Hadamard and minimal renormalizations

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A common language is introduced to study two, well-known, different methods for the renormalization of the energy-momentum tensor of a scalar neutral quantum field in curved space-time. Different features of the two renormalizations are established and compared.

I. INTRODUCTION

This paper is a comparative study of two alternative methods of renormalization for the energy-momentum tensor of a scalar quantum field in a curved background. Reference 1 could be considered as a previous attempt on the same line focused mainly on the massless case and in Hadamard vacua (vacua related to Green's functions with a Hadamard structure). We will study the massive case and unrestricted vacua in this work, and also compare the different physical results that can be obtained with the two methods.

Energy-momentum tensor renormalization is based on the subtraction, from the unrenormalized tensor, of another tensor with the same divergences, to obtain a renormalized quantity. Different criteria to choose this second tensor yield different renormalization methods that we can classify into two sets.

(1) The state-independent renormalizations, where the subtraction is done using a geometrical object, a function only of the local geometry at each point, and independent of any quantum state and of course of the observer or coordinate system. These methods must be covariant in the sense that they are observer independent and can be used for every geometrical background. We will study the canonical renormalization method (as explained in Ref. 2) in the generalized version given in Ref. 3, where the second tensor that we subtract is obtained from a Hadamard elementary solution and it is a function of the local geometry only. It can be proved that this renormalization in unique and that it coincides with the canonical one, where a DeWitt-Schwinger elementary solution is used. We shall call this recipe the "Hadamard renormalization."

(2) The state-dependent renormalizations. To develop a completely covariant geometrical method is in the best tradition of relativistic physicists, but realistically there is no compelling physical reason to choose the former class. In fact, we can as well suppose that the renormalization is state dependent, i.e., that somehow we choose a quantum state $|q\rangle$ and that we subtract $\langle q|T_{\mu\nu}|q\rangle$. In Minkowski unbounded space-time, if we use inertial observers, there is a unique vacuum $(0\rangle$ and a unique renormalization method (the one with $|q\rangle=|0\rangle$). However, we know that we have different vacua in curved space-time and also that the vacuum turns out to be observer dependent. Thus, it is possible that the renormalization could be observer dependent, in the sense that $|q\rangle$ could be observer dependent. In this second class of renormalization methods we have a greater freedom than in the first set. In fact the Wald axioms[4] restrict the differences between possible renormalizations to a conserved geometrical term. Other requirements imposed to Hadamard renormalization make this renormalization unique (cf. Ref. 3).

Even if the first set of renormalizations is, perhaps, the more aesthetic one, the second class could have some important advantages. In fact, with Hadamard renormalization we will always have the phenomenon of trace anomaly, and we must introduce in the right-hand side (RHS) of Einstein's equation second-order curvature tensors ${}^{(1)}H_{\mu\nu}$ and ${}^{(2)}H_{\mu\nu}$ that yield problems with the causality principle, at the classical level,[5] and ghosts in the completely quantized gravity theory.[6] Using the freedom of this new kind of state-dependent renormalization we can overcome these problems. We will use as an example of this second-class method introduced in Refs. 7 and 8 (see also Refs. 9—12) for the Robertson-Walker universe with comoving observers, that we shall call minimal renormalization. This method will be developed in this particular geometry and it will be generalized to all conformally flat geometries. Perhaps it can be generalized to a wider class of geometries. Reference 1 is an attempt to generalize this kind of renormalization to an arbitrary geometry in the massless case, but this renormalization is not continuous if we go from one geometry to another. Perhaps the correct generalization could be found if the observer is introduced in the renormalization method in a covariant way. However, for the moment, we will focus our study on Robertson-Walker geometry.

In Secs. II and III we review each method separately. In Sec. IV we see how we can obtain the minimal renormalization from the Hadamard renormalization formalism and generalize the minimal renormalization to conformally flat geometries. In Secs. V and VI we see that both methods yield different physical results. We can find different self-consistent de Sitter cosmological solutions (Sec. V), and the behavior of Minkowski space for small conformal fluctuations is different (Sec. VI). In Sec. VII we state our principle conclusions.

II. HADAMARD RENORMALIZATION

Let us first review the, by now, usual formalism (cf. Ref. 2 and bibliography therein, Refs. 3, 13, and 14). We will use the following conventions: The signature is $(-,+,+,+)$,

$$R^{\mu}_{\nu\theta\rho}=\Gamma^{\mu}_{\nu\rho,\theta}-\Gamma^{\mu}_{\nu\theta,\rho}+\Gamma^{\mu}_{\tau\theta}\Gamma^{\tau}_{\nu\rho}-\Gamma^{\mu}_{\tau\rho}\Gamma^{\tau}_{\nu\theta},$$
$$R_{\mu\nu}=g^{\rho}_{\theta}R^{\theta}_{\mu\rho\nu}.$$

We will study a scalar neutral field with due action

$$S[\phi]=\tfrac{1}{2}\int d^4x\sqrt{-g}\,(\phi_{,\mu}\phi^{,\mu}+m^2\phi^2+\xi R\phi^2)\,, \tag{2.1}$$

where, as usual, the action turns out to be invariant under a conformal transformation if $m=0$, $\xi=\frac{1}{6}$ (conformal coupling). The field equation is

$$(\Box-m^2-\xi R)\phi(x)=0 \tag{2.2}$$

and the energy-momentum tensor operator is

$$\begin{aligned}T_{\mu\nu}(x)&=\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}}\\&=\tfrac{1}{2}(1-2\xi)\{\phi_{,\mu},\phi_{,\nu}\}+\tfrac{1}{2}(2\xi-\tfrac{1}{2})\{\phi_{,\rho},\phi^{,\rho}\}g_{\mu\nu}-\xi\{\phi_{;\mu\nu},\phi\}+\xi g_{\mu\nu}\{\Box\phi,\phi\}\\&\quad+\tfrac{1}{2}\xi(R_{\mu\nu}-\tfrac{1}{2}Rg_{\mu\nu}-\tfrac{1}{4}m^2g^{\mu\nu})\{\phi,\phi\}\,,\end{aligned} \tag{2.3}$$

where $\{\phi,\psi\}=\phi\psi+\psi\phi$ and $T_{\mu\nu}{}^{;\nu}=0$.

Let us define the symmetric Green's function as

$$G_1(x,x')=\langle 0\,|\,\{\phi(x),\phi(x')\}\,|\,0\rangle\,. \tag{2.4}$$

Obviously this G_1 satisfies the field equation (2.2). If we symbolize the coincidence limit as

$$[G_1]=\lim_{x\to x'}G_1(x,x') \tag{2.5}$$

it can be proved that (cf. Ref. 3)

$$\begin{aligned}\langle 0|T_{\mu\nu}|0\rangle=&-\tfrac{1}{2}[G_{1;\mu\nu}]-\tfrac{3}{4}(\xi-\tfrac{1}{3})[\Box G_1]g_{\mu\nu}\\&+\tfrac{1}{2}(\tfrac{1}{2}-\xi)[G_1]_{;\mu\nu}+\tfrac{1}{2}(\xi-\tfrac{1}{4})\Box G_1g_{\mu\nu}\\&+\{\tfrac{3}{4}(\xi-\tfrac{1}{6})(m^2+\xi R)g_{\mu\nu}+\tfrac{1}{2}\xi R_{\mu\nu}\}[G_1]\,.\end{aligned} \tag{2.6}$$

We can also use this equation to compute a tensor from a symmetric G_1 that satisfies the field equation even if this G_1 is not related to a vacuum by Eq. (2.4). We shall call this tensor $\langle T_{\mu\nu}\rangle^{(G_1)}$.

We shall say that G_1 is an Hadamard-type elementary solution of the field equation (2.2) when $G_1(x,x')$ has the form

$$G_1(x,x')=\frac{\Delta^{1/2}(x,x')}{8\pi^2}\left[\frac{2}{\sigma}+v\ln\mu^2\sigma+w\right], \tag{2.7}$$

where σ is half the square of the geodesic distance between x and x', μ is an arbitrary mass scale (that normally we shall make equal to the mass in the massive case), $\Delta(x,x')$ is the Van Vleck–Morette determinant, and $v(x,x')$ and $w(x,x')$ are regular functions of σ that can be developed as

$$\begin{aligned}v(x,x')&=\sum_{n=0}^{\infty}v_n(x,x')\sigma^n\,,\\w(x,x')&=\sum_{n=0}^{\infty}w_n(x,x')\sigma^n\,,\end{aligned} \tag{2.8}$$

where v_n and w_n are regular functions in the coincidence limit. In order that G_1 satisfy the field equation (2.2) it is necessary that

$$\begin{aligned}&v_0+v_0{}^{;\mu}\sigma_{;\mu}=V-\Delta^{-1/2}\Box(\Delta^{1/2})\,,\\&v_n+v_n{}^{;\mu}\sigma_{;\mu}=\frac{1}{2n(n+1)}[Vv_{n-1}-\Delta^{-1/2}\Box(\Delta^{1/2}v_{n-1})]\,,\quad n\geq 1\,,\\&w_n+w_n{}^{;u}\sigma_{;\mu}=\frac{1}{2n(n+1)}[Vw_{n-1}-\Delta^{-1/2}\Box(\Delta^{1/2}w_{n-1})]-\frac{2n+1}{n(n+1)}v_n-\frac{1}{n(n+1)}v_{n;\mu}\sigma^{;\mu}\quad(V\equiv m^2+\xi R)\,.\end{aligned} \tag{2.9}$$

Thus, not only $\Delta(x,x')$, but also $v(x,x')$ is determined by the background geometry and the choice of $w_0(x,x')$ is the only thing that remains arbitrary. In fact the differential equation (2.9) plus the initial conditions stated under Eq. (2.8) fix $v(x,x')$ completely and also $w_n(x,x')$ $(n>1)$ if $w_0(x,x')$ is known.

Therefore, the two first terms of any Hadamard elementary solution (2.7) are always the same and we shall call them $G_1^{(v)}(x,x')$ (this term has a part which is divergent in the coincidence limit). The third term is determined by $w_0(x,x')$;

we shall call it $G_1^{(w)}(x,x')$, and it is always finite in the coincidence limit.

Thus $\langle T_{\mu\nu}\rangle^{(G_1)}$ is determined by w_0 or more precisely by the coincidence limits $[w_0]$, $[w_{0,\mu}]$, and $[w_{0;\mu\nu}]$. The coincidence limits of higher order are irrelevant because they do not appear in Eq. (2.6). From $G_1^{(v)}(x,x')$ using Eq. (2.6), we can compute $\langle T_{\mu\nu}\rangle^{(v)}$, which is the same for every Hadamard elementary solution (and has a divergent part). From $G_1^{(w)}(x,x')$ we can compute $\langle T_{\mu\nu}\rangle^{(w)}$ as a functional of $[w_0]$, $[w_{0,\mu}]$, $[w_{0;\mu\nu}]$ and the result is always finite:

$$16\pi^2\langle T_{\mu\nu}\rangle^{(w)}=-\{[w_{0;\mu\nu}]-\tfrac{1}{4}g_{\mu\nu}[\Box w_0]\}+\tfrac{1}{3}\{[w_0]_{;\mu\nu}-\tfrac{1}{4}g_{\mu\nu}\Box[w_0]\}-(\xi-\tfrac{1}{6})\{[w_0]_{;\mu\nu}-g_{\mu\nu}\Box[w_0]\}$$
$$+(\xi-\tfrac{1}{6})(R_{\mu\nu}-\tfrac{1}{2}Rg_{\mu\nu})[w_0]-\tfrac{1}{4}m^2[w_0]g_{\mu\nu}+9(\xi-\tfrac{1}{6})[v_1]g_{\mu\nu}\ . \tag{2.10}$$

Finally

$$\langle T_{\mu\nu}\rangle^{(G_1)}=\langle T_{\mu\nu}\rangle^{(w)}+\langle T_{\mu\nu}\rangle^{(v)}\ . \tag{2.11}$$

If we want this tensor to satisfy the conservation equation

$$\langle T_{\mu\nu}\rangle^{(G_1);\nu}=0 \tag{2.12}$$

the coincidence limits $[w_{0;\alpha}]$ and $[w_{0;\alpha\beta}]$ must satisfy the equation (cf. Ref. 3)

$$[w_{0;\alpha}]=[w_0]_{;\alpha}\ ,$$
$$[w_{0;\alpha\beta}]^{\beta}-\tfrac{1}{4}[\Box w_0]_{,\alpha}=\tfrac{1}{4}\Box([w_0]_{,\alpha})+\tfrac{1}{12}R_{\alpha\beta}[w_0]^{,\beta}+\tfrac{1}{4}(\xi-\tfrac{1}{6})R_{,\alpha}[w_0]-\tfrac{1}{4}\{m^2+(\xi-\tfrac{1}{6})R\}[w_0]_{,\alpha}+\tfrac{1}{2}[v_1]_{,\alpha}\ , \tag{2.13}$$

where v_1 is the coefficient $n=1$ in Eq. (2.8) and

$$[v_1]=\frac{m^4}{4}+\frac{m^2}{2}(\xi-\tfrac{1}{6})R+\tfrac{1}{4}(\xi-\tfrac{1}{6})R^2-\tfrac{1}{12}(\xi-\tfrac{1}{6})\Box R+\tfrac{1}{360}(R_{\theta\rho\tau\epsilon}R^{\theta\rho\tau\epsilon}-R_{\theta\rho}R^{\theta\rho}-\Box R)\ .$$

To see what remains arbitrary if we impose these conditions let us call

$$X\equiv[w_0]\ ,$$
$$Z_{\alpha\beta}+\tfrac{1}{4}g_{\alpha\beta}Y\equiv[w_{0;\alpha\beta}]\ , \tag{2.14}$$
$$Z^{\alpha}_{\alpha}\equiv 0\ .$$

Then the scalars X and Y are arbitrary; $[w_{0;\alpha}]$ is determined by Eq. (2.13) and the traceless tensor $Z_{\alpha\beta}$ must satisfy the equation

$$Z_{\alpha\beta}{}^{;\beta}=\tfrac{1}{4}\Box(X_{,\alpha})+\tfrac{1}{12}R_{\alpha\beta}X^{,\beta}+\tfrac{1}{4}(\xi-\tfrac{1}{6})R_{,\alpha}X^{,\alpha}-\tfrac{1}{4}[m^2+(\xi-\tfrac{1}{6})R]X_{,\alpha}+\tfrac{1}{2}[v_1]_{,\alpha}\ . \tag{2.15}$$

With these new definitions Eq. (2.10) reads

$$16\pi^2\langle T_{\mu\nu}\rangle^{(w)}=-Z_{\mu\nu}+\tfrac{1}{3}(X_{;\mu\nu}-\tfrac{1}{4}g_{\mu\nu}\Box X)-\tfrac{1}{4}m^2Xg_{\mu\nu}+(\xi-\tfrac{1}{6})(X_{;\mu\nu}-g_{\mu\nu}\Box X)$$
$$+(\xi-\tfrac{1}{6})(R_{\mu\nu}-\tfrac{1}{2}Rg_{\mu\nu})+9(\xi-\tfrac{1}{6})[v_1]g_{\mu\nu}\ . \tag{2.16}$$

We see that the trace Y is completely irrelevant to compute $\langle T_{\mu\nu}\rangle^{(w)}$. It is interesting to compute the trace and the divergence of this tensor:

$$16\pi^2\langle T^{\mu}_{\mu}\rangle^{(w)}=-m^2X-\xi(\xi-\tfrac{1}{6})\Box X+36(\xi-\tfrac{1}{6})[v_1]\ ,$$
$$16\pi^2\langle T_{\mu\nu}\rangle^{(w);\nu}=(-2+9\xi)[v_1]_{,\mu}\ . \tag{2.17}$$

We see that the trace only depends on X and that the divergence is independent of all the parameters in Eq. (2.13).

To implement the Hadamard renormalization we subtract from the unrenormalized vacuum expectation value (VEV) of the energy-momentum tensor $\langle 0|T_{\mu\nu}|0\rangle$, a tensor $\langle T_{\mu\nu}\rangle^{(G_1)}$ where the corresponding G_1 must be the more general Hadamard elementary solution that can be constructed using geometric quantities of the background at point x: $g_{\mu\nu},R,R_{\mu\nu},\ldots$ up to the fourth adiabatic order (i.e., the fourth order in the metric and its derivatives). Below we shall consider only terms up to this order because this is the minimal order that we need to eliminate all the divergences. G_1 is determined by X, Y, and $Z_{\mu\nu}$; the most general choice of these quantities with the correct dimension and correct Minkowski limit turns out to be (cf. Ref. 14)

$$X=X^{(M)}+AR+\frac{1}{m^2}[T+(3C_1-C_2)\Box R]\ , \tag{2.18a}$$

$$Z_{\mu\nu}=-m^2AR_{\mu\nu}+(X^{(M)}+m^2)(\xi-\tfrac{1}{6})R_{\mu\nu}$$
$$+C_1RR_{\mu\nu}+\left[\frac{A}{3}+C_2-C_1\right]R_{;\mu\nu}$$
$$-2C_2(R^{\theta\rho}R_{\mu\theta\nu\rho}+\tfrac{1}{2}\Box R_{\mu\nu})-\tfrac{1}{4}g_{\mu\nu}(\text{trace})\ , \tag{2.18b}$$

where the last tensor is traceless

$$T=\tfrac{1}{180}(R_{\theta\rho\tau\epsilon}R^{\theta\rho\tau\epsilon}-R_{\theta\rho}R^{\theta\rho})$$
$$+\tfrac{1}{2}(\xi-\tfrac{1}{6})R^2-\tfrac{1}{6}(\xi-\tfrac{1}{5})\Box R\ .$$

A, C_1, and C_2 are arbitrary real coefficients, $X^{(n)}$ is the Minkowski value $[X^{(n)}=m^2(2\gamma-\ln 2-1)]$, and Y is arbitrary. Now we can compute the $\langle T_{\mu\nu}\rangle^{(w)}$ from this w:

$$16\pi^2\langle T_{\mu\nu}\rangle^{(w)}=-m^2(\xi-\tfrac{1}{6})G_{\mu\nu}+9(\xi-\tfrac{1}{6})[v_1]g_{\mu\nu} -\tfrac{1}{4}g_{\mu\nu}T+m^2AG_{\mu\nu} +[C_1-(\xi-\tfrac{1}{6})A]\frac{{}^{(1)}H_{\mu\nu}}{2}-C_2^{(2)}H_{\mu\nu}\,, \tag{2.19}$$

where

$$G_{\mu\nu}=R_{\mu\nu}-\tfrac{1}{2}Rg_{\mu\nu}\,,$$
$${}^{(1)}H_{\mu\nu}=2R_{;\mu\nu}-2RR_{\mu\nu}+\tfrac{1}{2}g_{\mu\nu}(R^2-4\Box R)\,,$$
$${}^{(2)}H_{\mu\nu}=R_{;\mu\nu}-\Box R_{\mu\nu} -2R^{\theta\rho}R_{\mu\theta\nu\rho}+\tfrac{1}{2}g_{\mu\nu}(R_{\theta\rho}R^{\theta\rho}-\Box R)\,.$$

The DeWitt-Schwinger $G_1^{\rm DS}(x,x')$ (written up to its fourth adiabatic order only) is a particular case of G_1 obtained with the choice

$$A=(\xi-\tfrac{1}{6})(2\gamma-\ln 2)\,,$$
$$C_1=\tfrac{2}{3}C_2=-\frac{(2\gamma-\ln 2)}{2}\,. \tag{2.20}$$

The ordinary DeWitt-Schwinger renormalization method consists in the following subtraction procedure:

$$\langle T_{\mu\nu}\rangle^{\rm ren\ DS}=\langle 0\,|\,T_{\mu\nu}\,|\,0\rangle-\langle T_{\mu\nu}\rangle^{(G_1^{\rm DS})}\,, \tag{2.21}$$

where $\langle T_{\mu\nu}\rangle^{\rm ren\ DS}$ is the renormalized energy-momentum tensor and $\langle T_{\mu\nu}\rangle^{(G_1^{\rm DS})}$ is the DeWitt-Schwinger energy-momentum tensor computed up to the fourth adiabatic order because these are the orders that appear in the general quantities of Eq. (2.18). Of course a regularization procedure must be used to make the subtraction of the infinite quantities. A generic Hadamard renormalization will be defined by the subtraction

$$\langle T_{\mu\nu}\rangle^{\rm ren\ Had}=\langle 0\,|\,T_{\mu\nu}\,|\,0\rangle-\langle T_{\mu\nu}\rangle^{(G_1)}\,, \tag{2.22}$$

where now G_1 is an Hadamard elementary solution with parameters given by Eq. (2.18) (so the DeWitt-Schwinger renormalization is a particular case of Hadamard renormalization).

Let us answer three questions: Will Hadamard renormalization always produce a finite result? Are different Hadamard renormalizations physically different? What happens in the massless case?

$\langle 0\,|\,T_{\mu\nu}\,|\,0\rangle$ and $\langle T_{\mu\nu}\rangle^{(v)}$ have both (in the coincidence limit) a finite and an infinite term:

$$\langle 0\,|\,T_{\mu\nu}\,|\,0\rangle=\langle 0\,|\,T_{\mu\nu}\,|\,0\rangle_\infty+\langle 0\,|\,T_{\mu\nu}\,|\,0\rangle_f\,,$$
$$\langle T_{\mu\nu}\rangle^{(v)}=\langle T_{\mu\nu}\rangle_\infty^{(v)}+\langle T_{\mu\nu}\rangle_f^{(v)}\,. \tag{2.23}$$

Hadamard renormalization will produce a finite result if the vacuum $|\,0\rangle$ is such that

$$\langle 0\,|\,T_{\mu\nu}\,|\,0\rangle_\infty=\langle T_{\mu\nu}\rangle_\infty^{(v)}\,. \tag{2.24}$$

This circumstance must be proved case by case using a regularization procedure. If this is so (2.22) reads

$$\langle T_{\mu\nu}\rangle^{\rm ren\ Had}=\langle 0\,|\,T_{\mu\nu}\,|\,0\rangle_f-\langle T_{\mu\nu}\rangle_f^{(v)}-\langle T_{\mu\nu}\rangle^{(w)}\,. \tag{2.25}$$

This result is finite and covariantly conserved because $T_{\mu\nu}{}^{;\nu}=0$ and Eqs. (2.22) and (2.12).

From Eq. (2.19) we see that the three last terms of $\langle T_{\mu\nu}\rangle^{(w)}$ have arbitrary coefficients. In addition, if we change the scale $\mu\to\mu'$ we will have a change

$$G_1\to G_1'=G_1-\frac{\Delta^{1/2}}{4\pi^2}V(x,x')\ln\frac{\mu'}{\mu}$$

and therefore

$$16\pi^2\langle T'_{\mu\nu}\rangle^{(w)}=16\pi^2\langle T_{\mu\nu}\rangle^{(w)}-\ln\left[\frac{\mu'}{\mu}\right]^2[\tfrac{1}{180}(3\,{}^{(2)}H_{\mu\nu}-{}^{(1)}H_{\mu\nu})+\tfrac{1}{2}(\xi-\tfrac{1}{6})^2\,{}^{(1)}H_{\mu\nu}-(\xi-\tfrac{1}{6})m^2G_{\mu\nu}+\tfrac{1}{4}m^2g_{\mu\nu}]\,. \tag{2.26}$$

From Eqs. (2.19) and (2.26) we can see that two different Hadamard renormalizations, with different coefficients A, C_1, C_2, and different scales μ yield two different renormalized energy-momentum tensors related by

$$\langle T'_{\mu\nu}\rangle^{\rm ren\ Had}=\langle T_{\mu\nu}\rangle^{\rm ren\ Had}+aG_{\mu\nu} +b\,{}^{(1)}H_{\mu\nu}+c\,{}^{(2)}H_{\mu\nu}+dg_{\mu\nu}\,. \tag{2.27}$$

As the renormalized energy momentum tensor is used in the right-hand side (rhs) of Einstein's equation:

$$G_{\mu\nu}+\Lambda g_{\mu\nu}+\kappa\alpha\,{}^{(1)}H_{\mu\nu}+\kappa\beta\,{}^{(2)}H_{\mu\nu}=-\kappa\langle T_{\mu\nu}\rangle^{\rm ren}\,,$$
$$\kappa\equiv 8\pi G\,, \tag{2.28}$$

the coefficients a, b, c, d can be set together with the coupling constants G, Λ, α, and β. Thus all Hadamard renormalized tensors yield the same physical results. This conclusion was obvious because the tensors on the rhs of Eq. (2.27) are all the conserved tensors that can be constructed up to the fourth adiabatic order, but it is important for a complete understanding of the method. Therefore as the DeWitt-Schwinger renormalization belongs to the Hadamard class, no physical difference appears if we use a different Hadamard renormalization.

Finally it is evident that we cannot use Eq. (2.18) in the massless case; thus, we cannot find a Hadamard elementary solution using this method if $m=0$. Formally the renormalized trace can be computed as the limit $m\to 0$. In this limit we have

$$\langle T_\mu^\mu\rangle^{\rm ren\ Had}=\langle 0\,|\,T_\mu^\mu\,|\,0\rangle-\langle T_\mu^\mu\rangle^{(v)}-\langle T_\mu^\mu\rangle^{(w)}\,. \tag{2.29}$$

The first term is formally zero if $m=0$ (and $\xi=\frac{1}{6}$). In Ref. 15 it is proved that the second one is also zero in this case. We will compute the last one as the limit

$$\langle T^{\mu}_{\mu}\rangle^{\text{ren Had}}_{m=0} = -\lim_{m\to 0}\langle T^{\mu}_{\mu}\rangle^{(w)}$$
$$= \lim_{m\to 0}\frac{m^2}{16\pi^2}\left[X^{(M)} + AR + \frac{1}{m^2}[T + (3C_1 - C_2)\Box R]\right]$$
$$= \frac{1}{16\pi^2}[T + (3C_1 - C_2)\Box R]\ , \tag{2.30}$$

where we have used Eqs. (2.17) and (2.18). This is the usual trace anomaly if we neglect the arbitrary $\Box R$ term that can be arbitrarily modified introducing a R^2 counterterm in the Lagrangian. Working directly in the massless case the same trace anomaly is found in Ref. 16. Thus, a trace anomaly is unavoidable in this renormalization scheme. Also the presence of tensors ${}^{(1)}H_{\mu\nu}$ and ${}^{(2)}H_{\mu\nu}$ in Eq. (2.28) could produce unpleasant consequences at the classical level[5] and ghosts at the quantum level.[6]

III. MINIMAL RENORMALIZATION

We introduce a new renormalization method (cf. Refs. 7 and 8) that we shall apply only to Robertson-Walker universes that solve the problems stated at the end of the last paragraph. The metric can be written as

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2)\ ,$$
$$ds^2 = a^2(\eta) = (-d\eta^2 + dx^2 + dy^2 + dz^2)\ . \tag{3.1}$$

For each t we can find a vacuum $|0\rangle^{\text{ME}}_t$(ME=minimal energy) that minimizes the Hamiltonian:

$$H = \int_{S(t)} d^3x\sqrt{-g}\,T_{00} \tag{3.2}$$

computed at time t; i.e., it turns out that ${}^{\text{ME}}_{t}\langle 0|H|0\rangle^{\text{ME}}_t$ is the minimum of the set of the quantities $\langle 0|H|0\rangle$, where $|0\rangle$ is an arbitrary vacuum of the set of vacua that corresponds to positive- and negative-frequency solutions that can be written as a product $f(t)g(\mathbf{x})$ (cf. Ref. 17). This set of vacua is naturally related to the metric (3.1) and to the comoving reference system where it is written. Now we can introduce an energy-momentum tensor,

$$\langle T_{\mu\nu}\rangle^{\text{ME}}_t = {}^{\text{ME}}_{t}\langle 0|T_{\mu\nu}(x)|0\rangle^{\text{ME}}_t\ , \tag{3.3}$$

and the corresponding G_1

$$G_1^{\text{ME}(t)}(x,x') = {}^{\text{ME}}_{t}\langle 0|\{\phi(x),\phi(x')\}|0\rangle^{\text{ME}}_t\ , \tag{3.4}$$

that, in fact, are related by Eq. (2.6). We can define the minimal renormalization prescription as

$$\langle T_{\mu\nu}(t,\mathbf{x})\rangle^{\text{ren min}}_{\tau} = {}^{\text{ME}}_{\tau}\langle 0|T_{\mu\nu}(t,\mathbf{x})|0\rangle^{\text{ME}}_{\tau} - {}^{\text{ME}}_{t}\langle 0|T_{\mu\nu}(t,\mathbf{x})|0\rangle^{\text{ME}}_t\ , \tag{3.5}$$

where $|0\rangle_\tau$ is the minimal energy vacuum defined at time τ. The renormalization is state dependent in the sense that we subtract a tensor computed at a quantum state $|q\rangle = |0\rangle^{\text{ME}}_t$: the vacuum at time t. This state, of course, changes with time. Using the results of Ref. 14 we have, in the case $\xi = \frac{1}{6}$,

$$\begin{aligned}
{}^{\text{ME}}_{\tau}\langle 0|T_{00}(t)|0\rangle^{\text{ME}}_{\tau} &= \int\frac{d^3k}{(2\pi a)^3}\frac{\omega_k}{2}[1 + 2|\beta_k(t,\tau)|^2]\ ,\\
{}^{\text{ME}}_{\tau}\langle 0|T_{ij}|0\rangle^{\text{ME}}_{\tau} &= \tfrac{1}{3}g_{ij}\int\frac{d^3k}{(2\pi a)^3}\frac{\omega_k}{2}\left[\left[1 - \frac{m^2}{\omega_k{}^2}\right][1 + 2|\beta_k(t,\tau)|^2] - \frac{2}{H}\frac{d}{dt}[|\beta_k(t,\tau)|^2]\right]\ ,
\end{aligned} \tag{3.6}$$

where the Bogoliubov coefficients α_k and β_k can be computed solving the system

$$\begin{aligned}
\dot\alpha_k &= \frac{m^2H}{2\omega_k{}^2}\beta_k\exp\left[-2i\int^t\omega_k(t')dt'\right]\ ,\\
\dot\beta_k &= \frac{m^2H}{2\omega_k{}^2}\alpha_k\exp\left[2i\int^t\omega_k(t')dt'\right]\ ,\\
\omega_k{}^2 &\equiv \left[\frac{k^2}{a^2} + m^2\right]\ ,
\end{aligned} \tag{3.7}$$

with the initial conditions

$$\alpha_k(\tau,\tau) = 1,\quad \beta_k(\tau,\tau) = 0\ , \tag{3.8}$$

and where $H = \dot a/a$. Thus we can obtain

$$\begin{aligned}
{}^{\text{ME}}_{t}\langle 0|T_{00}(t,\mathbf{x})|0\rangle^{\text{ME}}_t &= \int\frac{d^3k}{(2\pi a)^3}\frac{\omega_k}{2}\ ,\\
{}^{\text{ME}}_{t}\langle 0|T_{ij}(t,\mathbf{x})|0\rangle^{\text{ME}}_t &= \frac{g_{ij}}{3}\int\frac{d^3k}{(2\pi a)^3}\frac{\omega_k}{2}\left[1 - \frac{m^2}{\omega_k{}^2}\right]\ .
\end{aligned} \tag{3.9}$$

It can be proved that these quantities are divergent and, in fact, are the only divergent terms of Eq. (3.6) because in the case $\xi = \frac{1}{6}$ for regular evolutions $a(t)$ we have $|\beta_k|^2 \sim k^{-6}$ when $k\to\infty$. Thus Eq. (3.5) is

$$\begin{aligned}
\langle T_{00}\rangle^{\text{ren min}}_{\tau} &= \int\frac{d^3k}{(2\pi a)^3}\omega_k|\beta_k(\tau,t)|^2\ ,\\
\langle T_{ij}\rangle^{\text{ren min}}_{\tau} &= \frac{g_{ij}}{3}\int\frac{d^3k}{(2\pi a)^3}\omega_k\left[|\beta_k(\tau,t)|^2\left[1 - \frac{m^2}{\omega_k{}^2}\right] - \frac{1}{H}\frac{d}{dt}[|\beta_k(\tau,t)|^2]\right]\ .
\end{aligned} \tag{3.10}$$

We call this renormalization "minimal" because we subtracted only the infinite term of the unrenormalized expression (3.6). The corresponding results computed with the Hadamard renormalization method reads

$$\langle T_{\mu\nu}(t)\rangle_\tau^{\text{ren Had}}=\langle T_{\mu\nu}\rangle_\tau^{\text{ren min}}+P_{\mu\nu}\ , \tag{3.11}$$

where $P_{\mu\nu}$ is called the "vacuum-polarization term" (that makes Hadamard renormalization a nonminimal one, cf. Ref. 14)

$$288\pi^2 P_{\mu\nu}=-m^2 G_{\mu\nu} -\tfrac{1}{60}[{}^{(1)}H_{\mu\nu}+\tfrac{1}{2}g_{\mu\nu}(R^2-6R^{\theta\rho}R_{\mu\theta\nu\rho})]\ . \tag{3.12}$$

It can be proved that $G_1^{\text{ME}(t)}$ is not a Hadamard elementary function (cf. Refs. 13 and 18) even so the vacuum expectation value (VEV) of the energy-momentum tensor computed in a vacuum that minimizes the energy, as those we use in this paragraph, can be renormalized with a finite result as it is shown explicitly in Ref. 14 [i.e., $G_1^{\text{ME}(t)}$ yields a tensor that satisfies Eq. (2.24)]. In Ref. 18 the authors arrive at a different conclusion because they find a term that makes the $G_1^{\text{ME}(t)}$ different from a Hadamard structure, but they do not verify whether this term produces an infinite difference between the corresponding energy-momentum tensors. In fact it does not, as can be verified using Eq. (78) of Ref. 15.

IV. MINIMAL RENORMALIZATION USING HADAMARD FORMALISM

To formulate minimal renormalization in Hadamard language let us consider the set of geometries which are conformal to Minkowski space-time: $g_{\mu\nu}=\Omega^2\eta_{\mu\nu}$ and let us repeat the construction of quantities X, Y, $Z_{\mu\nu}$ but now using the geometric object of these geometries where, e.g., the tensor $R_{\mu\theta\nu\rho}$ can be written as a function of $R_{\theta\rho}$, R, and $g_{\mu\nu}$. We keep the Minkowskian limit but now we add the condition that we must also have a finite massless limit, thus m must appear only in the numerator. The computation can be found in Ref. 14 where

$$\begin{aligned} X&=X^{(M)}+AR\ ,\\ Z_{\mu\nu}&=m^2BR_{\mu\nu}+C_1RR_{\mu\nu}+C_2R_{;\mu\nu}+C_3R_{\mu\theta}R^\theta_\nu-\tfrac{1}{4}g_{\mu\nu}(\text{trace})\ ,\end{aligned} \tag{4.1}$$

where A, B, C_1, C_2, and C_3, are real parameters. In the case $\xi=\frac{1}{6}$ in order to satisfy Eq. (2.13) we must have

$$\begin{aligned} &B=-A,\quad C_1=-C_3=-\tfrac{1}{180}\ ,\\ &C_2=\tfrac{1}{540}+\frac{A}{3}\ .\end{aligned} \tag{4.2}$$

Then Eq. (2.16) gives

$$2880\pi^2\langle T_{\mu\nu}\rangle_{\text{CF}}^{(w)}=180Am^2G_{\mu\nu}+RR_{\mu\nu}-\tfrac{1}{3}RR_{;\mu\nu}-R_{\mu\theta}R^\theta_\nu +\tfrac{1}{4}g_{\mu\nu}(R_{\theta\rho}R^{\theta\rho}-R^2+\tfrac{1}{3}\Box R)\ , \tag{4.3}$$

where CF denotes conformally flat. We can see that only the parameter A remains. We can write

$$\langle T_{\mu\nu}\rangle^{(G_1^{\text{CF}})}=\langle T_{\mu\nu}\rangle^{(v)}+\langle T_{\mu\nu}\rangle_{\text{CF}}^{(w)} \tag{4.4}$$

that is covariantly conserved because Eq. (2.14) is satisfied and it has a trace

$$\langle T^\mu_\mu\rangle^{(G_1^{\text{CF}})}=-m^2(X^{(M)}+AR)\ . \tag{4.5}$$

We can now define the minimal renormalization in this flat conformal geometry as

$$\langle T_{\mu\nu}\rangle^{\text{ren min}}=\langle 0|T_{\mu\nu}|0\rangle-\langle T_{\mu\nu}\rangle^{(G_1^{\text{CF}})} \tag{4.6}$$

from (4.5) we see that in the massless case $\langle T^\mu_\mu\rangle^{\text{ren min}}=0$; i.e., there is no trace anomaly. This renormalization is state dependent because we have chosen a particular state $|q\rangle$ to make the subtraction (a vacuum defined using the conformally flat geometry). Notice, that we used different vacua in Sec. III, and now we use only one. Even so we will obtain the same renormalized values.

From Eqs. (2.22) and (4.6) we have

$$\begin{aligned}\langle T_{\mu\nu}\rangle^{\text{ren Had}}&=\langle T_{\mu\nu}\rangle^{\text{ren min}}-\langle T_{\mu\nu}\rangle^{(G_1^{\text{CF}})}+\langle T_{\mu\nu}\rangle^{(G_1)}\\ &=\langle T_{\mu\nu}\rangle^{\text{ren min}}-\langle T_{\mu\nu}\rangle^{\text{ren Had}}_{(G_1^{\text{CF}})}\ ,\end{aligned} \tag{4.7}$$

where

$$2880\pi^2\langle T_{\mu\nu}\rangle^{\text{ren Had}}_{(G_1^{\text{CF}})}=180Am^2G_{\mu\nu}+RR_{\mu\nu}-\tfrac{1}{3}R_{;\mu\nu}-R_{\mu\theta}R^\theta_\nu+\tfrac{1}{4}g_{\mu\nu}(2R_{\theta\rho}R^{\theta\rho}+\tfrac{4}{3}\Box R-\tfrac{4}{3}R^2)\ . \tag{4.8}$$

Now if we compute this tensor in Robertson-Walker universes we obtain

$$\begin{aligned} 2880\pi^2\langle T_{00}\rangle^{\text{ren Had}}_{(G_1^{\text{CF}})}&=540Am^2\alpha_1+3\beta_1-3\beta_2+6\beta_3+6\beta_4\ ,\\ 2880\pi^2\langle T_{ij}\rangle^{\text{ren Had}}_{(G_1^{\text{CF}})}&=\tfrac{1}{3}g_{ij}[540Am^2(\alpha_1-2\alpha_2)+15\beta_1-3\beta_2-6\beta_3-12\beta_4-6\beta_5]\ ,\end{aligned} \tag{4.9}$$

where the coefficients α_i and β_i are listed in the Appendix. Thus if we compare Eqs. (3.11), (3.12), (4.7), and (4.9) we see that the minimal renormalization introduced in this section is equivalent to the one in Sec. III if

$$A=-\tfrac{1}{18} \tag{4.10}$$

and therefore from Eq. (4.1) in the massless case we have $X=-R/18$. This is precisely the value $[w_0]=X$ that has to be taken to produce the propagator:

$$\mathscr{D}_1(x,x')=\frac{a^{-1}(\eta)a^{-1}(\eta')}{2\pi^2[(\eta'-\eta)^2+(\mathbf{x}'-\mathbf{x})^2]}\ , \tag{4.11}$$

i.e., the propagator of the conformal vacuum in Robertson-Walker universes (cf. Ref. 3). Thus minimal renormalization is the natural renormalization in a conformally flat geometry: the parameters of its propagator are built with the objects of the geometry, and the only coefficient left undetermined is fixed by the condition that the $G_1^{\rm CF}$ turns out to be the one that corresponds to the conformal vacuum in the massless limit. The result arises naturally if, as in Ref. 19, the renormalization scheme is performed after the rescaling of the matter field using this conformal factor. Furthermore, the rescaling seems to be fundamental for the quantum analysis, as was shown in Ref. 20. Also, this rescaling appears to be necessary in this framework of the self-consistent approach described in Refs. 6 and 7.

Now we can reobtain these results using the language of Ref. 1. First let us observe that in the case $m=0$, $\xi=\frac{1}{6}$:

$$\begin{aligned}
&16\pi^2\langle T_{\mu\nu}\rangle^{(G_1);\nu}=0\ ,\\
&16\pi^2\langle T^\mu_\mu\rangle^{(G_1)}=-T\\
&\qquad=-\tfrac{1}{180}(R_{\theta\rho\tau\epsilon}R^{\theta\rho\tau\epsilon}-R_{\theta\rho}R^{\theta\rho}+\Box R)\ ,
\end{aligned}\tag{4.12}$$

where we have used Eqs. (2.21), (2.17), (2.18), and Ref. 15. Thus our solution of Sec. II is a solution of the problem stated in Ref. 1, Eq. (2.22) with

$$t_{\mu\nu}=-(8\pi^2)\langle T_{\mu\nu}\rangle^{(G_1)}\ . \tag{4.13}$$

On the contrary the $\langle T_{\mu\nu}\rangle^{(G_1^{\rm CF})}$ of this paragraph is not a solution because

$$\begin{aligned}
&16\pi^2\langle T_{\mu\nu}\rangle^{(G_1^{\rm CF});\nu}=0\ ,\\
&16\pi^2\langle T^\mu_\mu\rangle^{(G_1^{\rm CF})}=0\ .
\end{aligned}\tag{4.14}$$

Our conformal, anomaly-free, minimal renormalization recipe consists in the subtraction (4.6); therefore it is based in the use of the traceless Hadamard tensor $\langle T_{\mu\nu}\rangle^{(G_1^{\rm CF})}$. This tensor can be obtained using the results of Ref. 1 from its solution for the conformally flat case [see the Appendix of Ref. 1, Eq. (A3)]:

$$\begin{aligned}
t_{\mu\nu}&=\tfrac{1}{360}S_{\mu\nu}\\
&=\tfrac{1}{360}[-R_{\mu\lambda}R^{\lambda}_{\ \nu}+RR_{\mu\nu}-\tfrac{1}{2}R_{;\mu\nu}\\
&\qquad+\tfrac{1}{6}g_{\mu\nu}(3R_{\theta\rho}R^{\theta\rho}-2R^2+2\Box R)]
\end{aligned}\tag{4.15}$$

which, using $T_{\mu\nu}=(8\pi^2)^{-1}t_{\mu\nu}$, yields our tensor $\langle T_{\mu\nu}\rangle^{\rm ren\ Had}_{(G_1^{\rm CF})}$ in the case $m=0$, $\xi=\frac{1}{6}$. Then we can obtain

$$\langle T_{\mu\nu}\rangle^{(G_1^{\rm CF})}$$

[see Eq. (4.7)] as

$$\langle T_{\mu\nu}\rangle^{(G_1^{\rm CF})}=\langle T_{\mu\nu}\rangle^{\rm ren\ Had}_{(G_1^{\rm CF})}-\langle T_{\mu\nu}\rangle^{(G_1)}\ . \tag{4.16}$$

Thus, the traceless Hadamard tensor $\langle T_{\mu\nu}\rangle^{(G_1^{\rm CF})}$ can be considered as the difference between the solution of Ref. 1 and our solution. This idea could be generalized for a general metric, obtaining a traceless Hadamard tensor for every background but as the tensor of Ref. 1 is not continuous when we pass from one type of metric to the other we will obtain an unpleasant noncontinuous renormalization method.

Another interesting example of a state-dependent renormalization is the renormalization of $\langle\phi^2\rangle$ in the de Sitter space-time done in Ref. 21. Even if it is done with an *ad hoc* procedure, that can only be used in this space, it has relevant physical consequences (cf. Ref. 22).

Therefore we cannot disregard the state-dependent renormalization methods just because they have no covariant known generalization to all kinds of geometries. We must rather use the physical consequences of different renormalizations as a criterion to see which one is better.

Finally the absence of the trace anomaly in the minimal renormalization was clearly stated in this paragraph. However we can find in the literature some interpretations that talk about the appearance of an anomalous term in $\langle T^\mu_\mu\rangle^{\rm ren\ min}$ (see Refs. 23 and 24). This term is found by doing a perturbative expansion of the unrenormalized $\langle T^\mu_\mu\rangle$ in terms of m^{-1}. The result after the subtraction (4.6) is

$$\langle T^\mu_\mu\rangle^{\rm ren\ min}=-P^\mu_\mu+O(m^{-4})\ . \tag{4.17}$$

This equation which is valid for large values of the mass is interpreted in Refs. 23 and 24 as the origin of some antigravitational effects that occur when the mass of the scalar field is $m^2>288\pi^2\kappa^{-1}$. Using the usual Hadamard renormalization Eq. (4.17) can never be obtained. In this case the anomaly takes place when we work in the massless limit and the anomaly is given by

$$\langle T^\mu_\mu\rangle^{\rm ren\ Had}=P^\mu_\mu(m=0) \tag{4.18}$$

[note the difference in the sign of P^μ_μ in (4.17)].

V. SELF-CONSISTENT DE SITTER SOLUTIONS

With both renormalization methods the de Sitter universe turns out to be a self-consistent cosmological solution of Einstein's equations (2.28), namely, a solution to the back-reaction problem. In each case the curvature of the de Sitter space is a function of the mass, but the function that we obtain using one method is different from the one we obtain using the other, showing that the difference between the methods is not only mathematical.

For the Hadamard renormalization, Eqs. (2.28) are equivalent to (cf. Ref. 2)

$$\begin{aligned}
R=\frac{m^4G}{\pi}&\left[\frac{\psi(\frac{3}{2}+\nu)+\psi(\frac{3}{2}-\nu)}{2}-\ln M\right]\\
&-\frac{m^2RG}{36\pi}+\frac{R^2G}{12\pi 360}\ ,
\end{aligned}\tag{5.1}$$

where $R=12/r^2$, r is the radius of the de Sitter universe, $M=mr$, $\nu^2=(\frac{1}{4}-m^2r^2)$, and $\psi(z)$ is the digamma function. In this case all the curvatures satisfy

$$R\geq 4320\pi G^{-1}\equiv R_0\ \ \forall m \tag{5.2}$$

and the solution exists for every value of the mass. The term $m^2RG/36\pi$ can be eliminated from Eq. (5.1) by a finite renormalization of the gravitational constant. If we do so the result is rather different: R is always lower than $4320\pi G^{-1}$ and the de Sitter solution exists only if the mass is lower than a threshold value. This result was obtained by Anderson (cf. Ref. 25). We think that this "massive polarization term" can be absorbed into a renormalization of the gravitational constant only if it is present in all the possible background metrics. We are not sure about this point: up to now the term $m^2G_{\mu\nu}/288\pi^2$ has been detected in $P_{\mu\nu}$ when one works in Robertson-Walker or Kasner metrics only.

On the contrary for the minimal renormalization the equation which relates R and m is (cf. Refs. 8 and 26)

$$R=\frac{m^4G}{\pi}\left[\frac{\psi(\frac{3}{2}+\nu)+\psi(\frac{3}{2}-\nu)}{2}-\ln M\right] . \tag{5.3}$$

In this case, in order to have a self-consistent solution the mass of the field must satisfy

$$m\geq 6\sqrt{\pi}G^{-1/2}\equiv m_0 \tag{5.4}$$

(here the curvature R can take values in the interval $[0,+\infty))$.

The differences between the two solutions are shown in Fig. 1, where we use dimensionless axis m/m_0 and R/R_0. We see that the two methods are not different prescriptions of the same renormalization but, in fact, they are physically different renormalizations.

VI. THE STABILITY OF THE SELF-CONSISTENT SOLUTIONS

In both cases the Minkowski space is a trivial solution of Eqs. (2.28). There are some works that study the behavior of this solution when we introduce a global conformal perturbation in the metric (cf. Refs. 27 and 28).

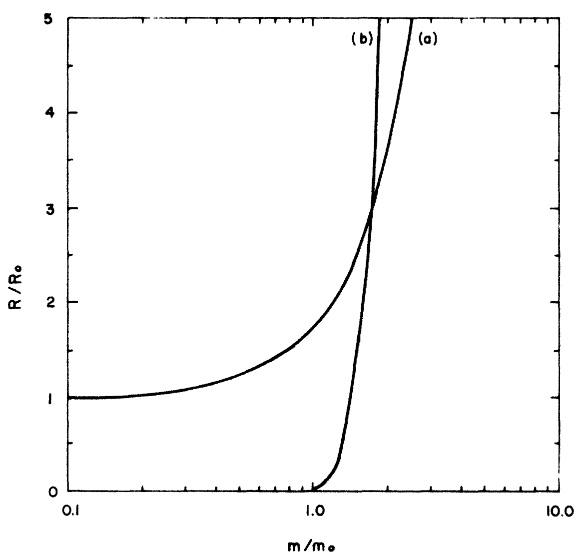

FIG. 1. The curvature of the de Sitter self-consistent solution vs the mass of the scalar field. The curve (a) corresponds to the Hadamard renormalization and (b) to the minimal renormalization. We have $R_0=4320\pi M_{\text{Planck}}{}^2$ and $m_0=6(\pi)^{1/2}M_{\text{Planck}}$.

The result of the analysis depends on the renormalization that we adopt.

Let us suppose that we write the metric as

$$\begin{aligned} &ds^2=a^2(\eta)(-d\eta^2+dx^2+dy^2+dz^2) ,\\ &a(\eta)=1+\delta(\eta) ,\\ &\delta(\eta)=0,\quad \eta<\eta_0 . \end{aligned} \tag{6.1}$$

We will first treat the massless case in which we can see how the different results appear. In this case if we use the minimal renormalization procedure we have $\langle T_{\mu\nu}\rangle^{\text{ren min}}=0$ and the Minkowski space remains stable as in the classical case. On the contrary if we study the same problem using the usual Hadamard renormalization method the result is that $\langle T_{\mu\nu}\rangle^{\text{ren Had}}=P_{\mu\nu}(m=0)$. If we write the (0,0) component of this equation we obtain the expression

$$\begin{aligned} &H\ddot{H}=\frac{k}{2k'}H^4-\frac{1}{2k'}H+\tfrac{1}{2}(\dot{H}^2-6\dot{H}H)+\frac{\Lambda}{6k'} ,\\ &k=\frac{\kappa}{2880\pi^2},\quad k'=-\left[6\alpha+2\beta+\frac{1}{2880\pi^2}\right]\kappa , \end{aligned} \tag{6.2}$$

where an overdot denotes d/dt.

As it was noted in Ref. 29 (see also Ref. 30) the Minkowski space is an unstable solution of this equation [in a phase space $(H,\dot{H})$ diagram we can see that all the trajectories that pass through $H=0$ must satisfy $\dot{H}=0$ and then the oscillations around the Minkowski space are not allowed]. This point is not clear in the literature where the Minkowski space is seen as a stable solution of (6.2) [for example in Refs. 22 and 31 this conclusion is obtained linearizing Eq. (6.2) in a wrong way].

Thus the Mikowski space turns out to be stable or unstable (in the massless case) if we use one renormalization or the other. This fact shows again that the two renormalizations are physically different.

The same problem has been studied for the massive case (see Refs. 27 and 28) but we think that the results presented in those papers could be erroneous (including ours). In both articles authors study the stability of the Minkowski space using the Einstein equation for the trace which is one order higher than the one for the (0,0) component. The stability behavior should be studied using the lowest-order equation. Otherwise we are not sure that the solutions we obtained are solutions for all Einstein equations (in fact, the same problem appears when we study the Minkowski stability classically). For the minimal renormalization it is proved that the trace equation and the (00) equation give the same result, i.e., that Minkowski space is unstable if the mass is bigger than a threshold mass $[KM_0{}^2\geq 288\pi^2$ (Refs. 32 and 33)]. But with the Hadamard renormalization the analysis is rather complicated if we work with the (0,0) component (we will discuss this problem elsewhere). However, these considerations are sufficient in order to show the physical inequivalence of the two renormalizations.

The behavior of the fluctuations around the de Sitter solution is an interesting problem. The result again depends on the renormalization procedure that we adopted.

With the minimal renormalization approach the result is that if the mass of the scalar field is higher than m_0 the de Sitter solution exists and is stable under global conformal fluctuations of the metric.[24] With the Hadamard approach the de Sitter solution exists for every value of the mass and its stability depends only on the value of the coefficients α and β of (2.28) (at least in the massless case).[30]

VII. CONCLUSIONS

We have studied two renormalizations using a common language. We have established their virtues and problems. We have reached the conclusion that the two renormalizations are physically different because they yield a different behavior in several physical problems. We cannot tell which one is the "good" renormalization. Nevertheless we can foresee that we will obtain different scenarios for the early Universe if we use one or the other renormalization. Thus a close study of the primordial phases in the Universe evolution critically dependent on the choice of the renormalization scheme, will eventually allow us to solve this problem in the near future.

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APPENDIX

The vacuum-polarization tensor can be written as

$$288\pi^2 P_{\mu\nu} = -m^2 G_{\mu\nu} - \tfrac{1}{60}[{}^{(1)}H_{\mu\nu} + \tfrac{1}{2} g_{\mu\nu}(R^2 - 6R^{\theta\rho}R_{\theta\rho})] . \quad \text{(A1)}$$

If we work in a Robertson-Walker (spatially flat) metric this tensor can be expressed as

$$2880\pi^2 P_{00} = -30m^2\alpha_1 + 3\beta_1 - 3\beta_2 + 6\beta_3 + 6\beta_4 ,$$

$$2880\pi^2 P_{ij} = \tfrac{1}{3} g_{ij}[-30m^2(\alpha_1 - 2\alpha_2) + 15\beta_1 - 3\beta_2 - 6\beta_3 - 12\beta_4 - 6\beta_5] ,$$

$$\alpha_1 = H^2, \quad \alpha_2 = R/6 ,$$

$$\beta_1 = \alpha_1{}^2, \quad \beta_2 = \alpha_2{}^2, \quad \beta_3 = \alpha_1\alpha_2 ,$$

$$\beta_4 = H\dot{\alpha}_2, \quad \beta_5 = \ddot{\alpha}_2 ,$$

where an overdot denotes d/dt.

[1] M. R. Brown and A. C. Ottewill, Proc. R. Soc. London **A389**, 379 (1983).

[2] N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, England, 1982).

[3] M. Castagnino and D. Harari, Ann. Phys. (N.Y.) **152**, 85 (1984).

[4] R. M. Wald, Commun. Math. Phys. **54**, 1 (1984).

[5] G. T. Horowitz and R. M. Wald, Phys. Rev. D **17**, 414 (1978).

[6] P. van Nieuwenhuizen, Phys. Rep. **68**, 189 (1981).

[7] R. Brout, F. Englert, and E. Gunzig, Ann. Phys. (N.Y.) **115**, 78 (1978); Gen. Relativ. Gravit. **10**, 1 (1979).

[8] R. Brout, F. Englert, J-M. Frere, E. Gunzig, P. Nardone, and C. Truffin, Nucl. Phys. **B170**, 228 (1980).

[9] A. A. Grib, S. G. Mamaeev, and V. M. Mostepanenko, Fortschr. Phys. **28**, 173 (1980).

[10] A. A. Grib, S. G. Mamaeev, and V. M. Mostepanenko, J. Phys. A **13**, 2057 (1980).

[11] S. G. Mamaeev and V. M. Mostepanenko, Zh. Eksp. Teor. Fiz. **78**, 20 (1980) [Sov. Phys. JETP **51**, 9 (1980)].

[12] A. A. Grib, S. G.Mamaeev, and V. M. Mostepanenko, in *Quantum Gravity,* edited by M. A. Markow and P. C. West (Plenum, New York, 1984).

[13] D. Harari, Ph.D. thesis, Buenos Aires University, 1983 (unpublished).

[14] M. Castagnino, D. Harari, and C. Nunez (unpublished).

[15] S. L. Adler, J. Liberman, and Y. J. Ng, Ann. Phys. (N.Y.) **106**, 279 (1977).

[16] R. M. Wald, Phys. Rev. D **17**, 1477 (1978).

[17] M. Castagnino, Gen. Relativ. Gravit. **15**, 1149 (1983).

[18] A. G. Najmi and A. C. Ottewill, Phys. Rev. D **30**, 1733 (1984).

[19] M. Castagnino, D. Harari, and J. P. Paz (unpublished).

[20] K. Fujikawa, Phys. Rev. D **23**, 2262 (1981).

[21] A. Vilenkin and L. H. Ford, Phys. Rev. D **26**, 1231 (1982).

[22] R. H. Brandenberger, Rev. Mod. Phys. **57**, 1 (1985).

[23] E. Gunzig and P. Nardone, Phys. Lett. **134B**, 412 (1984).

[24] E. Gunzig and P. Nardone, Class. Quantum Gravit. **2**, L47 (1985).

[25] P. Anderson, Florida University report, 1985 (unpublished).

[26] P. Spindel, Phys. Lett. **107B**, 361 (1981); E. Gunzig and P. Nardone, Gen. Relativ. Gravit. **16**, 305 (1984).

[27] M. Castagnino and J. P. Paz, Phys. Lett. **164B**, 274 (1985).

[28] E. Gunzig and P. Nardone, Phys. Lett. **118B**, 324 (1982); B. Biran, R. Brout, and E. Gunzig, *ibid.* **125B**, 339 (1983).

[29] T. Azuma and S. Wada, UT-Komaba report, 1983 (unpublished).

[30] P. Castagnino and J. P. Paz, in Proceedings of the Summer Meeting on Quantum Mechanics of Fundamental Systems, edited by C. Teitelboim (Plenum, New York, to be published).

[31] A. Vilenkin Phys. Rev. D **32**, 2511 (1985).

[32] P. Nardone, Class. Quantum Gravit. **3**, 453 (1986).

[33] A. A. Starobinski, in *Quantum Gravity,* Ref. 12.