \left[\mathbf{x} + \frac{\mathbf{z}}{m}\right]^2 . \quad \text{(C2)}$$

The singularities are situated only on the two light cones defined by $x'^2=0$ and $x''^2=0$. These two light cones intersect along a hyperbola, which lies in a plane orthogonal to $\mathbf{z}$. Outside these two light cones it falls off exponentially at infinity in spatial directions.[57] When m goes to zero the two light cones move away each from the other at infinite distance. For m infinite they collapse in one light cone.

APPENDIX D: EXTENSION OF APPENDIX B TO THE TWO-PARTICLE CASE

In this appendix we give the extension of the transformation of variables studied in Appendix B to the two-particle case.

Starting from the variables defined in Eqs. (A17) we give the following transformation to the new variables $T,\varepsilon,T_R,\varepsilon_R,\mathbf{z},\mathbf{k},\bar{\mathbf{r}},\bar{\mathbf{q}}$, which is again a point canonical transformation in the momentum p, and then linear in the other variables:

$$T = \tfrac{1}{2}(T_1+T_2) = \eta\frac{(p,\bar{x})}{M} = \eta\frac{(p,x)}{M} ,$$

$$\varepsilon = \varepsilon_1+\varepsilon_2 = \eta M, \quad T_R = T_1 - T_2 = \eta\frac{(p,r)}{M} , \quad \text{(D1a)}$$

$$\varepsilon_R = \frac{1}{2}(\varepsilon_1 - \varepsilon_2) = \eta\frac{(p,q)}{M} ,$$

where $M=(p^2)^{1/2}$, and more

$$z^i = \eta M\left[\bar{x}^i - \frac{p^i}{p_0}\bar{x}^0\right] = \eta M\left[x^i - \frac{p^i}{p_0}x^0 + \eta\frac{S^{i0}}{M} + \left[S^{ik} - \frac{p^i}{p^0}S^{0k}\right]\frac{p_k}{M(M+\eta p_0)}\right] ,$$

$$k^i = \eta\frac{p^i}{M}, \quad k^0 = (1+\mathbf{k}^2)^{1/2} ,$$

$$\bar{\mathbf{r}} = \mathbf{r} - \frac{\mathbf{p}}{M}\left[\eta r^0 - \frac{\mathbf{p}\cdot\mathbf{r}}{M+\eta p_0}\right] , \quad \text{(D1b)}$$

$$\bar{\mathbf{q}} = \mathbf{q} - \frac{\mathbf{p}}{M}\left[\eta q^0 - \frac{\mathbf{p}\cdot\mathbf{q}}{M+\eta p_0}\right] .$$

We have also

$$T_1 = T + \tfrac{1}{2}T_R = \eta\frac{(p,x_1)}{M} ,$$

$$T_2 = T - \tfrac{1}{2}T_R = \eta\frac{(p,x_2)}{M} ,$$

$$\varepsilon_1 = \tfrac{1}{2}\varepsilon + \varepsilon_R = \eta\frac{(p,p_1)}{M} , \quad \text{(D2)}$$

$$\varepsilon_2 = \tfrac{1}{2}\varepsilon - \varepsilon_R = \eta\frac{(p,p_2)}{M} .$$

The inverse transformation, from the new variables to the original variables x^μ, r^μ, p^μ, and q^μ, is

$$x^0 = \frac{k^0}{\varepsilon}(\varepsilon T + \mathbf{k}\cdot\mathbf{z}) + \frac{1}{\varepsilon}(T_R\mathbf{k}\cdot\bar{\mathbf{q}} - \varepsilon_R\mathbf{k}\cdot\bar{\mathbf{r}}) ,$$

$$x^i = \frac{1}{\varepsilon}[z^i + (\varepsilon T + \mathbf{k}\cdot\mathbf{z})k^i] + \frac{1}{\varepsilon}(T_R\bar{q}^i - \varepsilon_R\bar{r}^i) - \frac{\bar{S}^{ik}k^k}{\varepsilon(1+k_0)} + \frac{k^i}{\varepsilon(1+k_0)}(T_R\mathbf{k}\cdot\bar{\mathbf{q}} - \varepsilon_R\mathbf{k}\cdot\bar{\mathbf{r}}) ,$$

$$p^\mu = \varepsilon k^\mu, \quad r^0 = k^0 T_R + \mathbf{k}\cdot\bar{\mathbf{r}} ,$$

$$\mathbf{r} = \bar{\mathbf{r}} + \mathbf{k}\left[T_R + \frac{\mathbf{k}\cdot\bar{\mathbf{r}}}{1+k_0}\right] , \quad \text{(D3)}$$

$$q^0 = k^0\varepsilon_R + \mathbf{k}\cdot\bar{\mathbf{q}}, \quad \mathbf{q} = \bar{\mathbf{q}} + \mathbf{k}\left[\varepsilon_R + \frac{\mathbf{k}\cdot\bar{\mathbf{q}}}{1+k_0}\right] .$$

We have also

$$r_\perp{}^2 = -\bar{\mathbf{r}}^2, \quad q^2 = \varepsilon_R{}^2 - \bar{\mathbf{q}}^2, \quad (p,q) = \varepsilon\varepsilon_R .$$

The generators of the Poincaré group become

$$p^\mu = \varepsilon k^\mu ,$$

$$M^{ij} = z^i k^j - z^j k^i + \bar{S}^{ij}, \quad \bar{S}^{ij} = \bar{r}^i\bar{q}^j - \bar{r}^j\bar{q}^i , \quad \text{(D4)}$$

$$M^{i0} = k^0 z^i + \frac{\bar{S}^{ik}k^k}{1+k_0} .$$

Since the transformation (D1a) and (D1b) is canonical, that is, we have

$$\{\varepsilon,T\}=\{\varepsilon_R,T_R\}=1\ ,\qquad \{z^i,k^j\}=\delta^{ij}\ ,\qquad \{\bar r^{\,i},\bar q^{\,j}\}=\delta^{ij}\ , \tag{D5}$$

with the other Poisson brackets equal to zero, the generators (D4) satisfy the Poincaré algebra.

The original symplectic structure $\{x_i^\mu,p_j^\nu\}=\delta_{ij}\eta^{\mu\nu}$ is invariant under a Poincaré transformation; as a consequence a Poincaré transformation on the new variables is again represented by a canonical transformation, which is however a nonlinear transformation. That is, a finite Poincaré transformation (a,Λ) is now represented by the following (nonlinear) canonical transformation:

$$\begin{aligned}&k^\mu\to\Lambda^\mu_{\cdot\nu}k^\nu,\quad T\to T+a^\mu\Lambda_\mu^{\cdot\nu}k_\nu,\quad \bar r^{\,i}\to\bar r^{\,j}R_j^{\cdot i}(\Lambda,\mathbf{k}),\quad \bar q^{\,i}\to\bar q^{\,j}R_j^{\cdot i}(\Lambda,\mathbf{k})\ ,\\ &z^i\to\left[\Lambda^i_{\cdot j}-\frac{\Lambda^i_{\cdot\nu}k^\nu}{\Lambda^0_{\cdot\lambda}k^\lambda}\Lambda^0_{\cdot j}\right]z^j+\frac{1}{2}\left[\Lambda^i_{\cdot\mu}-\frac{\Lambda^i_{\cdot\nu}k^\nu}{\Lambda^0_{\cdot\lambda}k^\lambda}\Lambda^0_{\cdot\mu}\right]\bar S^{\,ij}R_i^{\cdot k}(\Lambda,\mathbf{k})\frac{\partial}{\partial k_\mu}R_j^{\cdot k}(\Lambda,\mathbf{k})+\varepsilon a^\nu\Lambda_\nu^{\cdot\mu}\left[\Lambda^i_{\cdot\mu}-\frac{\Lambda^i_{\cdot\nu}k^\nu}{\Lambda^0_{\cdot\lambda}k^\lambda}\Lambda^0_{\cdot\mu}\right]\ ,\end{aligned} \tag{D6}$$

where

$$\begin{aligned}&R^{\,i}_{\cdot j}(\Lambda,\mathbf{k})=\Lambda_j^{\cdot i}-\frac{\Lambda_0^{\cdot i}\Lambda_j^{\cdot\mu}k_\mu}{1+\Lambda^0_{\cdot\lambda}k^\lambda}-\frac{k^i}{1+k_0}\left[\Lambda_j^{\cdot 0}-\frac{(\Lambda^0_{\cdot 0}-1)\Lambda_j^{\cdot\mu}k_\mu}{1+\Lambda^0_{\cdot\lambda}k^\lambda}\right]\ ,\\ &\frac{1}{2}\bar S^{\,ij}R(\Lambda,\mathbf{k})\frac{\partial}{\partial k_\mu}R^{\,j}_{\cdot k}(\Lambda,\mathbf{k})=\frac{\bar S^{\,ij}}{1+\Lambda^0_{\cdot\lambda}k^\lambda}\left[\eta_i^\mu\left[\Lambda^0_{\cdot j}-\frac{(\Lambda^0_{\cdot 0}-1)k_j}{1+k_0}\right]-(\eta_0^\mu+k^\mu)\frac{k_i\Lambda^0_{\cdot j}}{1+k_0}\right]\ .\end{aligned}$$

The quantization is performed as in Appendix B. The commutation relations for the original variables are

$$[\hat p^{\,\mu},\hat x^{\,\nu}]=[\hat q^{\,\mu},\hat r^{\,\nu}]=i\eta^{\mu\nu}\ . \tag{D7a}$$

For the variables of Eq. (A17) we have

$$[\hat p^{\,\mu},\hat{\tilde x}^{\,\nu}]=i\eta^{\mu\nu},\quad [\hat\varepsilon_R,\hat T_R]=i,\quad [\bar r^{\,i},\bar q^{\,j}]=i\delta^{ij}\ . \tag{D7b}$$

Finally, for the new variables we have

$$[\hat\varepsilon,\hat T]=[\hat\varepsilon_R,\hat T_R]=i,\quad [\hat z^{\,i},\hat k^{\,j}]=[\hat{\bar r}^{\,i},\hat{\bar q}^{\,j}]=i\delta^{ij}\ . \tag{D7c}$$

The ordering of the operators is the same as in Appendix B, and the same inversion formulas hold.

The operators representing the Poincaré generators are

$$\hat p^{\,\mu}=\hat\varepsilon\hat k^{\,\mu},\quad \hat M^{\,ij}=\hat z^{\,i}\hat k^{\,j}-\hat z^{\,j}\hat k^{\,i}+\hat{\bar S}^{\,ij},\quad \hat M^{\,i0}=\hat z^{\,i}\hat k^{\,0}-i\frac{\hat k^{\,i}}{2\hat k^{\,0}}+\frac{\hat{\bar S}^{\,ij}\hat k^{\,j}}{1+\hat k_0}\ , \tag{D8}$$

and satisfy the Poincaré commutator algebra.

The relations analogous to (B11) and (B12) are now

$$\begin{aligned}&\langle\varepsilon',\mathbf{k}',\varepsilon_R',\bar{\mathbf{q}}'\mid\varepsilon,\mathbf{k},\varepsilon_R,\bar{\mathbf{q}}\rangle=\frac{(1+\mathbf{k}^2)^{1/2}}{|\varepsilon|^3}\delta(\varepsilon-\varepsilon')\delta^3(\mathbf{k}-\mathbf{k}')\delta(\varepsilon_R-\varepsilon_R')\delta^3(\bar{\mathbf{q}}-\bar{\mathbf{q}}')\ ,\\ &\int|\varepsilon,\mathbf{k},\varepsilon_R,\bar{\mathbf{q}}\rangle\frac{|\varepsilon|^3 d\varepsilon\, d^3k\, d\varepsilon_R d^3\bar q}{(1+\mathbf{k}^2)^{1/2}}\langle\varepsilon,\mathbf{k},\varepsilon_R,\bar{\mathbf{q}})=\mathbb{1}\ ,\end{aligned} \tag{D9}$$

and

$$\begin{aligned}&\langle T',\mathbf{z}',T_R',\bar{\mathbf{r}}'\mid T,\mathbf{z},T_R,\bar{\mathbf{r}}\rangle=\frac{|\hat\varepsilon|^3}{(1+\mathbf{k}^2)^{1/2}}\delta(T-T')\delta^3(\mathbf{z}-\mathbf{z}')\delta(T_R-T_R')\delta^3(\bar{\mathbf{r}}-\bar{\mathbf{r}}')\ ,\\ &\int|T,\mathbf{z},T_R,\bar{\mathbf{r}}\rangle\frac{|\hat\varepsilon|^3}{(1+\hat{\mathbf{k}}^2)^{1/2}}\langle T,\mathbf{z},T_R,\bar{\mathbf{r}}\,|\,dT\,d^3z\,dT_R d^3\bar r=\mathbb{1}\ .\end{aligned} \tag{D10}$$

The transformation coefficients from one basis to the other are

$$\langle T,\mathbf{z},T_R,\bar{\mathbf{r}}\mid\varepsilon,\mathbf{k},\varepsilon_R,\bar{\mathbf{q}}\rangle=(2\pi)^{-4}e^{-i(\varepsilon T-\mathbf{k}\cdot\mathbf{z}+\varepsilon_R T_R-\bar{\mathbf{q}}\cdot\bar{\mathbf{r}})}\ . \tag{D11}$$

We have again the following transformation coefficients from the old to the new basis:

$$\langle p,q\,|\,\varepsilon,\mathbf{k},\varepsilon_R,\overline{\mathbf{q}}\rangle=\frac{(1+\mathbf{k}^2)^{1/2}}{|\varepsilon|^3}\delta(\varepsilon-\eta M)\delta^3\left[\mathbf{k}-\frac{\eta\mathbf{p}}{M}\right]\delta\left[\varepsilon_R-\eta\frac{(p,q)}{M}\right]\delta^3\left[\overline{\mathbf{q}}-\mathbf{q}+\frac{\mathbf{p}}{M}\left[\eta q^0-\frac{\mathbf{p}\cdot\mathbf{q}}{M+\eta p_0}\right]\right],$$
$$\langle x,r\,|\,\varepsilon,\mathbf{k},\varepsilon_R,\overline{\mathbf{q}}\rangle=(2\pi)^{-4}\exp\left\{-i\left[\varepsilon(k,x)+\varepsilon_R(k,r)-\overline{\mathbf{q}}\left[\mathbf{r}-\mathbf{k}\frac{r^0+(k,r)}{1+k^0}\right]\right]\right\}, \tag{D12}$$

$$\langle p,q\,|\,T,\mathbf{z},T_R,\overline{\mathbf{r}}\rangle=(2\pi)^{-4}\exp\left[-i\left\{\eta\left[MT-\frac{\mathbf{p}\cdot\mathbf{z}}{M}+\frac{(p,q)}{M}T_R\right]-\left[\mathbf{q}-\frac{\mathbf{p}}{M}\left[\eta q^0-\frac{\mathbf{p}\cdot\mathbf{q}}{M+\eta p_0}\right]\right]\overline{\mathbf{r}}\right\}\right],$$
$$\langle x,r\,|\,T,\mathbf{z},T_R,\overline{\mathbf{r}}\rangle=\int_{(p^2>0)}\frac{d^4p}{(2\pi)^4}e^{-i(p,x)}\exp\left[i\eta\left[(p^2)^{1/2}T-\frac{\mathbf{p}\cdot\mathbf{z}}{(p^2)^{1/2}}\right]\right]$$
$$\times\delta\left[T_R-\eta\frac{(p,r)}{(p^2)^{1/2}}\right]\delta^3\left[\overline{\mathbf{r}}-\mathbf{r}-\frac{\mathbf{p}}{(p^2)^{1/2}}\left[\eta r^0-\frac{\mathbf{p}\cdot\mathbf{r}}{(p^2)^{1/2}+\eta p_0}\right]\right]. \tag{D13}$$

Between $\Psi(z,\overline{\mathbf{r}},T,T_R)=\langle\mathbf{z},\overline{\mathbf{r}},T,T_R\,|\,\Psi\rangle$ and $\Psi(\mathbf{k},\overline{\mathbf{q}},\varepsilon,\varepsilon_R)=\langle\mathbf{k},\overline{\mathbf{q}},\varepsilon,\varepsilon_R\,|\,\Psi\rangle$ we have the relations

$$\Psi(\mathbf{z},\overline{\mathbf{r}},T_R,T)=\int\frac{|\varepsilon|^3d\varepsilon\,d^3k\,d\varepsilon_R d^3\overline{q}}{(2\pi)^4(1+\mathbf{k}^2)^{1/2}}e^{-i(\varepsilon T-\mathbf{k}\cdot\mathbf{z}+\varepsilon_R T_R-\overline{\mathbf{q}}\cdot\overline{\mathbf{r}})}\Psi(\mathbf{k},\overline{\mathbf{q}},\varepsilon,\varepsilon_R),$$
$$\Psi(\mathbf{k},\overline{\mathbf{q}},\varepsilon,\varepsilon_R)=\frac{(1+\mathbf{k}^2)^{1/2}}{|\varepsilon|^3}\int\frac{dT\,d^3z\,dT_R d^3\overline{r}}{(2\pi)^4}e^{i(\varepsilon T-\mathbf{k}\cdot\mathbf{z}+\varepsilon_R T_R-\overline{\mathbf{q}}\cdot\overline{\mathbf{r}})}\Psi(\mathbf{z},\overline{\mathbf{r}},T,T_R). \tag{D14}$$

Finally we have the following properties under Poincaré transformations:

$$(U(a,\Lambda)\Psi)(\mathbf{k},\overline{\mathbf{q}},\varepsilon,\varepsilon_R)=e^{i\varepsilon(k,a)}\Psi((\Lambda^{-1}\mathbf{k}),\overline{q}^{\,j}R_j^{\;i}(\Lambda^{-1},\mathbf{k}),\varepsilon,\varepsilon_R),$$
$$(U(a,\Lambda)\Psi)(\mathbf{z},\overline{\mathbf{r}},T,T_R)=\int\frac{d^3z'd^3k}{(2\pi)^3}e^{i[(\Lambda\mathbf{k})\cdot\mathbf{z}-\mathbf{k}\cdot\mathbf{z}']}\Psi(\mathbf{z}',\mathbf{R}(\Lambda,\mathbf{k})\overline{\mathbf{r}},T-(\Lambda k,a),T_R), \tag{D15}$$

which are of course nonlocal transformations.

Equation (D15) restricted to the elementary solutions gives the same transformation law as that of Eq. (5.8) in terms of the new variables, that is

$$(U(a,\Lambda)\Psi_{(\eta,\mathbf{k},pnl,m)})(\mathbf{z},\overline{\mathbf{r}},T,T_R)=e^{i\eta M_{pnl}(\Lambda k,a)}\sum_{m'}\Psi_{(\eta,\Lambda\mathbf{k},pnl,m')}(\mathbf{z},\overline{\mathbf{r}},T,T_R)D^{(l)}_{m'm}(R^{-1}(\Lambda,\mathbf{k})). \tag{D16}$$

In the nonrelativistic limit we put $x_i^\mu\equiv(ct_i,\mathbf{x}_i)$ and $p_i^\mu\equiv(m_ic+E_i/c,\mathbf{p}_i)$, and the new variables become

$$T_i=ct_i+O(c^{-1}),\quad \varepsilon_i=m_ic+\frac{1}{c}\left[E_i+\frac{m_i}{2m^2}\mathbf{p}^2-\frac{\mathbf{p}\cdot\mathbf{p}_i}{m}\right]+O(c^{-3}),\quad \mathbf{z}=mc\left[\frac{m_1\mathbf{x}_1+m_2\mathbf{x}_2}{m}-\frac{\mathbf{p}_1t_1+\mathbf{p}_2t_2}{m}\right]+O(c^{-1}),$$
$$\mathbf{k}=\frac{\mathbf{p}}{mc}+O(c^{-3}),\quad \overline{\mathbf{r}}=\mathbf{r}-\frac{\mathbf{p}}{m}t_R+O(c^{-2}),\quad \overline{\mathbf{q}}=\frac{m_2}{m}\mathbf{p}_1-\frac{m_1}{m}\mathbf{p}_2+O(c^{-2}), \tag{D17}$$

where $m=m_1+m_2$, $t_R=t_1-t_2$.

In this limit the plane (T_1,T_2) becomes the plane (t_1,t_2) and the line $T_1=T_2$ [i.e., $(p,r)=0$] becomes the line $t_1=t_2$.

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[1] *The Theory of Action-at-a-Distance in Relativistic Particle Mechanics*, edited by E. H. Kerner (Gordon and Breach, New York, 1972).

[2] *Relativistic Action-at-a-Distance: Classical and Quantum Aspects*, edited by J. Llosa (Lecture Notes in Physics, Vol. 162) (Springer, Berlin, 1982).

[3] I. T. Todorov, Sissa report, Trieste, 1980 (unpublished); in *Quantum Theory, Groups, Fields and Particles*, edited by A. O. Barut (Reidel, Dordrecht, 1983).

[4] T. Takabayasi, Suppl. Prog. Theor. Phys. **67**, 1 (1979).

[5] L. Lusanna, Nuovo Cimento **65B**, 135 (1981).

[6] L. Lusanna, in *Relativistic Mechanics toward Green's Functions: Multitemporal Dynamics*, proceedings of the VII Seminar on Problems of High Energy Physics and Quantum Field Theory (Protvino, USSR, 1984), Vol. I, p. 123; in *Hadron Spectroscopy—1985*, edited by S. Oneda (AIP Conf. Proc. No. 132) (AIP, New York, 1985), p. 294.

[7] I. T. Todorov, Report No. Comm. JINR E2-10125, Dubna, 1976 (unpublished); Ann. Inst. Henri Poincaré **28A**, 207 (1978).

[8] A. Komar, Phys. Rev. D **18**, 1881 (1978); **18**, 1887 (1978); **18**, 3017 (1978); **19**, 2908 (1979).

[9] Ph. Droz-Vincent, Lett. Nuovo Cimento **1**, 839 (1969); **7**, 206 (1973); **23**, 184 (1978); **28**, 166 (1980); **33**, 383 (1982); **35**, 83

(1982); Rep. Math. Phys. **8**, 79 (1975); Ann. Inst. Henri Poincaré **18A**, 57 (1975); **27A**, 407 (1977); **32A**, 377 (1980); Phys. Rev. D **19**, 702 (1979); Nuovo Cimento **12B**, 1 (1972); **58A**, 355 (1980); Collège de France, report, 1984 (unpublished); Collège de France, report, 1985 (unpublished).

[10]F. Rohrlich, Ann. Phys. (N.Y.) **117**, 292 (1979); Phys. Rev. D **25**, 2576 (1982); A. King and F. Rohrlich, Ann. Phys. (N.Y.) **130**, 350 (1980).

[11]H. W. Crater and P. Van Alstine, Phys. Rev. D **30**, 2585 (1984).

[12]H. W. Crater and P. Van Alstine, J. Math. Phys. **23**, 1697 (1982); Ann. Phys. (N.Y.) **148**, 57 (1983).

[13]H. Sazdjian, Phys. Rev. D **33**, 3401 (1986); **33**, 3425 (1986); **33**, 3435 (1986); Orsay Report No. IPNO/TH 85-34, 1985 (unpublished).

[14]The following models have also second-class constraints: T. Takabayasi, Prog. Theor. Phys. **54**, 563 (1975); **57**, 331 (1977); M. Kalb and P. Van Alstine, Yale Report No. COO-3075-146, 1976 (unpublished); K. Kamimura and T. Shimizu, Prog. Theor. Phys. **58**, 383 (1977); D. Dominici, J. Gomis, and G. Longhi, Nuovo Cimento **48B**, 152 (1978); **56A**, 263 (1980); L. Castellani, D. Dominici, and G. Longhi, *ibid.* **48A**, 91 (1978); A. Barducci, R. Casalbuoni, and L. Lusanna, *ibid.* **54A**, 340 (1979); J. A. Llosa, F. Marqués, and A. Molina, Ann. Inst. Henri Poincaré, **32A**, 303 (1980); D. Dominici, J. Gomis, J. A. Lobo, and J. M. Pons, Nuovo Cimento **61B**, 306 (1981).

[15]Bilocal models related to the models with first-class constraints: I. Sogami, Prog. Theor. Phys. **50**, 1729 (1973); Nuovo Cimento **46A**, 78 (1978); Lett. Nuovo Cimento **24**, 411 (1979); J. Lukierski and M. Oziewicz, Bull. Acad. Pol. Sci. **22**, 711 (1974); Phys. Lett. **69B**, 339 (1977); G. Preparata and N. S. Craigie, Nucl. Phys. **B102**, 97 (1976); **B102**, 478 (1976); C. C. Chiang, Phys. Lett. **62B**, 419 (1976); A. Z. Capri, Max Planck Institute report, 1976 (unpublished); Phys. Rev. D **25**, 2118 (1982); G. Pocsik, ITP-Budapest Report No. 367, 1976 (unpublished).

[16]Models with extensions to more than two particles: H. Sadzjian, Lett. Math. Phys. **5**, 319 (1981); F. Rohrlich, Phys. Rev. D **23**, 1305 (1981); A. Komar, *ibid.* **24**, 2330 (1981); L. Lusanna, Nuovo Cimento **64A**, 65 (1981); J. Gomis, J. A. Lobo, and J. M. Pons, *ibid.* **64B**, 316 (1981); V. Iranzo, J. A. Llosa, F. Marqués, and A. Molina, Ann. Inst. Henri Poincaré **35A**, 1 (1981); J. Samuel, Phys. Rev. D **26**, 3475 (1982); **26**, 3482 (1982); A. P. Balachandran, D. Dominici, G. Marmo, N. Mukunda, I. S. Nilsson, A. Simoni, E. C. G. Sudarshan, and F. Zaccaria, *ibid.* **26**, 3492 (1982); T. Biswas and F. Rohrlich, Lett. Math. Phys. **6**, 325 (1982); T. Biswas, F. Rohrlich, and M. King, Nuovo Cimento **77B**, 49 (1983); Ph. Droz-Vincent, Lett. Nuovo Cimento **38**, 177 (1983); C. R. Acad. Sci. Paris **299**, 139 (1984); E. Sorace and M. Tarlini, Nuovo Cimento **82B**, 29 (1984); J. Gomis, J. A. Lobo, A. Poch, and J. M. Pons, Ann. Inst. Henri Poincaré **40A**, 59 (1984).

[17]H. W. Crater and P. Van Alstine, Phys. Lett. **100B**, 166 (1981); Phys. Rev. Lett. **53**, 1527 (1984); H. W. Crater, in *Hadron Spectroscopy—1985* (Ref. 6), p. 308.

[18]T. Biswas and F. Rohrlich, Nuovo Cimento **88A**, 125 (1985); T. Biswas, *ibid.* **88A**, 145 (1985).

[19]D. G. Currie, T. F. Jordan, and E. C. G. Sudarshan, Rev. Mod. Phys. **35**, 350 (1965); H. Leutwyler, Nuovo Cimento **37**, 556 (1965).

[20]E. E. Salpeter and H. Bethe, Phys. Rev. **84**, 1232 (1951); M. Gell-Mann and F. Low, *ibid.* **84**, 350 (1951); G. C. Wick, *ibid.* **96**, 1124 (1954); M. Nakanishi, Suppl. Prog. Theor. Phys. **43**, 1 (1969); A. A. Arkhipov and I. V. Savrin, Serpukhov Report No. 83-50, 1983 (unpublished).

[21]H. Sazdjian, Phys. Lett. **156B**, 381 (1985).

[22]L. P. Horwitz and F. Rohrlich, Phys. Rev. D **24**, 1528 (1981); **26**, 3452 (1982); **31**, 932 (1985).

[23]I. T. Todorov, in *Properties of the Fundamental Interactions*, edited by A. Zichichi (Editrice Compositori, Bologna, 1973), Vol. 9, Part C, p. 953; Phys. Rev. D **3**, 2351 (1971).

[24]M. B. Halpern and W. Siegel, Phys. Rev. D **16**, 2486 (1977).

[25]P. A. M. Dirac, Can. J. Math. **2**, 129 (1950); *Lectures on Quantum Mechanics*, Belfer Graduate School of Science, Monographs Series (Yeshiva University, New York, 1964); J. L. Anderson and P. G. Bergmann, Phys. Rev. **83**, 1018 (1951); P. G. Bergmann and J. Goldberg, *ibid.* **98**, 531 (1955); E. C. G. Sudarshan and N. Mukunda, *Classical Mechanics: a Modern Perspective* (Wiley, New York, 1974); A. Lichnerowicz, C. R. Acad. Sci. Paris **A280**, 523 (1975); A. J. Hanson, T. Regge, and C. Teitelboim, *Constrained Hamiltonian Systems* (Contributi del Centro Linceo Interdisciplinare di Scienze Matematiche, Fisiche e loro Applicazioni, n. 22) (Accademia Nazionale dei Lincei, Roma, 1975); K. Sundermeyer, *Constrained Dynamics with Applications to Yang-Mills Theory, General Relativity, Classical Spin, Dual String Model* (Lecture Notes in Physics, Vol. 169) (Springer, Berlin, 1982).

[26]J. Gomis, K. Kamimura, and J. M. Pons, Prog. Theor. Phys. **71**, 1051 (1984).

[27]P. A. M. Dirac, Rev. Mod. Phys. **21**, 392 (1949).

[28]L. D. Faddeev and A. A. Slavnov, *Gauge Fields: Introduction to Quantum Theory* (Benjamin/Cummings, Reading, MA, 1980).

[29]D. Dominici, G. Longhi, J. Gomis, and J. M. Pons, J. Math. Phys. **25**, 2439 (1984).

[30]D. G. Currie, Phys. Rev. **142**, 817 (1956); R. N. Hill, J. Math. Phys. **8**, 201 (1967); L. Bel, Ann. Inst. Henri Poincaré **12A**, 307 (1970); Ph. Droz-Vincent, Phys. Scr. **2**, 129 (1970); J. G. Wray, Phys. Rev. D **1**, 2212 (1970); L. Bel, Ann. Inst. Henri Poincaré **14A**, 189 (1971); R. Arens, Arch. Rat. Mech. Anal. **47**, 255 (1972); L. Bel and J. Martin, Ann. Inst. Henri Poincaré **22A**, 173 (1975); L. Bel, Departamento de Fisica Teorica de la Universidad Autonoma de Barcelona, Report No. UAB FT-34, 1977 (unpublished); F. Marqués, V. Iranzo, A. Molina, A. Montoto, and J. Llosa, Phys. Rev. D **31**, 314 (1985); V. Iranzo, J. A. Llosa, and F. Marquès, Ann. Phys. (N.Y.) **150**, 114 (1983); X. Jaon, A. Molina, and V. Iranzo, J. Math. Phys. **27**, 512 (1986); J. Pons, Ann. Phys. (N.Y.) **148**, 192 (1983); J. Gomis, M. Novell, and J. M. Pons, *ibid.* **153**, 389 (1984).

[31]G. Longhi, L. Lusanna, and J. M. Pons (in preparation).

[32]G. Longhi, talk given at the Encuentros Relativisticos Espanolos, St. Ander, 1984 (unpublished).

[33]L. Lusanna, talk given at the 4th M. Grossmann Meeting, Roma, 1985 (unpublished); talk given at the 1st Torino Meeting on Superunification and Extra Dimensions, ISI, Torino, 1985 (unpublished); G. Longhi, talk given at the Encuentros Relativisticos Espanolos, Maó, Menorca, 1985 (unpublished).

[34]M. H. L. Pryce, Proc. R. Soc. London **A195**, 62 (1948); T. D. Newton and E. P. Wigner, Rev. Mod. Phys. **21**, 400 (1949); L. L. Foldy and S. A. Wouthuysen, Phys. Rev. **78**, 29 (1950); T. F. Jordan and N. Mukunda, *ibid.* **132**, 1842 (1953); E. H. Kerner, J. Math. Phys. **6**, 1218 (1965); A. J. Kalnay, *The Localization Problem in Studies in the Foundations*, Methodology and Phylosophy of Science, Vol. 4, edited by M. Bunge (Springer, Berlin, 1971); M. Lorente and P. Roman, J. Math. Phys. **15**, 70 (1974); J. Niederle, in *Localizability of Particles in Hadron Constituents and Symmetries, Smolenice, 1976*, edited by J. Lonik and M. Noga (Veda, Bratislavia, 1978),

Vol. 3; H. Sazdjian, Nucl. Phys. **B161**, 469 (1979).

[35]N. N. Bogolubov, A. A. Logunov, and I. T. Todorov, *Introduction to Axiomatic Quantum Field Theory* (Benjamin, Reading, MA, 1975).

[36]Ph. Droz-Vincent, Phys. Rev. D **29**, 687 (1984).

[37]V. Iranzo, J. A. Llosa, F. Marqués, and A. Molina, Ann. Inst. Henri Poincaré **40A**, 1 (1984).

[38]V. A. Rizov, H. Sazdjian, and I. T. Todorov, Ann. Phys. (N.Y.) **165**, 59 (1985).

[39]R. Weder, Helv. Phys. Acta **50**, 105 (1977); J. Funct. Anal. **27**, 100 (1978).

[40]U. H. Niederer and L. O'Raifeartaigh, Fortschr. Phys. **22**, 111 (1974).

[41]C. Palmieri and B. Vitale, Nuovo Cimento **66A**, 299 (1970); E. Candotti, C. Palmieri, and B. Vitale, *ibid.* **70A**, 233 (1970); **7A**, 271 (1972).

[42]R. P. Feynman, M. Kislinger, and F. Ravndal, Phys. Rev. D **3**, 2706 (1971).

[43]Y. S. Kim and M. E. Noz, Phys. Rev. D **8**, 3521 (1973); **12**, 129 (1975).

[44]H. Yukawa, Phys. Rev. **77**, 219 (1950); **80**, 1047 (1950); **91**, 416 (1953).

[45]S. Ishida, Prog. Theor. Phys. **46**, 1570 (1971); **46**, 1905 (1971); in *Hadron Spectroscopy—1985* (Ref. 6), p. 277; S. Ishida, K. Takeuchi, S. Tsuruta, and M. Watanabe, Phys. Rev. D **20**, 2906 (1979); S. Ishida, S. Hinata, M. Oda, K. Takeuchi, and K. Yamada, Prog. Theor. Phys. **71**, 806 (1984).

[46]A. Z. Capri and C. C. Chiang, Nuovo Cimento **36A**, 331 (1976); **38A**, 191 (1977); Prog. Theor. Phys. **59**, 996 (1978).

[47]H. Leutwyler and J. Stern, Ann. Phys. (N.Y.) **112**, 94 (1978); S. Mallik and E. Hugentobler, Phys. Rev. D **20**, 1849 (1979); C. Kunz, Helv. Phys. Acta **53**, 159 (1980).

[48]See, for instance, K. L. Nagy, *State Vector Spaces with Indefinite Metric in Quantum Field Theory* (P. Noordkopf Ltd., Groningen, The Netherlands, 1966).

[49]D. Dominici, J. Gomis, and G. Longhi, Nuovo Cimento **48A**, 257 (1978).

[50]S. Weinberg, Phys. Rev. **133**, B1318 (1964).

[51]S. Weinberg, Phys. Rev. **181**, 1893 (1969); A. O. Barut and R. Raczka, *Theory of Group Representations and Applications* (Polish Scientific Publishers, Warszawa, 1977).

[52]T. Aaberge, Helv. Phys. Acta **50**, 331 (1977); **50**, 917 (1977).

[53]P. Havas, in *Statistical Mechanics of Equilibrium and Non Equilibrium*, edited by J. Meixner (North-Holland, Amsterdam, 1965); in *Problems in the Foundations of Physics*, edited by M. Bunge (Springer, New York, 1971).

[54]O. L. Brill and B. Goodman, Am. J. Phys. **35**, 832 (1967).

[55]K. B. Wolf, in *Group Theory and Its Applications*, edited by E. M. Loebl (Academic, New York, 1975), Vol. III.

[56]L. Castellani, Nuovo Cimento **47B**, 1 (1978).

[57]N. N. Bogolubov and D. V. Shirkov, *Introduction to the Theory of Quantized Fields*, 3rd ed. (Wiley, New York, 1976).